THREE QUANTIFIERS AND A MODULUS

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n > N, |x_n - l| < \epsilon$$

Real Analysis for those who find it hard going

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Chapter 1

Introduction

Why the strange title? What is the meaning of

$$\text{“}\forall \epsilon > 0, \exists N \text{ s.t. } \forall n > N, |x_n - l| < \epsilon\text{” ?}$$

And why do most people find Real Analysis difficult?

I'll start by trying to answer the third question, and that will take us back to the other two questions. Real Analysis has been described as "Calculus with attitude". The "attitude" is "argumentative", best summed up in the response "prove it". Sometimes in Real Analysis we spend a lot of effort to prove things that look pretty obvious. It isn't initially clear why we do this. It only starts to become clearer when we start to prove things that aren't at all obvious, or disprove things that look as though they should be obvious. Unfortunately we don't get to that point very quickly. I can give you an example (without proof at this stage): when you add up a finite number of numbers, the order doesn't matter, you can take the numbers in any order and the total is always the same. But this isn't always the case with an infinite collection of numbers: the order in which you add the numbers can make a difference.

So Real Analysis is about trying to be precise and and avoiding appeals to intuition in proofs. There's nothing wrong with intuition and drawing diagrams to illustrate what is happening. But we don't accept these things as proofs, and that is where the trouble starts for most people. Precise proofs also require precise definitions, so that we know what we are trying to prove. We are going to start by studying sequences of numbers and what it means to say that an infinite sequence of numbers converges to a limit. Sequences probably provide the simplest types of convergence behaviour, so this is a good place to start before getting on to continuity, differentiation and integration. An infinite sequence is an ordered list of numbers, one number for each positive integer. We denote the sequence $x_1, x_2, x_3, \ldots$ as (x_n). For example, if $x_n = \frac{n}{n+1}$ then $x_2 = \frac{2}{3}$.

In theory you could do a course of Real Analysis before ever encountering things like differentiation, but that wouldn't be a good starting point for most people. So I will assume that you have already had some exposure to Calculus, and that you have some familiarity with the usual properties and types of numbers: Integers, Rationals (fractions), and Real Numbers (decimals). The reason Real Analysis is called *Real* Analysis is that it is focused on the Real Numbers. By contrast, Complex Analysis is focused on the Complex Numbers. Paradoxically, despite its title, many people seem to find Complex Analysis easier than Real Analysis, perhaps because many of the definitions and proofs are similar to what they have already seen in Real Analysis, so the methods and attitude are no longer such a culture shock.

So Real Analysis emphasises precision and that starts with precision in definitions. If I say that $\frac{n}{n+1}$ tends to 1 as n tends to ∞, you will probably (I hope) agree with me, particularly if I illustrate the statement with the picture in Figure 1.1. Here we've put the positive integers $1, 2, 3, \ldots$ along the horizontal n-axis, and represented the corresponding values of $\frac{n}{n+1}$ by drawing dots at the appropriate vertical heights. The picture (suggestively) also shows a horizontal line at a vertical height of 1 unit. You can see that the dots get closer and closer to the line as n increases, i.e. as we move our view towards the right-hand side of the picture. If you went to $n > 100$, the dots would appear to be *on* the line, but of course $\frac{n}{n+1} < 1$ for every positive integer n, so the dots are always below the line.

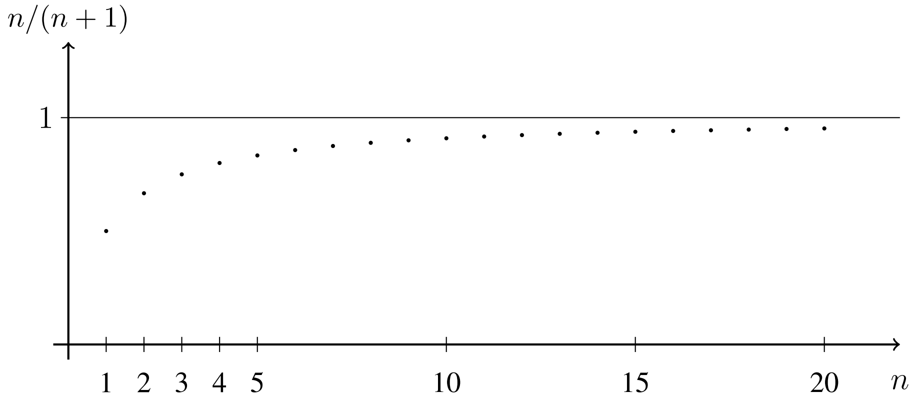


Figure 1.1: $\frac{n}{n+1}$ tends to 1 as n tends to ∞.

The expression $\frac{n}{n+1}$ generates a very "smooth" sequence, making it easy to draw a diagram and see what is happening. But it isn't so easy with less well-behaved sequences. Imagine having to deal with $\frac{5n^2-7n+99}{2n^2-n+1}$. Even that isn't too bad but it would require a lot of computation to draw the diagram and be really convinced that the expression tends to $5/2$ as n tends to infinity. Now imagine trying the same thing with $\pi(n)\log(n)/n$, where $\pi(n)$ is the number of prime numbers less than n and $\log(n)$ is the natural logarithm of n. The behaviour of $\pi(n)$ is rather erratic as n increases, and it isn't obvious whether or not there is any limiting value of $\pi(n)\log(n)/n$ as n tends to ∞. Even if I tell you it tends to 1 (and the result is called the prime number theorem), how would we prove it? What's needed is a precise definition of convergence for sequences.

A precise definition has to satisfy at least three criteria. First, it must not use vague terms like "small" and "large" because 10^{-100} may look small, but compared with 10^{-1000} it is enormous. So "small" and "large" are too vague. Even more vague are terms like "infinity" and "infinitesimal" because we will be dealing with Real Numbers and no Real Numbers have these attributes (whatever they may mean). A second criterion is that the definition must accord with our intuition in normal circumstances. A definition of convergence which asserted that the sequence formed from $\frac{n}{n+1}$ did not converge, or converged to 2, would not be acceptable. Finally, as a third criterion, the definition should provide us with a way to determine if a sequence does or does not converge to a particular limit.

A way around the problem of "small" positive numbers is to refer to *all* positive numbers, which certainly includes all small ones, however you personally choose to define "small". The same idea works for "large" as well. Funnily enough, although "small" and "large" are fairly meaningless, "smaller" and "larger" are quite respectable, since we can compare two positive Real Numbers a and b. If $a < b$ we say that a is smaller than b, and that b is larger than a. While we will avoid the use of "small" and "large" in proofs and definitions, intuition is a different matter. So if you find it helpful to *think* in terms of small and large numbers (and most of us do), feel free to continue.

Now look again at Figure 1.1. What do we mean when we say the dots get closer and closer to the line? Suppose we want to get to within 0.01 of the line. Clearly there will be some point on the horizontal axis, say N, beyond which all the remaining dots (i.e. the ones to the right of N) lie within 0.01 of the line. In algebraic terms we are looking for a value N such that for every $n > N$ the difference between 1 and $\frac{n}{n+1}$ is less than 0.01. It is very easy to find such a value N because the difference between 1 and $\frac{n}{n+1}$ is just

$$1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1}.$$

So if we want this difference to be less than 0.01, all we have to do is make sure that $n + 1 > 100$. We can take our value N to be anything that ensures that this is the case for every $n > N$. For example, we could take $N = 99$, but $N = 1000$ would also do the job. Indeed, if we did take $N = 1000$ and $n > N$, then the difference between 1 and $\frac{n}{n+1}$ would be less than 0.001. In the case of this sequence it seems we can get as close as we like to the line representing the limit. More precisely, if we want to get within ϵ of the line (where $\epsilon > 0$), then we can find a corresponding N, beyond which all the remaining dots lie within ϵ of the line, i.e. $\frac{1}{n+1} < \epsilon$. Clearly $N = \frac{1}{\epsilon}$ will do this job because if $n > N = \frac{1}{\epsilon}$ then $\frac{1}{n} < \epsilon$ and consequently $\frac{1}{n+1} < \epsilon$.

Before considering an arbitrary sequence, look at the logical structure of what we just did in the previous paragraph. We proved that

for every $\epsilon > 0$, *there exists* some value N such that *for every* $n > N$, the difference between 1 and $\frac{n}{n+1}$ is less than ϵ.

The phrases "for every" and "there exists" are known as *quantifiers*. The sentence is logically complicated because it contains three quantifiers, and their order is important. So N can depend on ϵ, and n can depend on N. The definition of convergence for an arbitrary sequence follows the same pattern. It contains the same three quantifiers in the same positions. In place of $\frac{n}{n+1}$ we have the general term x_n of the sequence in question, and in place of 1 we have the proposed limit l. The difference between l and x_n is given by $|x_n - l|$, where I remind you that $|a|$ denotes the *modulus* or *absolute value* of a, which may be defined as

$$|a| = +\sqrt{a^2} = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Figure 1.2 attempts to illustrate the general situation. We have a sequence of numbers $x_1, x_2, x_3, \ldots$ represented as dots on the diagram. (In the previous example x_n was $\frac{n}{n+1}$.) These dots gradually settle around the line at height l as we move to the right-hand side. The shaded strip has lower boundary at height $l - \epsilon$ and upper boundary at height $l + \epsilon$. From some point N onwards (i.e. for every $n > N$) the points of the sequence lie inside the shaded strip (i.e. $|x_n - l| < \epsilon$). If we were to reduce the value of ϵ, thereby making the shaded strip narrower, we might reasonably expect to have to move N to the right. Roughly speaking we are saying that the difference between x_n and l can be made as small as we like (i.e. less than any $\epsilon > 0$) by taking n sufficiently large (i.e. greater than N, for some N).

Our definition of what we mean by saying that "$x_n \to l$ as $n \to \infty$" is that for any sized strip (characterized by $\epsilon > 0$) there exists a number N such that for every $n > N$, $|x_n - l| < \epsilon$.

That then is the explanation for the title of this book: "Three quantifiers and a modulus". The symbols $\forall$ and $\exists$ that appear in the subtitle are just shorthand for the quantifiers. The symbol $\forall$ can be read as "for every" or "for all" and it is called a *universal* quantifier - in typographic terms it is an A rotated by 180 degrees and stands for ALL. The symbol $\exists$ can be read as "there exists" or "there is" and it is called an *existential* quantifier - in typographic terms it is an E rotated by 180 degrees and stands for EXISTS. So the string of symbols

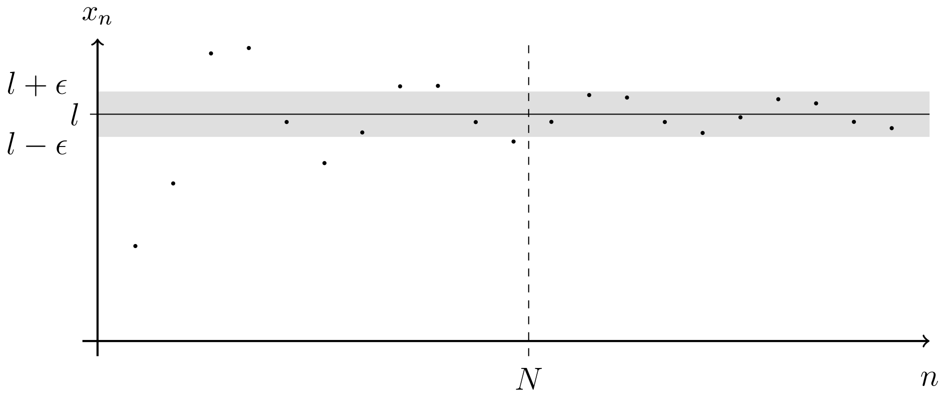


Figure 1.2: x_n tends to l as n tends to ∞.

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n > N, |x_n - l| < \epsilon$$

reads as

for every $\epsilon > 0$, there exists N such that for every $n > N$, $|x_n - l| < \epsilon$.

In either form this is our definition of what we mean by saying:

the sequence x_n converges to the limit l as n tends to infinity,

or, in symbols:

$$x_n \to l \text{ as } n \to \infty.$$

This can also be worded as "x_n tends to l as n tends to infinity" and written as $\lim_{n\to\infty} x_n = l$. If the sequence S is given implicitly without explicit use of n, we can write $\lim S = l$: for example, $\lim(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots) = 1$.

There is no attempt in this definition to attach a meaning to "infinity" and certainly no acceptance of ∞ as any sort of number. It might have been better to speak of n getting "arbitrarily large", although even that is somewhat vague. But we are stuck with the commonly accepted terminology.

The development of this definition of convergence of a sequence, and related definitions covering infinite series, continuity and differentiability took a very long time. Well over 2,000 years ago Greek mathematicians, such as Archimedes, were already using notions of convergence to obtain formulas for areas and volumes of geometrical shapes including the circle and the sphere. There were many attempts over the following two millennia to make the definitions precise. It seems that the earliest versions of what we use today were given by Bernard Bolzano in the 1810s, but these were not generally used until taken up by Karl Weierstrass in the 1870s. The fact it took so long to develop indicates that it is far from easy or obvious. So if you find it hard, don't be discouraged.

There is, of course, nothing magic about the symbols used such as ϵ, N, n, x_n and l. Here is the definition in another form:

$$\forall \delta > 0, \exists M \text{ s.t. } \forall r > M, |y_r - Y| < \delta$$

And this defines what we mean by saying that $y_r \to Y$ as $r \to \infty$. It is just a convention to use ϵ for a number that can be imagined to be arbitrarily small and N for a number imagined to be correspondingly large (but bear in mind we never attach any mathematical consequences to such imaginings).

We also emphasised that we will be precise in definitions and proofs. But we will avoid being pedantic. To explain the difference, look again at the definition in its symbolic form. We wrote $\forall \epsilon > 0 \ldots$. We didn't say that ϵ can be any (positive) *Real Number*: that was understood. Similarly, we wrote $\forall n > N \ldots$. We didn't say that n has to be a *positive Integer*: again that was understood because we are referring to an infinite sequence (x_n) where n is restricted to positive Integer values. What about N, should that be an Integer or will any suitable Real Number suffice? Actually it doesn't matter - if you really want an integer value and the N you have isn't an integer, just increase N to the next integer above the existing value. In summary, we won't be pedantic when using a quantifier when it's reasonably obvious to which class of numbers the quantified object must belong.

We will get on to *using* the definition of convergence of sequences in Chapter 3. But first we will cover some preliminary items. These form our next Chapter and include the use of quantifiers and how they can make life easier, properties of the Real Numbers especially the completeness property, proof by induction, and some useful inequalities.

Chapter 2

Preliminaries

2.1 Notation

Here we collect together the symbols used throughout this book and examples of their usage. If you got as far as a Calculus course before attempting Real Analysis, you will probably be familiar with most of these. We start with some terms borrowed from logic and set theory.

We use curly brackets to denote a set. For example $\{1, 2, 3\}$ denotes the set that contains the numbers $1, 2$ and 3. If we give this set a name, say $S = \{1, 2, 3\}$, then we use the symbol $\in$ to denote membership. We could write $2 \in S$ and read this as saying "2 is a member of S". We also use strike-through to negate membership as in $5 \notin S$, read as "5 is not a member of S". Sets aren't restricted to numbers. We might write "Everest $\in$ set of all mountains". The members of a given set are often called its *elements*. Rather than listing the elements, a set is often defined by some property that describes its members. For example, $S = \{x : x \text{ is a prime number and } x < 1000\}$ is the set of prime numbers less than 1000.

If S and T are two sets and every member of S is also a member of T, then we say that S is a *subset* of T and write this as $S \subseteq T$. The symbol $\subseteq$ denotes subset inclusion. The set without any members is called the *empty set* and is usually denoted by $\emptyset$. The empty set is a subset of every set, and every set is a subset of itself. Other subsets of S are called *proper* subsets, while $\emptyset$ and S are called *improper* subsets of S.

For two sets S and T, we can form their *union* $S \cup T$, and their *intersection* $S \cap T$ by defining

$$S \cup T = \{x : x \in S \text{ or } x \in T\}, \qquad S \cap T = \{x : x \in S \text{ and } x \in T\}.$$

For example, if $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5, 6\}$ then $S \cup T = \{1, 2, 3, 4, 5, 6\}$ and $S \cap T = \{3, 4\}$. If we have a collection $\mathcal{C}$ of sets we can similarly form their union and their intersection:

$$\bigcup_{S \in \mathcal{C}} S = \{x : x \in S \text{ for some } S \in \mathcal{C}\}, \qquad \bigcap_{S \in \mathcal{C}} S = \{x : x \in S \text{ for every } S \in \mathcal{C}\}.$$

Some specific sets of numbers have specific symbols to act as their names. The set of all Natural Numbers is denoted by $\mathbb{N}$, so $\mathbb{N} = \{1, 2, 3, \ldots\}$. Some people like to include 0 as a Natural Number. I won't do that, instead I will denote the set $\{0, 1, 2, \ldots\}$ by the symbol $\mathbb{N}_0$. The set of all Integers is denoted by $\mathbb{Z}$. So $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$. This means that we can identify the Natural Numbers as the set of positive Integers. The use of the letter $\mathbb{Z}$ comes from the German word Zahl for number. The set of all Rational Numbers (i.e. fractions, both positive and negative) is denoted by $\mathbb{Q}$. The letter $\mathbb{Q}$ relates to "quotient", reflecting the fact that fractions are quotients of integers. These are called Rational Numbers because they are *ratios* of Integers. Finally for us, the set of all Real Numbers (which turns out to be the set of all decimals) is denoted by $\mathbb{R}$. Every Natural Number is also an Integer, every Integer n can be expressed as a Rational Number $n/1$, and every Rational Number has a decimal representation. So we have the inclusion relationship $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Implication symbols are sometimes useful. We may write $x = 2 \implies x^2 = 4$, read as "$x = 2$ implies $x^2 = 4$" (which is true). But don't use the symbol carelessly. It is definitely NOT true that $x^2 = 4 \implies x = 2$, since there is also the possibility that $x = -2$. Sometimes two statements are equivalent and then we might use $\iff$, read as "implies and is implied by", or more succinctly as "if and only if" (which some authors abbreviate to the horrible "iff"). An example of correct usage is: $x = 2 \iff x - 2 = 0$.

Another logical symbol is $\neg$, read as "not:". It is used to negate the statement which follows it. For example, $x^2 = 4 \implies \neg(x = 17)$ (if $x^2 = 4$ then x certainly isn't 17). We won't make extensive use of this symbol but, as we will see, it can be helpful when negating a complicated statement.

Remaining abbreviations are the universal and existential quantifiers that we met in the previous chapter: $\forall$ and $\exists$. We will play some games with them in the next section of this chapter. As you may have noticed already we also use s.t. as an abbreviation for "such that".

2.2 Sentence negation

Many proofs in analysis are proofs by contradiction - we assume the opposite of what we are trying to prove (its negation) and then go on to derive a contradiction. Such a process involves the problem of negating a given sentence. Negating a sentence that contains several quantifiers can be a daunting prospect. You need a cool head, but it can be done systematically. First, the definition.

Definition 2.1. The negation of a sentence S is the sentence $\neg S$. It is the statement which is necessarily true if S is false and necessarily false if S is true.

Whether or not S is true or false plays no role in constructing $\neg S$.

Example 2.1. S = "All swans are white".

The negation of S is the sentence "Not: all swans are white", or in acceptable English "It is false that all swans are white". Although this is the negation of S, it is not in its simplest or most useful form. The original sentence begins "All swans...", it is saying that "For all swans... such-and-such is true". To negate this we have to show that there is a swan for which such-and-such is NOT true. Thus the negation can be put in the form "There is a swan which is not white".

Quantifiers may be used to simplify the business of sentence negation. In the above example S may be written as: $\forall x \in$ the set of swans, $x \in$ the set of white objects. The negation says: $\exists x \in$ the set of swans s.t. $x \notin$ the set of white objects.

Roughly speaking "$\forall$" becomes "$\exists$" (and vice-versa) while "," becomes "s.t." (and vice-versa) - this depends on careful punctuation - and the statement following the quantifier is itself negated. Thus if $P(x)$ is a proposition about a Real Number x, the sentence "$\forall x, P(x)$" would have negation "$\exists x$ s.t. $\neg P(x)$".

Similarly "$\exists x$ s.t. $Q(x)$" negates to "$\forall x, \neg Q(x)$". Note that (since $\neg\neg\ P(x) = P(x)$) the negation of a negation simply results in the original sentence - this is quite a good way of checking that a negation is correct.

The sentence "All swans are black" in NOT the negation of "All swans are white", Since while the former is false if the latter is true, the former is not necessarily true if the latter is false. Likewise the sentence "There is a swan which is black" is not the negation because even if this were false, it would not prove that all swans were white (there might be blue swans).

Next we try a more complicated mathematical example involving two quantifiers.

Example 2.2. $S = \forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $y < x$.

To negate such a sentence we apply the rules for dealing with $\forall, \exists, ,$, s.t. outlined above. The sentence is of the form

"$\forall x \in \mathbb{R}, P(x)$", where $P(x) = \exists y \in \mathbb{R}$ s.t. $y < x$.

The negation may therefore be written as "$\exists x \in \mathbb{R}$ s.t. $\neg P(x)$" - of course this involves us with the sentence "$\neg P(x)$", i.e. the negation of "$P(x)$".

But $P(x) = \exists y \in \mathbb{R}$ s.t. $y < x$ and applying the rules once again we obtain $\neg P(x) = \forall y \in \mathbb{R}, \neg(y < x)$. Overall therefore we may write the negation of S

as $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, \neg(y < x)$. As a final simplification the final statement "$\neg(y < x)$" (i.e. "not: y is less than x" or "it is false that y is less than x") may be written as "$y \geq x$". We therefore obtain
$\neg S = \exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, y \geq x$.

In general we do not have to go through the process of defining an intermediate proposition $P(x)$, as we did above, provided we can visualise some brackets inserted in appropriate places. Thus

$$\begin{aligned} S &= \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } y < x \\ \neg S &= \exists x \in \mathbb{R} \text{ s.t. } \neg[\exists y \in \mathbb{R} \text{ s.t. } y < x] \\ &= \exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R}, \neg(y < x) \\ &= \exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R}, y \geq x \end{aligned}$$

Here is another example involving three quantifiers.

Example 2.3.

$$S = \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, y \text{ satisfying } |x - y| < \delta, |\sin x - \sin y| < \epsilon.$$

To negate this we insert brackets (mentally if possible):

$$S = \forall \epsilon > 0, [\exists \delta > 0 \text{ s.t. } \{\forall x, y \text{ satisfying } |x - y| < \delta, |\sin x - \sin y| < \epsilon\}].$$

The negation is then arrived at step-by-step.

$$\begin{aligned} \neg S &= \exists \epsilon > 0 \text{ s.t. } \neg[\exists \delta > 0 \text{ s.t. } \{\forall x, y \text{ satisfying } |x - y| < \delta, |\sin x - \sin y| < \epsilon\}] \\ &= \exists \epsilon > 0 \text{ s.t. } \forall \delta > 0, \neg\{\forall x, y \text{ satisfying } |x - y| < \delta, |\sin x - \sin y| < \epsilon\} \\ &= \exists \epsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x, y \text{ satisfying } |x - y| < \delta \text{ s.t. } \neg(|\sin x - \sin y| < \epsilon) \\ &= \exists \epsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x, y \text{ satisfying } |x - y| < \delta \text{ s.t. } |\sin x - \sin y| \geq \epsilon \end{aligned}$$

Note that the original statement really contains an abbreviation. When it says "$\forall x, y$ satisfying ..." should really say: "$\forall$ pairs (x, y) satisfying ...". The negation should likewise say "$\exists$ a pair (x, y) satisfying ...". Such abbreviations are common.

Example 2.3 also enables us to emphasise another point: the order of the quantifiers matters. The quantity δ in the statement S depends only on ϵ. If we were to alter the order of quantifiers, placing $\forall(x, y)$ *before* $\exists\delta$, then δ could vary with x and y. Tidying up the English a bit, the revised statement T reads as

$$T = \forall \epsilon > 0 \text{ and } \forall (x, y), \exists \delta > 0 \text{ s.t. if } |x - y| < \delta, \text{ then } |\sin x - \sin y| < \epsilon.$$

Statement T is a weaker statement than S. You'll have to wait until we discuss continuity and the trigonometric functions to see if T and S are true or false.

The purpose of analysis is to rid mathematics of contentious arguments, particularly those dealing with the concept of infinity. Thus phrases such as "For a very large $x \ldots$" or "For infinitesimal values of $z \ldots$" are unacceptable. They are imprecise and we will not use them. Our aim in this book is to build up, in a rigorously logical fashion, the basic results of Calculus. But we have to start somewhere. So in our development of the material we will accept *as given* the basic axioms of the Real Number system. These are set out in the next section, with particular attention given to the most significant and unfamiliar of the axioms. We will also accept normal logical reasoning such as proof by induction and proof by contradiction. However, to fulfil the aim of eliminating contentious arguments we will lean over backwards to avoid claiming that any result is "obvious".

Exercises for Section 2.2

1. Negate this statement: "There exist positive Integers p and q such that $p^2 = 2q^2$."
This statement is false. It is equivalent to saying that $\sqrt{2}$ is a rational number. In the next Section we provide a proof that $\sqrt{2}$ is not a rational number.

2. Negate the following statement P.
P= "For every positive Integer n, there exists an Integer $p > n$ such that p and $p + 2$ are both prime numbers."
P is known as the twin prime conjecture. We do not know (at the time of writing) which of P and $\neg P$ is true.

3. Negate the following statement Q.
Q= "There exists a positive Integer D such that for every positive integer n there exist prime numbers p and q greater than n such that the difference between p and q is at most D."
The case $D = 2$ corresponds to the twin prime conjecture. Whether Q is true or false was not known until a breakthrough made by Yitang Zhang in 2013 who proved that Q is true, with $D = 70\,000\,000$. Subsequently it has been shown that D can be taken as 246. Hopefully it will eventually get whittled down to $D = 2$.

2.3 Real Numbers

Axioms for the set of Real Numbers, $\mathbb{R}$
A. Field Axioms
$\mathbb{R}$ is a *field*. It has defined within it sum and product operations. If $a, b \in \mathbb{R}$ then the sum of a and b is written as $a + b$ and the product is written as ab. These operations have the following properties.

1. $\forall a, b \in \mathbb{R}, a + b \in \mathbb{R}$ and $ab \in \mathbb{R}$ (closure)
2. $\forall a, b \in \mathbb{R}, a + b = b + a$ and $ab = ba$ (commutativity)
3. $\forall a, b, c \in \mathbb{R}, a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$ (associativity)
4. $\forall a, b, c \in \mathbb{R}, a(b+c) = ab+ac$ (multiplication is distributive over addition)
5. $\exists 0 \in \mathbb{R}$ s.t. $\forall a \in \mathbb{R}, a + 0 = a$ (additive identity)
6. $\exists 1 \in \mathbb{R}$ $(1 \neq 0)$ s.t. $\forall a \in \mathbb{R}, a1 = a$ (multiplicative identity)
7. $\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R}$ s.t. $a + (-a) = 0$ (additive inverse)
8. $\forall a \in \mathbb{R}, (a \neq 0), \exists(a^{-1}) \in \mathbb{R}$ s.t. $a(a^{-1}) = 1$ (multiplicative inverse)

(Any structure satisfying Axioms A1 to A8 is called a field.)

B. Order Axioms
$\mathbb{R}$ is an *ordered* field. It has defined within it an order relation $<$ which satisfies:

1. If $a \in \mathbb{R}$ then either i) $a > 0$ (we say a is positive), or ii) $a = 0$, or iii) $(-a) > 0$ (we say a is negative)
2. $a > 0, b > 0 \implies a + b > 0$
3. $a > 0, b > 0 \implies ab > 0$

N.B. Subtraction is defined by taking $a - b$ to mean $a + (-b)$ and division is defined by taking a/b to mean $a(b^{-1})$ (when $b \neq 0$). Then $a > b$ is defined to mean $(a - b) > 0$. The other order relations $<, \ \geq, \ \leq$ may be defined in terms of $>$. So we define $a < b$ to mean that $b > a$ (i.e. $(b - a) > 0$). We define $a \geq b$ to mean that either $a > b$ or $a = b$, and we define $a \leq b$ to mean that either $a < b$ or $a = b$.

C. The Completeness Axiom
A non-empty collection of Real Numbers, S, which is bounded above, possesses a least upper bound.

An explanation of the Completeness Axiom is given below. The field and order axioms (A and B) are referred to as algebraic axioms, while C is analytical. From Axioms A and B it is possible to derive all the familiar algebraic results concerning Real Numbers. For example we can prove that if $a > b$ and $b > c$ then $a > c$. It's easy once you recognise that $a > b$ means $(a - b) > 0$ and likewise $b > c$ means that $(b - c) > 0$. Then by Axiom B2, $(a - b) + (b - c) > 0$, which simplifies to $(a - c) > 0$, and this can be expressed as $a > c$.

Most people don't have any great problems with the field axioms once they have learned that 0 doesn't have a multiplicative inverse, meaning that you cannot (must not!) divide by 0. But manipulating inequalities is a more common cause of errors. The thing to be most careful about is multiplying or dividing an inequality. If you multiply or divide the inequality $a > b$ by a negative number, you must be careful to change the sign $>$ to $<$. The reason is that $a > b$ if and only if $-a < -b$. This is easy to prove since $a > b$ means that $(a - b) > 0$, while $-a < -b$ means that $(-b - (-a)) > 0$, and the latter expression simplifies to $(a - b) > 0$. So both $a > b$ and $-a < -b$ mean the same thing. It is perhaps easy to recognise that $3 > 2$ since 3 lies to the right of 2 on the number line, while $-3 < -2$ because -3 lies to the left of -2. If we multiply $3 > 2$ by 5 we get $15 > 10$ which is correct, but if we multiply by -5 we must change the inequality sign to get $-15 < -10$. What is not so easy to recognise is that if we multiply $3 > 2$ by x, we cannot say that $3x > 2x$, unless we know that $x > 0$. Another warning is that if you multiply $a > b$ by 0, then you certainly don't get $0 > 0$ or $0 < 0$!!! The moral is to be very careful.

Sometimes we need to "reciprocate" inequalities. By this I mean that if $a > b$ then, provided that $ab > 0$, we may divide by ab and deduce that $\frac{1}{b} > \frac{1}{a}$. But the proviso is vitally important because if $ab < 0$ then $\frac{1}{b} < \frac{1}{a}$. An easy mistake to make can leave you 100% wrong! Be very, very careful. We'll come back to how inequalities are used in analysis, and they are used a lot. But before that we will have a closer look at the Completeness Axiom, which may well convey absolutely nothing to you unless you have seen it before.

2.3.1 The Completeness Axiom

You met the Natural Numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ when you first learned to count. Subsequently you will have been taught about fractions and about negative numbers. Each of the latter two classes of numbers arises from consideration of simple equations.

The equation $x + 3 = 5$ has the solution $x = 2$, which lies in $\mathbb{N}$. However, the similar looking equation $x + 5 = 3$ has no solution in $\mathbb{N}$. This points the need for negative numbers, and zero. Such considerations lead us to the Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Similarly, equations such as $5x = 3$ point the need for the Rational Numbers (fractions), $\mathbb{Q}$. It is not hard to see that the set of Rational Numbers satisfies all of the axioms A and B set out above. But, as we will show, Axiom C is not satisfied by the set of Rational Numbers. The purpose of Axiom C is to enable us to study the concept of a limit. Although it is not immediately obvious, this axiom ensures that we have enough numbers at our disposal to talk sensibly about limits - this would be impossible if we restricted ourselves to the Rational Numbers.

Let us look at the step from the Rational Numbers to the Real Numbers. Again we consider equations - for example $x^2 = 2$. This has no solution if we require x to be a Rational Number, a fact that was established by the Pythagoreans around 2,500 years ago.

Theorem 2.1. The positive square root of 2 cannot be expressed as the ratio of two Natural Numbers. So $\sqrt{2} \notin \mathbb{Q}$. We say that $\sqrt{2}$ is *irrational.*
[Note that *irrational* refers to a number not being a ratio of two Integers, rather than the more common English usage of the word as a synonym for "crazy".]

Proof. (Proof by contradiction.) The claim that $\sqrt{2}$ is irrational can be expressed as: $\forall p, q \in \mathbb{N}, (p/q)^2 \neq 2$. The negation of this sentence (expressing the claim that $\sqrt{2}$ is a Rational Number) is: $\exists p, q \in \mathbb{N}$ s.t. $(p/q)^2 = 2$. If this latter statement were true then we could certainly ensure that p and q have no common factors, simply by cancelling. So the strategy is to take the sentence

$$S : \exists p, q \in \mathbb{N} \text{ (without common factors) s.t. } (p/q)^2 = 2$$

and show that it leads to a contradiction.

Suppose that $x^2 = 2$ and that $x = p/q$ for Natural Numbers (positive integers) p and q that have no common factors, i.e. they are fully cancelled. Then $x^2 = p^2/q^2$ and so $p^2 = x^2q^2 = 2q^2$. Since $2q^2$ is divisible by 2, so is p^2 and so therefore is p. Hence $p = 2r$ where r is some integer. It follows that $4r^2 = 2q^2$ or $2r^2 = q^2$. It now follows that q is divisible by 2. Hence both p and q have a common factor, namely 2, and so p/q is not fully cancelled. But this is a contradiction. Thus our initial assumption is false and x cannot be a Rational Number if $x^2 = 2$. $\square$

Solution of such equations require the creation of a further set of numbers. Of course we can get closer and closer to $\sqrt{2}$ by considering the sequence of rounded down decimal approximations $1, 1.4, 1.41, 1.414, \ldots$. Each of these is a Rational Number, for example $1.41 = 141/100$. An algebraist might identify the new type of number, $\sqrt{2}$, with the defining sequence, $(1, 1.4, 1.41, 1.414, \ldots)$. Similarly they might identity π (which also turns out to be irrational), with the sequence $(3, 3.1, 3.14, \ldots)$.

Consider the sequence $(\frac{n}{n+1}) = (\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots)$. We feel we ought to be able to write $\lim(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots) = 1$ and, with a suitable definition of "lim", we have seen in the Introduction that we can do precisely this. But equally well we ought to be able to write $\lim(1, 1.4, 1.41, 1.414, ...) = \sqrt{2}$. However to write down such a statement we need to be able to talk about $\sqrt{2}$ - that is to say we need a system of numbers which doesn't have a gap at precisely the point where we would expect to find $\sqrt{2}$. Similarly we need π and all the other irrational numbers. Without them no sensible theory of limits is going to be possible. In this sense, the set $\mathbb{Q}$ of Rational Numbers is *incomplete*.

Informally, we can see that $\mathbb{Q}$ is very "incomplete" by considering decimal expansions. If we consider a fraction p/q with p and q Natural Numbers, the decimal expansion of p/q is obtained by dividing p by q. The possible remainders at each step in the division are $0, 1, \ldots, q-1$. At some stage past the point where all remaining carry-down digits are zeros in the division, you will encounter a remainder that has occurred previously, and at that point the decimal expansion will start to recur. In fact the length of the recurring section can be at most q because there are only q possible remainders. If the recurring section is simply a string of zeros, then we say that the decimal terminates. Including terminating decimals as recurring decimals that just happen to have an unending string of zeros, *every Rational Number has a recurring decimal expansion*. You might like to obtain the decimal expansion of $3/7$ by division to see how, what we have just described in general, works in practice.

So every Rational Number has a recurring decimal expansion. Conversely, and again *informally*, every recurring decimal expansion corresponds to a Rational Number. For example, suppose $x = 15.715343434\ldots$ with the "34" recurring. Since the recurring section is of length 2, we multiply by 10^2 to get $100x = 1571.5343434\ldots$. Now place x below $100x$ and line up the decimal points:

$$\begin{aligned} 100x &= 1571.5343434\ldots \\ x &= \quad\;\; 15.7153434\ldots \end{aligned}$$

Now subtract x from $100x$ so that the recurring section cancels out. We get $99x = 1571.534 - 15.715$, and this gives $x = (1571534 - 15715)/99000$, which is a Rational Number. It is not hard to see how to generalise this example to deal with any recurring decimal.

Why did I say *informally* in the previous two paragraphs? The problem is that an infinite decimal expansion is really an infinite sum. For example, $1.111\ldots$ really represents $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots$ and we haven't yet obtained the rules for dealing with such sums. So at this stage the above arguments are informal and illustrative, rather than definitive. Nevertheless they strongly suggest that "most" decimals do not represent Rational Numbers because "most" decimals do not recur. Imagine writing down infinitely long decimals at random, it seems pretty unlikely that you'd get recurrence (I said this was informal!).

The problem with infinite decimals not being respectable at this stage means that we mustn't *start* by identifying the Real Numbers as the set of all decimals. So instead we simply define the Real Numbers as a set that obeys Axioms A, B and C. The flaw in the decimal approach is that it is imprecise, while the flaw in the axiomatic approach is that there might be no set which satisfies all the axioms. Fortunately it is possible to construct $\mathbb{R}$. Starting with $\mathbb{N}$, *whose basic properties we assume*, we can construct $\mathbb{Z}$, then $\mathbb{Q}$, and finally $\mathbb{R}$. An outline of this process is given in Appendix A. We don't do it here and now because we want to get on to Analysis and not take a detour through the construction of $\mathbb{R}$. But at each stage all the appropriate properties can be verified, so that we arrive at a system of numbers which satisfies axioms A, B and C. It can also be shown that this set of numbers is essentially unique.

Honesty compels me to mention a further problem with the axiomatic approach - Axioms A, B and C do not explicitly mention the subsets $\mathbb{N}, \mathbb{Z}$ or $\mathbb{Q}$. Strictly speaking we should show how to identify these subsets of $\mathbb{R}$ using the axioms. Again this can be done and we indicate the process in Appendix B.

Now we will take a closer look at Axiom C (completeness). We start with the definition of an upper bound, which is needed to make sense of Axiom C.

Definition 2.2. If S is a non-empty set of Real Numbers and A is a number such that for every $x \in S$, $x \leq A$, then we say that S is bounded above by A and that A is an upper bound of the set S.

Example 2.4. The finite set $\{1, 6, -7, 8\frac{1}{2}, \pi, e\}$ is bounded above by 99. It is also bounded above by $8\frac{1}{2}$. (e is Euler's number, its value to 3 decimal places is 2.718.)

Example 2.5. The set $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ is bounded above by 2. It is also bounded above by 1.

Example 2.6. The set $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$ is bounded above by 2. It is also bounded above by 1.

In the first two examples above the sets have maximum elements. It is clear that an upper bound cannot be lower than the maximum element. In the third case, however, the set has no maximum element. Nevertheless the numbers do get closer and closer to 1, and it looks fairly obvious that no number less than 1 would qualify as an upper bound for the set. In such a case we say that 1 is a *least upper bound* or *supremum* of the set. Abbreviations are l.u.b., or $\sup$.

The precise definition of what we mean by saying that the set S has least upper bound B is as follows.

Definition 2.3. If S is a non-empty set of Real Numbers and B is a number such that

1. $\forall x \in S, x \leq B$,

2. $\forall \epsilon > 0, \exists y \in S$ s.t. $y > B - \epsilon$.

then we say that B is a least upper bound or supremum of the set S, and write $B = \sup(S)$.

Condition 1 of this definition says that B is *an* upper bound of S, and condition 2 says that no smaller number, $B - \epsilon$, is an upper bound of S.

If a set has a maximum element B then this element will be the least upper bound because it will satisfy both conditions 1 and 2.

Example 2.7. Prove that 1 is a least upper bound of $\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\}$.
Solution. We note that the set consists of all numbers of the form $1 - \frac{1}{2^n}$ for $n = 1, 2, 3, \ldots$.

1. Since $\frac{1}{2^n} > 0$, it follows that $1 - \frac{1}{2^n} < 1$ for all $n = 1, 2, 3, \ldots$. Hence condition (1) is verified.

2. Choose $\epsilon > 0$. Take a value of n for which $2^n > \frac{1}{\epsilon}$. Then $\frac{1}{2^n} < \epsilon$, and so $1 - \frac{1}{2^n} > 1 - \epsilon$. Taking $y = 1 - \frac{1}{2^n}$, condition (2) is also verified.

If you are exceptionally argumentative (good, you are getting the hang of Real Analysis) you might ask how we can be sure that we can choose a positive Integer n for which $2^n > \frac{1}{\epsilon}$. We can start to answer this by proving that for $n \geq 1$, $2^n > n$. This can be done by induction. I'm assuming that you have seen some use of induction in your previous mathematics courses, but in case you are a bit rusty or troubled by it we will take a brief refresher in Section 2.4 below.

In this case, use of induction is particularly simple. We want to prove that $2^n > n$ for every positive Integer n. For $n = 1$ it is certainly true that $2^1 > 1$ because $2^1 = 2$. Now assume that for a particular integer $k \geq 1$ we know that $2^k > k$. Then $2^{k+1} > 2k = k + k \geq k + 1$. So, by induction, it follows that $2^n > n$ for every positive integer n.

Having established that, hopefully to your satisfaction, if we can find a positive integer n such that $n > \frac{1}{\epsilon}$, it will follow that $2^n > \frac{1}{\epsilon}$. The assertion that for every $\epsilon > 0$ there exists a positive integer n such that $n > \frac{1}{\epsilon}$ is equivalent to what is known as the Axiom of Archimedes. There are lots of equivalent ways to phrase this axiom, one of which is to say that given any Real Number x there is a Natural Number (i.e. a positive integer) n such that $n > x$. In other words, $\mathbb{N}$ is not bounded above. If x is negative or zero then $x < 1$ and if $x > 0$ it is equivalent to our version by taking $\epsilon = \frac{1}{x}$. So is it true? Of course it is true. It is a consequence of the Completeness Axiom and so we don't need to include it as an additional axiom. Here's the proof.

Theorem 2.2 (The Archimedean Axiom)**.** The set of Natural Numbers $\mathbb{N}$ is not bounded above.

Proof. (By contradiction.) Suppose that $\mathbb{N}$ is bounded above and therefore, by the Completeness Axiom, it has a least upper bound B. There must exist a value $n \in \mathbb{N}$ such that $n > B - 1$. But then $n + 1 \in \mathbb{N}$ and $n + 1 > B$, a contradiction. We conclude that $\mathbb{N}$ is not bounded above (we say $\mathbb{N}$ is *unbounded* above). $\square$

We can now interpret Axiom C. It asserts that if the set S is bounded above, then it has a least upper bound. Of course the least upper bound may be irrational as in the case of the set $\{1, 1.4, 1.41, 1.414, \ldots\}$ of rounded down decimals approximating $\sqrt{2}$. If we restricted ourselves to the Rational Numbers then this set would have no least upper bound. Any Rational Number greater than $\sqrt{2}$ would be an upper bound, but there would be no least such Rational Number. We would have a most undesirable gap just where $\sqrt{2}$ should be.

To sum up, Axiom C ensures we have a number system which is suitable for the development of a rigorous and coherent theory of limiting processes. Amongst other things it ensures that every positive number has a positive square root, a positive cube root, a positive 4^{th} root, and so on; a proof of this is given in Appendix C. We will use the axiom to show (amongst other things) that any increasing sequence of numbers which is bounded above necessarily converges. We can then use this result to establish conditions for the convergence of series in general, and the properties of standard power series for $\sin, \cos, \exp, \log$, etc.

Exercises for Section 2.3

1. [This is a "fun" question - no-one is ever likely to ask you to do this again.] **Using the axioms for** $\mathbb{R}$, prove that $\forall a, b \in \mathbb{R}, (-a)(-b) = ab$. [Consider $(-a)(-b) - ab = (-a)(-b) - ab + (-a)b - (-a)b$.]
2. Suppose that n is a positive Integer. Prove that n^2 is divisible by 3 (i.e with remainder 0) if and only if n is divisible by 3.
3. Prove that $\sqrt{3}$ is irrational.
4. Why is this a false statement: "if n is a positive Integer then n^2 is divisible by 4 if and only if n is divisible by 4"?
5. Suppose that x is an irrational number. Prove that $\frac{1}{x}$ is irrational. Prove also that if a, b are Rational Numbers with $b \neq 0$, then $z = a + bx$ is irrational.

6. Obtain the decimal expansion of the Rational Number 3/7.

7. Find positive Integers p and q such that $p/q = 27.53272727\cdots = 27.53\overline{27}$. Here the 27 recurs indefinitely as indicated by the bar over the recurring section.

8. What's the length of the recurring section in the decimal expansion of $1/17$?

9. Find the least upper bound of the set $S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$ and prove that it is the least upper bound.

10. Find the least upper bound of the set $T = \{-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, -\frac{4}{5}, \ldots\}$ and prove that it is the least upper bound.

11. Suppose that $a, b \in \mathbb{R}$ and that $a < b$. Prove that there is a Rational Number r between a and b. [Hint: use the Archimedean Axiom to establish that there is some positive integer q such that $q(b - a) > 1$.]

12. Suppose that $a, b \in \mathbb{R}$ and that $a < b$. Prove that there is an irrational number z between a and b. [Hint: use the previous result to get an interval with Rational end points between a and b and then find an easy irrational number in this interval.]

2.4 The Method of Induction

Here we will give an informal description of the method of induction. The method relies on properties of the Natural Numbers and has been likened to climbing a ladder. First you prove that you can get on the first rung. Then you prove that from any rung you can climb to the next rung. The conclusion is that you can reach any rung. In mathematical terms we assume that we have some statement $P(n)$ about the Natural Number n and we wish to prove that $P(n)$ is true for each n. The statement $P(n)$ corresponds to the n^{th} rung on the ladder. So we first prove that $P(1)$ is true (we have got onto the first rung). (Of course if $P(1)$ isn't true then $P(n)$ isn't true for each Natural Number n and we have to revise what we are trying to prove.) Assuming we can prove that $P(1)$ is true, we then *assume* that $P(k)$ is true for an arbitrary Natural Number k and prove (*on this assumption*) that $P(k+1)$ is true. This proves that from any rung you can get to the next rung. The conclusion is that $P(n)$ holds true for every Natural Number n.

Example 2.8. Prove that $1 + 2 + \ldots + n = n(n+1)/2$ for every $n = 1, 2, \ldots$.
Solution. We start by giving the sum on the left-hand side a name such as $T(n)$ so that we have to prove the claim that $T(n) = n(n+1)/2$ for each Natural Number n. First we prove that $T(1) = (1 \times 2)/2$, and this is very easy since both $T(1)$ and $(1 \times 2)/2$ equal 1. Then we assume that $T(k) = k(k+1)/2$ for some Natural

Number k. We want to prove the corresponding statement when k is replaced by $k+1$, i.e. we want to prove that $T(k+1) = (k+1)(k+2)/2$. Assuming that $T(k) = k(k+1)/2$ we can add $(k+1)$ to both sides to get

$$T(k+1) = T(k) + (k+1) = k(k+1)/2 + (k+1) = (k+1)(k+2)/2,$$

and this is what we wanted to prove. So we conclude, by induction, that $T(n) = n(n+1)/2$ for every Natural Number n.

Example 2.9. Prove that $8^n - 7n + 6$ is divisible by 7 for every $n = 1, 2, \ldots$.
Solution. First we check the statement for $n = 1$. If $n = 1$ then $8^n - 7n + 4 = 8 - 7 + 6 = 7$. So the statement is certainly true for $n = 1$. Now assume that $8^k - 7k + 6$ is divisible by 7 for some positive integer k. If we multiply this by 8, it will remain divisible by 7. So $8^{k+1} - 56k + 48$ is divisible by 7. Now add $49k - 49$, which is clearly divisible by 7, to obtain that $8^{k+1} - 7k - 1$ is divisible by 7, i.e. $8^{k+1} - 7(k+1) + 6$ is divisible by 7. It follows, by induction, that $8^n - 7n + 6$ is divisible by 7 for every $n = 1, 2, \ldots$.

Although we described induction as starting with $n = 1$, it is legitimate to start with a larger value, and sometimes this is necessary.

Example 2.10. Prove that $(1.1)^n > n$ if $n \geq 39$ (where $n \in \mathbb{N}$).
Solution. First we check that $(1.1)^{39} > 39$, which it is (use a calculator). Next assume that $(1.1)^k > k$ for some integer $k \geq 39$. Multiplying this by 1.1 gives $(1.1)^{k+1} > (1.1) \times k = k + \frac{k}{10} \geq k + \frac{39}{10} > k + 1$. The result follows by induction.

In Appendix B we show how the set of Natural Numbers $\mathbb{N}$ can be identified as a subset of the set of Real Numbers $\mathbb{R}$. This enables us to put the principle of Induction on a firm footing within our axiomatic approach to the properties of $\mathbb{R}$.

Exercises for Section 2.4

1. Use induction to prove that $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$ for each positive Integer n. [Here $\sum_{i=1}^{n} x_i$ is shorthand for $x_1 + x_2 + \ldots + x_n$, so that $\sum_{i=1}^{n} i^2$ means $1^2 + 2^2 + \ldots + n^2$.]
2. Prove that $n! > 10^n$ if $n \geq 25$.
3. [Here is a "fun" question.] Find the error in the following "proof" that everyone in a room has the same birthday. Here is the "proof":
   The proposition is that all n people in the room have the same birthday. This is obviously true if $n = 1$ as there is only one person with one birthday in the room. Now suppose that whenever there are k people in the room, they all have the same birthday. We want to prove this is true for $k+1$ people. So take a room of $k+1$ people and send one person outside (call

this person X). By the inductive supposition, the remaining k people all have the same birthday, say day d. Now invite X back into the room and send another person out (call this person Y, and note that Y has birthday d). There are again k people in the room and $k-1$ have birthday d, so, by the inductive supposition all the people in the room (now including person X) have birthday d. Invite person Y back into the room and we find that all $k+1$ people in the room have birthday d. So, by induction, whenever there are n people in the room, they all have the same birthday.

2.5 Inequalities

The set of symbols below the title of this book contains three quantifiers and a modulus, but also three inequality signs. Inequalities play a vital role in Real Analysis, whole books are devoted to them. And it is perhaps something of a surprise that from weakness (inequalities) comes strength (precision). It really is the combination of quantifiers and inequalities that is so effective.

Let's start with a few reminders and it helps to think about Real Numbers represented along a straight line, often called the *Real Line*. So $a < b$ means that a is to the left of b, a fact that can also be expressed as $b > a$, meaning that b is to the right of a. As examples we have $1 < 2,\ 3 > 1,\ -1 < 3,\ -2 > -7$. The symbols $\leq$ and $\geq$ allow the possibility of equality. So it is true that $1 \leq 2$ and also that $1 \leq 1$. If $a \leq b$ and $b \leq a$ then $a = b$.

If you add (or subtract) the same thing on both sides of an inequality, it remains true. So if $a < b$ then $a + c < b + c$. But you must be much more careful with multiplication (and division). We already mentioned this problem in Section 2.3, but it deserves re-emphasis. If $a < b$ AND $c > 0$ then $ac < bc$, but if $c < 0$ you must reverse the inequality to get $ac > bc$. For example, if we start with $2 < 3$ and multiply by -5 the resulting two numbers are -10 and -15, and $-10 > -15$. This looks obvious when the multiplier is clearly negative, but when it is a variable such as c, you must check if it is positive or negative. And if it is zero, of course $ac = bc = 0$.

An inequality involving a modulus sign such as $|x| < 3$ is equivalent to saying that if x is positive (or zero) we must have $x < 3$, while if x is negative we must have $-3 < x$. So we get

$$|x| < 3 \iff -3 < x < 3.$$

There is nothing magic about x or 3. You will have seen the inequality $|x_n - l| < \epsilon$ and this is equivalent to saying that $-\epsilon < x_n - l < \epsilon$. By adding l to each term, this can also be expressed as $l - \epsilon < x_n < l + \epsilon$.

$$|x_n - l| < \epsilon \iff -\epsilon < x_n - l < \epsilon \iff l - \epsilon < x_n < l + \epsilon.$$

Inequalities involving modulus signs need to be handled with care, particularly if you feel an urge to get rid of the modulus signs. For example, it is true that $1 < |-2|$, but certainly not true that $1 < -2$. This looks obvious, but such mistakes are much easier to make with complicated algebraic expressions where it is not obvious whether they are positive or negative (or even 0). So modulus signs should be removed with care and not discarded willy-nilly because you simply don't like the look of them.

Enough of the sermonising for now. A result used extensively in Real Analysis, with which you may not already be familiar is the so-called triangle inequality.

Theorem 2.3. If $x, y \in \mathbb{R}$ then

$$\big||x| - |y|\big| \leq |x \pm y| \leq |x| + |y|.$$

The aspect we shall use most frequently is $|x + y| \leq |x| + |y|$, and this is called the triangle inequality.

Proof. (a) We have

$$\begin{aligned}|x+y|^2 &= (x+y)^2 \\ &= x^2 + 2xy + y^2 \\ &= |x|^2 + 2xy + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 \\ &= (|x| + |y|)^2.\end{aligned}$$

Since $|x + y|$ and $|x| + |y|$ are both non-negative this gives

$$|x + y| \leq |x| + |y|.$$

(b) Replacing y by $-y$ above, we obtain

$$\begin{aligned}|x - y| &\leq |x| + |-y| \\ &= |x| + |y|.\end{aligned}$$

So $|x \pm y| \leq |x| + |y|$.

(c) Replacing y by $x - y$ in part (b) we obtain

$$|x - (x - y)| \leq |x| + |x - y|.$$

Consequently $|y| \leq |x| + |x - y|$ and this can be written as

$$|y| - |x| \leq |x - y|. \tag{2.1}$$

Since this hold for all $x, y \in \mathbb{R}$ we may exchange x and y to obtain

$$|x| - |y| \leq |y - x| = |x - y|. \tag{2.2}$$

From 2.1 and 2.2 it follows that

$$\big||x| - |y|\big| \leq |x - y|,$$

and replacing y by $-y$ once again we also have

$$\big||x| - |y|\big| \leq |x + y|.$$

Thus we obtain

$$\big||x| - |y|\big| \leq |x \pm y|.$$

The result follows. □

The name "triangle inequality" arises from considering a triangle with sides given by vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$. The length of a vector $\mathbf{x}$ is denoted by $|\mathbf{x}|$. The length of the side given by $\mathbf{x} + \mathbf{y}$ cannot exceed the total of the lengths of the other two sides, so $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$. If you don't know anything about vectors, you can safely ignore this explanation of the name.

Sometimes people get confused by strings of equality and inequality signs positioned vertically as shown here to the right. The terms a, b, c, d stand for more complicated mathematical expressions. The conclusion from this string is that $a < d$. It reads that $a < b$, $b = c$ and $c \leq d$. It should not be read as saying that $a = c$. It is really the same as writing $a < b = c \leq d$ on one line, but it has been broken vertically for convenience.

$$\boxed{\begin{aligned} a &< b \\ &= c \\ &\leq d \end{aligned}}$$

Warning.
Do not mix inequalities that point in opposite directions like the example on the right here. If $a < b$ and $b > c$, that tells you nothing about the relationship between a and c. So don't do it.

This is horrible:

$$\begin{aligned} a &< b \\ &> c \end{aligned}$$

Exercises for Section 2.5

1. Suppose that x and y are positive Real Numbers. Prove that $\sqrt{xy} \leq (x + y)/2$. [$\sqrt{xy}$ is called the *geometric mean* of x and y, while $(x+y)/2$ is called the *arithmetic mean*. So the result says that the geometric mean is at most the arithmetic mean.] What is the condition on x and y that is both necessary and sufficient for $\sqrt{xy}$ to be equal to $(x + y)/2$?

2. [A "fun" question] Mr. Fixit, the demon mathematics teacher, gets advance notice of his pupils' mathematics scores in a national examination. He is upset to see that both Class X and Class Y have lower (arithmetic) mean scores than last year. He hastily moves pupil P from Class X to Class Y. Both mean scores are now better than last year and his performance bonus is secure. Can you explain?

Chapter 3

Sequences

3.1 Important items

In this section we collect together the salient points from our preliminary discussion and remind you of some useful algebraic results.

Theorem 3.1 (The triangle inequality)**.** If $x, y \in \mathbb{R}$ then

$$\big||x| - |y|\big| \leq |x \pm y| \leq |x| + |y|.$$

Definition 3.1. If S is a set of real numbers and A is a constant such that for every $x \in S$, $x \leq A$, then we say that S is bounded above by A and that A is an upper bound of the set S.

Definition 3.2. If S is a set of numbers and B is a constant such that

1. $\forall x \in S, x \leq B$,
2. $\forall \epsilon > 0, \exists y \in S$ s.t. $y > B - \epsilon$.

then we say that B is a least upper bound or supremum of the set S. We write $B = \sup S$. Condition 1 says that B is *an* upper bound of S, and condition 2 says that no smaller number $B - \epsilon$ is an upper bound of S.

Axiom C. (The Completeness Axiom)
A non-empty collection of real numbers, S, which is bounded above, possesses a least upper bound.

The Binomial Theorem
If $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ then

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2}{3!}a^{n-3}b^3 + \ldots + b^n.$$

This can be written as

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n}b^n \\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r,\end{aligned}$$

where $\binom{n}{r} = \dfrac{n!}{(n-r)!r!}$ is the number of ways of choosing r objects from n objects, so that $\binom{n}{r}$ is read as "n choose r". The Binomial Theorem can be proved by induction.

Geometric Series

If $a \in \mathbb{R}$ and $n \in \mathbb{N}$ then the expression $1 + a + a^2 + a^3 + \ldots + a^n = \sum_{r=0}^{n} a^r$ is called a *geometric series* with ratio a (meaning that each term in the sum is a times its predecessor) and length $n+1$. If $a = 1$ then the sum reduces to $n+1$. For $a \neq 1$ a formula for the sum can be obtained as follows. Put $S = \sum_{r=0}^{n} a^r$ and then $aS = \sum_{r=0}^{n} a^{r+1}$. Write these in full, line up equal powers of a and subtract:

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \ldots + a^n \\ aS &= \phantom{1 + {}} a + a^2 + a^3 + \ldots + a^n + a^{n+1} \\ \text{so } \ S - aS &= 1 - a^{n+1}. \end{aligned}$$

Hence $(1-a)S = 1 - a^{n+1}$, which gives $S = \dfrac{1-a^{n+1}}{1-a}$. Thus for $a \neq 1$ we have

$$\sum_{r=0}^{n} a^r = 1 + a + a^2 + a^3 + \ldots + a^n = \frac{1-a^{n+1}}{1-a}.$$

3.2 Bounds

A similar concept to "bounded above" is "bounded below".

Definition 3.3. If S is a set of numbers and A is a constant such that for every $x \in S$, $x \geq A$, then we say that S is bounded below by A and that A is a lower bound of the set S.

Definition 3.4. If S is a set of numbers bounded above and bounded below then we say that S is bounded.

Definition 3.5. If S is a set of numbers and A is a constant such that

1. $\forall x \in S, x \geq A$,
2. $\forall \epsilon > 0, \exists y \in S$ s.t. $y < A + \epsilon$.

then we say that A is a greatest lower bound (g.l.b.) or infimum (inf) of the set S. We write $A = \inf S$. Condition 1 says that A is *a* lower bound of S, and condition 2 says that no larger number $A + \epsilon$ is a lower bound of S.

Theorem 3.2. If S is a set of numbers which is bounded below, then S has a greatest lower bound.

Proof. We use Axiom C. We define $S' = \{x : -x \in S\}$. Since S is bounded below, $\exists A$ s.t. $\forall x \in S, x \geq A$. Therefore for each $x \in S, -x \leq -A$. It follows that S' is bounded above by $-A$. Hence, by the Completeness Axiom, S' has a least upper bound B. We show that $-B$ is a greatest lower bound of S.

1. We have that for all $x' \in S', x' \leq B$. Therefore for all $x' \in S', -B \leq -x'$. That is, $\forall x \in S, -B \leq x$.
2. Choose $\epsilon > 0$. We have that $\exists y' \in S'$ s.t. $y' > B - \epsilon$. Define $y = -y'$. Then $y \in S$ and $-y > B - \epsilon$. That is, $y < -B + \epsilon$.

From (1) and (2) it follows that $-B$ is a greatest lower bound of S. $\square$

Theorem 3.3. A least upper bound is unique. In other words, if S is a set of numbers bounded above and A, B are least upper bounds of S, then $A = B$. Similarly a greatest lower bound is unique.

Proof. Suppose S is a set of numbers bounded above and that A, B are least upper bounds and $A \neq B$. We may assume $A < B$. From the definition of a least upper bound. we have

1. $\forall x \in S, x \leq A$.

2. $\forall \epsilon > 0, \exists y \in S$ s.t. $y > B - \epsilon$.

Since $A < B$ we have $(B - A) > 0$. In (2) put $\epsilon = (B - A)$. Then $\exists y \in S$ s.t. $y > B - (B - A) = A$. But this contradicts (1). It follows that $A = B$.

The proof for greatest lower bound is similar. $\square$

Definition 3.6. Suppose we have an ordered collection of numbers x_n, one associated with each positive integer n:

$$x_1, x_2, x_3, \ldots, x_n, \ldots$$

Such a collection, arranged in order of increasing n, is called a *sequence* of numbers. Such a sequence would be denoted by (x_n). The individual numbers $x_1, x_2, \ldots$ are called the *terms* of the sequence.

Warning! In everyday English the words "sequence" and "series" are pretty well interchangeable. But in mathematics they mean something completely different. A (mathematical) sequence is an ordered list, but a (mathematical) series is what you get by adding up the terms in a sequence of numbers. We will deal with series after we have dealt with sequences. Be careful not to confuse the two.

Examples of sequences:

1. $(n) = (1, 2, 3, \ldots, n, \ldots)$.

2. $(\frac{1}{n^2}) = (1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots)$.

3. $(\frac{(-1)^n}{n!}) = (-1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, \ldots, \frac{(-1)^n}{n!}, \ldots)$.

4. $(5) = (5, 5, 5, \ldots, 5, \ldots)$.

5. $(\frac{z^n}{n}) = (z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots, \frac{z^n}{n}, \ldots)$.

Note. The sequence $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ is not the same sequence as $(\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{8}, \ldots)$ since we pay attention to the order in which the terms are written down. Of course the set of numbers is the same in both cases.

Definition 3.7. Suppose (x_n) is a sequence of real numbers. Let S be the *set* of numbers $\{x_1, x_2, x_3, \ldots\}$. If S is bounded above by A then we say that the sequence (x_n) is bounded above by A. If B is the least upper bound of S then we say B is the least upper bound or supremum of (x_n). We may write $B = \sup(x_n)$. Similarly, we define boundedness below, boundedness, and the concept of $\inf(x_n)$, in terms of the underlying set S.

Exercises for Section 3.2

1. Find the greatest lower bound of the sequence $\left(\frac{n}{n^2+1}\right)$

2. Prove that the sequence $\Big((-1)^n n\Big)$ is unbounded both above and below.

3.3 Convergence

Next we define what we mean by saying that a sequence of real numbers converges to a limiting value.

Definition 3.8. We say that "$x_n \to l$ as $n \to \infty$" if and only if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n > N, |x_n - l| < \epsilon.$$

The sentence "$x_n \to l$ as $n \to \infty$" is read as "x_n converges to (or tends to) l as n tends to infinity". We may also write: $\exists \lim_{n\to\infty} x_n = l$. We say that l is the limit of the sequence (x_n). [Note that we say "n tends to infinity"; we do NOT say "n converges to infinity".]

Notes.

1. In the above definition the sentence

$$\text{“}x_n \to l \text{ as } n \to \infty''$$

must be interpreted as a whole. We are not attempting to define the individual terms "$\to$" or "∞".

2. Many sequences do not have limits.

3. To show that $x_n \to l$ as $n \to \infty$ directly from the definition it is necessary to prove that for any given $\epsilon > 0$, a corresponding N can be found; i.e. we must find an expression for N in terms of ϵ. In that sense, the definition is an operational definition - it tells you what you have to do.

4. In general, the number N depends on ϵ: the closer ϵ is taken to zero, the larger the corresponding value of N. There is no necessity for N to be an Integer, although the values of n have to be positive Integers for the sequence (x_n) to make sense.

We will start using this definition of convergence with a very simple example.

Example 3.1. Prove that $\frac{1}{n} \to 0$ as $n \to \infty$.
Solution. Choose $\epsilon > 0$. Put $N = \frac{1}{\epsilon}$. Take any $n > N$ and consider $|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} = \epsilon$. Hence $\frac{1}{n} \to 0$ as $n \to \infty$.

In the same way we can easily prove things like $\frac{2}{n} \to 0$ as $n \to \infty$ - just replace $N = \frac{1}{\epsilon}$ by $N = \frac{2}{\epsilon}$. In Section 3.7 we will deal with a whole host of basic convergent sequences. But here we will just use the definition on a few more examples to get a feel for how it works.

Example 3.2. Prove that $\frac{2n}{3n-1} \to \frac{2}{3}$ as $n \to \infty$.
Solution. (You might find it helpful to look again at the example of the sequence $(\frac{n}{n+1})$ that we investigated in Chapter 1, the Introduction.)
Here the sequence (x_n) is $(\frac{2n}{3n-1})$, the first few terms of which are $1, \frac{4}{5}, \frac{3}{4}, \frac{8}{11}, \ldots$.
We begin by choosing an arbitrary $\epsilon > 0$. We make an inspired guess for N (how this is done will be explained later): put $N = \frac{1}{3\epsilon}$.

Now suppose that $n > N$ and consider the expression $|\frac{2n}{3n-1} - \frac{2}{3}|$. We have

$$\begin{aligned}
\left|\frac{2n}{3n-1} - \frac{2}{3}\right| &= \left|\frac{6n - 2(3n-1)}{3(3n-1)}\right| \\
&= \left|\frac{2}{3(3n-1)}\right| \\
&\leq \frac{2}{3(2n)} \qquad \text{note } 2n \leq 3n - 1 \\
&= \frac{1}{3n} \\
&< \frac{1}{3N}.
\end{aligned}$$

But $N = \frac{1}{3\epsilon}$ and so $\epsilon = \frac{1}{3N}$. It follows that $\forall n > N$,

$$\left|\frac{2n}{3n-1} - \frac{2}{3}\right| < \epsilon.$$

Thus the definition is satisfied - we have taken a perfectly arbitrary $\epsilon > 0$ and found a number N such that if $n > N$ then $|x_n - \frac{2}{3}| < \epsilon$.

Comment. We could have taken N to be any number greater than or equal to $\frac{1}{3\epsilon}$. So $N = \frac{1}{3\epsilon} + 5$ would have served our purposes just as well. In general, there is no need to choose the smallest possible N. We select the most convenient value to avoid excessive amounts of algebraic manipulation. We'll see something of how that works in the next example.

Example 3.3. Prove that the sequence $(\frac{2n^2+7}{n^2+n+1})$ converges and find its limit.
Solution. Dividing top and bottom of the fraction by n^2 we get

$$\frac{2n^2+7}{n^2+n+1} = \frac{2+\frac{7}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}}.$$

It seems likely that $\frac{7}{n^2} \to 0$ as $n \to \infty$ and that $\frac{1}{n}+\frac{1}{n^2} \to 0$ as $n \to \infty$. Thus, if there is a limit, its value is probably $\frac{2+0}{1+0} = 2$. Accordingly we attempt to prove that

$$\frac{2n^2+7}{n^2+n+1} \to 2 \text{ as } n \to \infty.$$

Choose $\epsilon > 0$.
Put $N = \frac{7}{\epsilon}$ (we'll explain this inspired "guess" later).
Take any $n > N$ and consider the expression $\left|\frac{2n^2+7}{n^2+n+1} - 2\right|$. We have

$$\begin{aligned}
\left|\frac{2n^2+7}{n^2+n+1} - 2\right| &= \left|\frac{2n^2+7-2(n^2+n+1)}{n^2+n+1}\right| \\
&= \left|\frac{-2n+5}{n^2+n+1}\right| \\
&\leq \left|\frac{2n}{n^2+n+1}\right| + \left|\frac{5}{n^2+n+1}\right| \quad \text{(by the triangle inequality)} \\
&= \frac{2n+5}{n^2+n+1} \\
&\leq \frac{7n}{n^2+n+1} < \frac{7n}{n^2} = \frac{7}{n} < \frac{7}{N} = \epsilon. \qquad (3.1)
\end{aligned}$$

It follows that if $n > N$ then $\left|\frac{2n^2+7}{n^2+n+1} - 2\right| < \epsilon$. Hence the sequence does converge and its limit is 2.

Comments.

1. Once we got to the expression $\left|\frac{-2n+5}{n^2+n+1}\right|$ we started to use inequalities. Why? It's because the numerator is, crudely speaking, of size n and the denominator is of size n^2, so the fraction will behave rather like $\frac{1}{n}$ for "large" n. So we may use inequalities to "simplify" the expression while retaining this crude estimate of its size. Thus we "simplify" the numerator to the somewhat larger $7n$ (which is just a multiple of n) and the denominator to the somewhat smaller n^2. This then cancels down to $\frac{7}{n}$.

2. You are not expected to guess in advance that we put

$$\text{“}N = 7/e\text{”}.$$

Lay out the proof as above but leave this line only partially completed at first:

$$\text{“}N = \text{”}$$

When you reach the stage labelled 3.1, a suitable definition for N is made for you. In this case we require $7/N \leq \epsilon$, i.e. $N \geq 7/\epsilon$. Thus we have to choose N greater than or equal to $7/\epsilon$, and so it may as well be $7/\epsilon$. Indeed, you can use the following as a kind of pro-forma for these proofs:

Choose $\epsilon > 0$. Put $N = ...$ Take any $n > N$ and consider $\lvert x_n - l \rvert ... < \epsilon$

3. The phraseology of the question "find its limit" suggests there can only be one limit of a particular sequence. We prove this below.

4. Returning to the definition of the sentence "$x_n \to l$ as $n \to \infty$" we see that altering a finite number of the terms of a sequence cannot alter the limit (if any). For if the last term to be altered is the m^{th}, it is only necessary to alter the value of N to $m + 1$, if the value of N is less than or equal to m.

   For similar reasons, removing, or adding, a finite number of terms cannot alter the convergence behaviour of a sequence. In particular if k is a constant (positive integer) then the sequences (x_n) and (x_{n+k}) either both converge or both do not converge.

5. Some people like to ensure that N is an integer. This is easy, just round up a non-integer N to the first integer above. If N works then any integer above it will also work. To save on words, here below is a useful definition: the *integer part* or *floor* function.

Definition 3.9. We denote by $\lfloor x \rfloor$ the integer part of x, i.e. the unique integer M Satisfying $M \leq x < M + 1$.

Examples: $\lfloor 3/2 \rfloor = 1, \ \lfloor 2 \rfloor = 2, \ \lfloor -1.7 \rfloor = -2, \ \lfloor -2 \rfloor = -2.$
This item of terminology enables us to shorten a sentence such as "$N =$ the first integer greater than $7/\epsilon$" to "$N = \lfloor 7/\epsilon \rfloor + 1$"

The definition of convergence can also be used to prove that some sequences do not converge to any limit.

Example 3.4. Show that the sequence $((-1)^n)$ does not converge.
Solution. The first few terms of the sequence are: $-1, 1, -1, 1, \ldots$. We will attempt a proof by contradiction. So suppose that $(-1)^n \to l$ as $n \to \infty$ for some number l. Choose $\epsilon = 0.1$. Then $\exists N$ s.t. if $n > N, \lvert (-1)^n - l \rvert < 0.1$.

(a) Choose n even s.t. $n > N$. Then $|1 - l| < 0.1$.
(b) Choose n odd s.t. $n > N$. Then $|(-1) - l| < 0.1$.
Of course $|1 - (-1)| = 2$, but using (a) and (b), and the triangle inequality gives

$$\begin{aligned} 2 = |1 - (-1)| &= |1 - l - ((-1) - l)| \\ &\leq |1 - l| + |(-1) - l| \\ &< 0.1 + 0.1 = 0.2 \end{aligned}$$

But $2 < 0.2$ is a contradiction and so we deduce that $((-1)^n)$ does not converge to any limit.

Note. Our choice for ϵ was rather extreme. We could have chosen $\epsilon = 1$ and obtained $2 < 2$, which is still a contradiction (just!).

Exercises for Section 3.3

1. Use the definition of convergence to prove that for any Real Number a, $\frac{a}{n} \to 0$ as $n \to \infty$.
2. Use the definition of convergence to prove that $\frac{n^2}{n^2+n+1} \to 1$ as $n \to \infty$.
3. Use the definition of convergence to prove that the sequence $(\frac{2n+7}{7n-3})$ converges, and find its limit.
4. Prove that the sequence $((-1)^n \frac{n}{n+1})$ does not converge.
5. Suppose that $x_n < A$ for every $n \in \mathbb{N}$ and that $x_n \to l$ as $n \to \infty$. Prove that $l \leq A$. Give an example to show that we cannot assert that $l < A$.

3.4 Some easy results on limits

Theorem 3.4. The limit of a sequence is unique. If $x_n \to l_1$ as $n \to \infty$ and $x_n \to l_2$ as $n \to \infty$, then $l_1 = l_2$.

Of course we all know intuitively that a sequence cannot have two distinct limits. The purpose of this theorem (and of many subsequent theorems) is to show that the formal definition of convergence that we adopted is consistent with our intuition.

Proof. (By contradiction.) Suppose $l_1 \neq l_2$. We may assume $l_1 < l_2$. From the definition of convergence, taking $\epsilon = (l_2 - l_1)/2$ we have
(a) $\exists N_1$ s.t. $\forall n > N_1, |x_n - l_1| < (l_2 - l_1)/2$,
(b) $\exists N_2$ s.t. $\forall n > N_2, |x_n - l_2| < (l_2 - l_1)/2$.

Choose $n > \max(N_1, N_2)$. Then we have

$$\begin{aligned}|l_2 - l_1| &= |l_2 - x_n + x_n - l_1| \\ &\leq |l_2 - x_n| + |x_n - l_1| \\ &< (l_2 - l_1)/2 + (l_2 - l_1)/2 \\ &= l_2 - l_1\end{aligned}$$

But this gives $l_2 - l_1 < l_2 - l_1$, an obvious contradiction. The result follows. $\square$

The next theorem also proves something that you may feel is pretty obvious: a constant sequence converges to its constant value.

Theorem 3.5. If $x_n = l$ for every $n \in \mathbb{N}$, then $x_n \to l$ as $n \to \infty$.

Proof. Choose $\epsilon > 0$. Put $N = 1$ then for any $n > N$, $|x_n - l| = 0 < \epsilon$. $\square$

The following result concerns a sequence "sandwiched" or "squeezed" between two other sequences that have a common limit.

Theorem 3.6 (The sandwich or squeeze rule)**.** Suppose that for every $n \in \mathbb{N}$, $y_n \leq x_n \leq z_n$, and that both sequences (y_n) and (z_n) converge to a common limit l. Then $x_n \to l$ as $n \to \infty$.

Proof. For every $n \in \mathbb{N}$ we have $y_n - l \leq x_n - l \leq z_n - l$. Choose $\epsilon > 0$. Since (y_n) converges to l, $\exists N_1$ s.t. $\forall n > N_1$, $|y_n - l| < \epsilon$. Similarly $\exists N_2$ s.t. $\forall n > N_2$, $|z_n - l| < \epsilon$. So if $N = \max(N_1, N_2)$, then $\forall n > N$

$$-\epsilon < y_n - l \leq x_n - l \leq z_n - l < \epsilon,$$

which gives $|x_n - l| < \epsilon$. Thus $x_n \to l$ as $n \to \infty$. $\square$

Comments. A simple consequences of this result is that if $0 \leq x_n \leq z_n$ for every $n \in \mathbb{N}$ and if $z_n \to 0$ as $n \to \infty$, then $x_n \to 0$ as $n \to \infty$ because (x_n) is sandwiched between the constant sequence (0), and the sequence (z_n), both of which converge to 0.

A somewhat similar result is that $|x_n| \to 0$ as $n \to \infty$ if and only if $x_n \to 0$ as $n \to \infty$. To see this, just to look at the definitions of the two statements. You will see that they say exactly the same thing once you realise that $\big||x_n|\big| = |x_n|$.

Maybe the next result isn't quite so obvious. It's really telling us that a convergent sequence can't get arbitrarily far from its limit.

Theorem 3.7. If (x_n) is a convergent sequence, then it is bounded.

Proof. Suppose $x_n \to l$ as $n \to \infty$. Taking $\epsilon = 1$ in the definition, we find $\exists N$ s.t. $\forall n > N, |x_n - l| < 1$. Take M to be the first integer greater than N. Put $A = \max(|x_1|, |x_2|, |x_3|, \ldots, |x_M|, 1 + |l|)$. If $n \leq M$ then $|x_n| \leq A$. By the triangle inequality $|x_n| - |l| \leq |x_n - l|$, and so $|x_n| \leq |x_n - l| + |l|$. Hence, if $n > M, |x_n| \leq 1 + |l| \leq A$. Thus for all values of $n, |x_n| \leq A$. It follows that (x_n) is bounded. $\square$

Comment. If $x_n \to l$ as $n \to \infty$ then for any number $k > l$, we will find that $x_n < k$ for all sufficiently large n. To see this take $\epsilon = (k - l) > 0$, so that there exists N such that for any $n > N$, $|x_n - l| < k - l$, and this entails $x_n - l < k - l$, which gives $x_n < k$. Similarly for any number $k < l$, we will find that $x_n > k$ for all sufficiently large n. This is a useful observation.

If you look back to Example 3.3 you will see that we said things like “ It seems likely that $\frac{7}{n^2} \to 0$ as $n \to \infty$” in order to *find* the limit before using the definition to *prove* that this was the limit. That proof involved some nasty algebra and inequalities. It would be impossibly tedious to deal separately with every conceivable sequence using the definition as we did in that example. So is there a better way? Yes there is, and to get us going we will now prove some rules for combining convergent sequences. The proof of these rules involves using the definition, but once we have them, along with some “standard” convergent sequences, we can deal with nasty expressions, such as the $\frac{2n^2+7}{n^2+n+1}$ of Example 3.3, much more easily.

Exercises for Section 3.4

1. Give an example of a bounded sequence that is not convergent. [This means that the converse of Theorem 3.7 is not true. This is hardly surprising. However, every bounded sequence does have a convergent subsequence. This is known as the Bolzano-Weierstrass Theorem and we will prove it in Section 3.10.]

2. Assuming that the sequence $(\frac{1}{n})$ converges to 0 (see Example 3.1), and assuming that the sine function has all its usual properties, determine the limits of the sequences $(\frac{|\sin(n)|}{n})$ and $(\frac{\sin(n)}{n})$.

3. Prove that $\dfrac{n!}{n^n} \to 0$ as $n \to \infty$.

3.5 Combination rules

Theorem 3.8 (Combination rules)**.** Suppose that $x_n \to x$ as $n \to \infty$ and $y_n \to y$ as $n \to \infty$. Then

(i) (multiple rule) if a is any constant, $ax_n \to ax$ as $n \to \infty$,

(ii) (sum rule) $x_n + y_n \to x + y$ as $n \to \infty$,

(iii) (product rule) $x_n y_n \to xy$ as $n \to \infty$,

(iv) (quotient rule) $x_n/y_n \to x/y$ as $n \to \infty$ provided that $y \neq 0$.

Proof.

(i) If $a = 0$ then (ax_n) is the constant sequence (0), which converges to $0 = ax$. So suppose that $a \neq 0$. Choose $\epsilon > 0$. Put $\epsilon' = \epsilon/(|a|)$. Since $x_n \to x$ as $n \to \infty$, there exists N such that for all $n > N$, $|x_n - x| < \epsilon'$. In other words, if $n > N$, $|x_n - x| < \epsilon/(|a|)$, giving $|ax_n - ax| < \epsilon$. Hence $ax_n \to ax$ as $n \to \infty$.

(ii) Choose $\epsilon > 0$. Put $\epsilon' = \epsilon/2$. Then $\exists N_1, N_2$ such that if $n > N_1$ then $|x_n - x| < \epsilon'$, while if $n > N_2$ then $|y_n - y| < \epsilon'$. Put $N = \max(N_1, N_2)$. Then if $n > N$,

$$\begin{aligned} |x_n + y_n - (x + y)| &\leq |x_n - x| + |y_n - y| \\ &\leq \epsilon' + \epsilon' = \epsilon \end{aligned}$$

Hence $x_n + y_n \to x + y$ as $n \to \infty$.

(iii) Since (y_n) converges it is a bounded sequence, and so $\exists A$ such that $|y_n| < A$ for all n. Choose $\epsilon > 0$. Put $\epsilon' = \epsilon/(A + |x|)$. Then $\exists N_1, N_2$ such that if $n > N_1$ then $|x_n - x| < \epsilon'$, while if $n > N_2$ then $|y_n - y| < \epsilon'$. Put $N = \max(N_1, N_2)$.
Then if $n > N$,

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \\ &\leq |y_n||x_n - x| + |x||y_n - y| \\ &< A\epsilon' + |x|\epsilon' \\ &= (A + |x|)\epsilon' = \epsilon. \end{aligned}$$

Hence $x_n y_n \to xy$ as $n \to \infty$.
Note. You are not expected to guess in advance that we put $\epsilon' = \epsilon/(A + |x|)$. This choice is really made for you at the end when we obtained $|x_n y_n - xy| < (A + |x|)\epsilon'$. A similar comment applies to part (iv) below.

(iv) In view of part (iii) it is only necessary to show that if $y_n \to y$ as $n \to \infty$ and $y \neq 0$, then $1/y_n \to 1/y$ as $n \to \infty$. Strictly speaking we ought also to assume that none of the y_n's are zero. However we will show (see inequality (3.2) below) that if $y \neq 0$ then only a finite number of the y_n's can be zero. These we imagine removed from the sequence.

First we obtain a positive lower bound for $|y_n|$. Taking $\epsilon = |y|/2$ in the definition, we obtain:

$$\exists N_1 \text{ s.t. } \forall n > N_1, |y_n - y| < |y|/2.$$

By the triangle inequality $|y| - |y_n| \leq |y_n - y|$, and so for $n > N_1$, $|y| - |y_n| < |y|/2$. Hence

$$|y_n| > |y|/2 \text{ if } n > N_1. \tag{3.2}$$

Next choose $\epsilon > 0$. Put $\epsilon' = y^2\epsilon/2$. Then $\exists N_2$ such that if $n > N_2$ then $|y_n - y| < \epsilon'$.
Put $N = \max(N_1, N_2)$. Then if $n > N$,

$$\begin{aligned}\left|\frac{1}{y_n} - \frac{1}{y}\right| &= \left|\frac{y_n - y}{y_n y}\right| \\ &= \frac{|y_n - y|}{|y_n||y|} \\ &< \frac{2|y_n - y|}{|y|^2} \quad \text{using inequality (3.2)} \\ &< \frac{2\epsilon'}{y^2} = \epsilon.\end{aligned}$$

Hence $1/y_n \to 1/y$ as $n \to \infty$ and part (iv), the quotient rule, follows.

□

Comment. You might well ask how to find proofs like the ones given above.

Consider the addition rule. If we know that x_n is close to x for large values of n and that y_n is close to y for large values of n, it is natural to anticipate that $x_n + y_n$ will be close to $x + y$ for large values of n. So we attempt to express $|(x_n + y_n) - (x + y)|$ in terms of $x_n - x$ and $y_n - y$. That is easy because $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$, and the triangle inequality can then be used to give $|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y|$.

The product rule is on the same lines but it isn't quite so clear how to write $|x_n y_n - xy|$ in terms of $x_n - x$ and $y_n - y$. Of course if x_n is close to x then $x_n y_n$ is close to xy_n, and this leads us to write $|x_n y_n - xy|$ in the form

$$|x_n y_n - xy_n + xy_n - xy|,$$

which can then be split into two, hopefully small, terms using the triangle inequality.

To illustrate how the combination rules can be used we give the following example.

Example 3.5. Prove that the sequence $(\frac{3n^2+5n-9}{10n^2-3n+7})$ converges, and determine its limit. You may assume that $\frac{1}{n} \to 0$ as $n \to \infty$, which we proved in Example 3.1.

Solution. By dividing top and bottom of $\frac{3n^2+5n-9}{10n^2-3n+7}$ by n^2, it can be written as $\dfrac{3+\frac{5}{n}-\frac{9}{n^2}}{10-\frac{3}{n}+\frac{7}{n^2}}$. Now the combination rules can be used; we deal first with the numerator. Since $\frac{1}{n} \to 0$ as $n \to \infty$, by the multiple rule $\frac{5}{n} \to 0$ as $n \to \infty$. By the product rule applied to $\frac{1}{n} \times \frac{1}{n}$, we deduce $\frac{1}{n^2} \to 0$ as $n \to \infty$ and, again by the multiple rule, $\frac{-9}{n^2} \to 0$ as $n \to \infty$. The constant sequence (3) converges to 3, so by the sum rule $3 + \frac{5}{n} - \frac{9}{n^2} \to 3 + 0 + 0 = 3$ as $n \to \infty$. Applying the same strategy to the denominator we get $10 - \frac{3}{n} + \frac{7}{n^2} \to 10$ as $n \to \infty$. Finally, by the quotient rule

$$\frac{3n^2+5n-9}{10n^2-3n+7} = \frac{3+\frac{5}{n}-\frac{9}{n^2}}{10-\frac{3}{n}+\frac{7}{n^2}} \to \frac{3}{10} \text{ as } n \to \infty.$$

Our explanation is given very fully, but it would generally suffice to say that the “combination rules” have been used on $\dfrac{3+\frac{5}{n}-\frac{9}{n^2}}{10-\frac{3}{n}+\frac{7}{n^2}}$ without specifying every individual step. You should ask your instructor what level of detail is required.

Warning. Here is a warning about what not to do. This will give your lecturer apoplexy, and it won't do your grades any good either. I'll put it in red as it is so horrible.

$$\begin{aligned}\frac{3n^2+5n-9}{10n^2-3n+7} &= \frac{3+\frac{5}{n}-\frac{9}{n^2}}{10-\frac{3}{n}+\frac{7}{n^2}} \\ &= \frac{3+0+0}{10+0+0} = \frac{3}{10} \text{ as } n \to \infty\end{aligned}$$

I am reduced to observing that (for example) $\frac{5}{n}$ is NOT 0 for ANY value of n. It may tend to zero as n increases, but that is not what appears in red above. The moral is to use $\to$ and $=$ as appropriate, they are not interchangeable symbols.

A related error is the (always erroneous) use of “variable” limits as in

$$3 + \frac{5}{n} - \frac{9}{n^2} \to 3 + 0 - \frac{9}{n^2} \text{ as } n \to \infty.$$

The limit l in "$x_n \to l$ as $n \to \infty$" is a CONSTANT, independent of n. So it MUST NOT contain any reference to n.

We remarked earlier that the convergence or otherwise of a sequence is unaffected by altering a finite number of terms. In the case of the quotient rule we also remarked that it can be convenient to ignore the fact that $\frac{x_n}{y_n}$ will be undefined if $y_n = 0$, provided that this only happens for a finite number of terms; we can imagine the sequence starting beyond the point where this problem arises. For example, we might write $\frac{n}{n-2} \to 1$ as $n \to \infty$ even though $\frac{n}{n-2}$ is undefined for $n = 2$. We simply imagine the sequence starting at $n = 3$, the first few terms being $1, 2, \frac{5}{3}, \frac{6}{4}, \frac{7}{5}, \ldots$. Indeed, if you look closely at the definition of convergence you will see that it only refers to values of n that exceed some number N. So the first few (million, billion?) terms are irrelevant to questions of convergence. Furthermore, it is sometimes convenient to allow sequences to be numbered starting at something other than 1, and it makes no difference to convergence properties. If the start is numbered 0 (for example) we can use the notation $(x_n)_0^\infty$ to indicate that the sequence is $(x_0, x_1, x_2, \ldots)$.

When dealing with expressions that involve n it is helpful to have some idea of the relative sizes of various terms for "large" values of n. Roughly speaking, in increasing orders of magnitude, we have

$$n^r \ (r > 0), \quad n^s \ (s > r), \quad a^n \ (a > 1), \quad n!, \quad n^n.$$

By saying this we really mean that (for example) $\frac{n!}{n^n} \to 0$ as $n \to \infty$. It is useful to keep these relative sizes in mind.

To get the full benefit from the combination rules we need some basic sequences. We've already seen that a constant sequence converges to its constant value. The multi-part Theorem 3.11 below provides a repertoire of what are generally called *basic null sequences*; "null" because they all tend to zero. Before getting to that result we have two preparatory theorems which are very helpful.

Theorem 3.9. Suppose that (x_n) is a bounded sequence and that (y_n) is a null sequence (meaning that $y_n \to 0$ as $n \to \infty$). Then $(x_n y_n)$ is a null sequence, i.e. $x_n y_n \to 0$ as $n \to \infty$.

Proof. Since (x_n) is bounded, $\exists A$ such that $|x_n| < A$ for every value of n. Now choose $\epsilon > 0$ and put $\epsilon' = \epsilon/A$. Since (y_n) is null, $\exists N$ s.t. $\forall n > N, |y_n - 0| < \epsilon'$, i.e. $|y_n| < \epsilon'$. So, if $n > N$,

$$|x_n y_n - 0| = |x_n y_n| \leq A|y_n| < A\epsilon' = \epsilon,$$

and it follows that $x_n y_n \to 0$ as $n \to \infty$. $\square$

A couple of comments about the inequality signs in the proof above may be appropriate here. First, since $|x_n|$ must be non-negative, and $|x_n| < A$, we know that A is strictly positive and cannot be zero. So division by A to form ϵ' is legitimate. (We did something similar in the proof of part (iii) of Theorem 3.8 - did you notice?) Second, y_n might be zero so we can't assume that $|x_n y_n| < A|y_n|$, the best we can do is $|x_n y_n| \leq A|y_n|$. But even so, since $A > 0$ and $|y_n| < \epsilon'$, we still have $A|y_n| < A\epsilon'$. It's this sort of thing I meant when I warned earlier about the need to be careful when using inequalities.

Theorem 3.9 is particularly useful for sequences involving expressions like $(-1)^n$. Since $|(-1)^n| = 1$ for all values of $n \in \mathbb{N}$, the sequence (x_n) with $x_n = (-1)^n$ is a bounded sequence. Accepting that $\frac{1}{n} \to 0$ as $n \to \infty$, it follows that $(-1)^n \frac{1}{n} \to 0$ as $n \to \infty$.

Exercises for Section 3.5

1. Prove that the sequence $\left(\dfrac{2n^2 + (-1)^n n + 7}{3n^2 - 7n + 1}\right)$ converges, and determine its limit.

2. Prove that the sequence $\left(\dfrac{n^2 + 5n - 3}{2n^3 + 5n^2 - n + 3}\right)$ converges, and determine its limit.

3.6 Monotonic sequences

Some sequences behave so nicely that we give names to these nice properties.

Definition 3.10.
If $x_{n+1} > x_n$ for every $n \in \mathbb{N}$, then we say that the sequence (x_n) is *strictly increasing* and we may write $(x_n) \uparrow\uparrow$ or just $x_n \uparrow\uparrow$.
If $x_{n+1} \geq x_n$ for every $n \in \mathbb{N}$, then we say that the sequence (x_n) is *monotonically increasing* and we may write $(x_n) \uparrow$ or just $x_n \uparrow$.
If (x_n) is strictly increasing, then it is certainly monotonically increasing, so the monotonic property is weaker than the strict property.

If $x_{n+1} < x_n$ for every $n \in \mathbb{N}$, then we say that the sequence (x_n) is *strictly decreasing* and we may write $(x_n) \downarrow\downarrow$ or just $x_n \downarrow\downarrow$.
If $x_{n+1} \leq x_n$ for every $n \in \mathbb{N}$, then we say that the sequence (x_n) is *monotonically decreasing* and we may write $(x_n) \downarrow$ or just $x_n \downarrow$.
Again, the monotonic property is weaker than the strict property.

Don't read more into $\uparrow$ and $\downarrow$ than in their definitions: a constant sequence is both $\uparrow$ and $\downarrow$.

Why are these properties "nice"? The answer is in the next theorem.

Theorem 3.10. If (x_n) is monotonically increasing and bounded above, then it converges to its least upper bound. Similarly if (x_n) is monotonically decreasing and bounded below, then it converges to its greatest lower bound.

Proof. We deal with the increasing case, the decreasing case is similar. So suppose $x_n \uparrow$ and that $B = \sup(x_n)$ (i.e. B is the least upper bound of (x_n)). Choose $\epsilon > 0$. By the definition of a least upper bound

1. $x_n \leq B$ for every $n \in \mathbb{N}$, and
2. $\exists N \in \mathbb{N}$ s.t. $x_N > B - \epsilon$.

Then if $n > N$, we have

$$B - \epsilon < x_N \leq x_n \leq B < B + \epsilon, \quad (x_N \leq x_n \text{ since } x_n \uparrow)$$

and so $|x_n - B| < \epsilon$. Hence $x_n \to B$ as $n \to \infty$. □

Here is an example illustrating the use of this theorem. It is certainly not "obvious".

Example 3.6. Prove that the sequence given by $x_n = (1 + \frac{1}{n})^n$ converges.

Solution. First we prove that the sequence is bounded above, then we prove that it is strictly increasing. By the binomial theorem (see Section 3.1)

$$\begin{aligned}\Big(1+\frac{1}{n}\Big)^n &= 1 + n\Big(\frac{1}{n}\Big) + \frac{n(n-1)}{2!}\Big(\frac{1}{n}\Big)^2 + \frac{n(n-1)(n-2)}{3!}\Big(\frac{1}{n}\Big)^3 + \dots \\ &\quad \dots + \Big(\frac{1}{n}\Big)^n \\ &= 1 + 1 + \frac{1}{2!}\Big(1-\frac{1}{n}\Big) + \frac{1}{3!}\Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big) + \dots \\ &\quad \dots + \frac{1}{n!}\Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\cdots\Big(1-\frac{n-1}{n}\Big) \end{aligned} \tag{3.3}$$

The terms in the brackets such as $\big(1 - \frac{2}{n}\big)$ are all positive and less than 1. Also $2! = 2$, $3! = 3 \times 2 > 2^2$, $4! = 4 \times 3 \times 2 > 2^3$, and so on until $n! > 2^{n-1}$. Consequently

$$\begin{aligned}\Big(1+\frac{1}{n}\Big)^n &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} \\ &= 1 + \left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right) \quad \text{(summing the geometric series as in Section 3.1)} \\ &= 1 + 2\big(1 - \frac{1}{2^n}\big) \\ &< 3.\end{aligned}$$

Hence the sequence is bounded above by 3 (we are not saying that this is the least upper bound).

To see that the sequence is strictly increasing, return to equation 3.3 and consider the effect of increasing n to $n+1$. Each bracketed term such as $\left(1-\frac{2}{n}\right)$ will increase since $\left(1-\frac{r}{n}\right) < \left(1-\frac{r}{n+1}\right)$. Also, the number of terms in the sum will increase by 1, and all these terms are strictly positive. So $(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1}$ and hence the sequence is strictly increasing.

We deduce that the sequence converges to some limit less than (or equal to) 3. This limit is known as *Euler's number*, denoted by e, whose approximate value is 2.718. In fact e is irrational, as we will prove in Chapter 7.

[Somewhat confusingly, *Euler's constant*, denoted by γ, is different from Euler's number. The value of γ is approximately 0.5772, and it is also defined as a limiting value: $\gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{n} - \log(n)\right)$. Interestingly, it is not known whether γ is rational or irrational.]

There are many techniques for determining if a particular sequence (x_n) is increasing or decreasing. The ones to try first are looking at the difference between consecutive terms $x_{n+1}-x_n$ or at their ratio $\frac{x_{n+1}}{x_n}$. Sometimes it is necessary to modify these slightly, for example to avoid problems with roots by considering $x_{n+1}^2 - x_n^2$ or $\frac{x_{n+1}^2}{x_n^2}$.

Exercises for Section 3.6

1. Let $y_n = \dfrac{n!a^n}{n^n}$ where $0 < a < 2$. Prove that (y_n) is a strictly decreasing sequence by considering the ratio y_n/y_{n+1} and using the lower bound on $\left(1+\dfrac{1}{n}\right)^n$ implied by Example 3.6. Deduce that (y_n) converges and determine its limit.
[In fact (y_n) converges for $0 < a < e$. The argument outlined in the question can be adapted to deal with the situation if $2 \le a < e$ because then the sequence is decreasing from some n onwards. However the following question shows that the situation is different if $a \ge e$.]

2. Prove that $\forall n \in \mathbb{N}$, $\dfrac{n!e^n}{n^n} > 2$. Hint: Use the fact that $\left(\dfrac{r+1}{r}\right)^r < e$ for $r = 1, 2, \ldots, n$, and multiply all these inequalities together.

3.7 Some basic sequences

Theorem 3.11 (Basic null sequences)**.** The following terms form null sequences (i.e. each converges to 0).

1. $\dfrac{1}{n^k}$ for any positive integer k,
2. $n^k a^n$ when $|a| < 1$ (equivalently $\frac{n^k}{b^n}$ when $|b| > 1$), where k is any integer,
3. $\dfrac{a^n}{n!}$ for any $a \in \mathbb{R}$,
4. $\dfrac{n^k}{n!}$ for any integer k,
5. $\left(\dfrac{1}{n!}\right)^{\frac{1}{n}}$.

Proof.

1. We start with $x_n = \frac{1}{n}$. Choose $\epsilon > 0$. Put $N = \frac{1}{\epsilon}$ Then for any $n > N$, $n > \frac{1}{\epsilon}$, so $\frac{1}{n} < \epsilon$. Consequently $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$. Hence $\frac{1}{n} \to 0$ as $n \to \infty$. By the product rule applied k times, we deduce that $\dfrac{1}{n^k} \to 0$ as $n \to \infty$ for any positive integer k.

2. If $a = 0$, the sequence formed from $n^k a^n$ is just the constant sequence (0), which converges to 0. So we can assume in the rest of the proof that $a \neq 0$. We present the rest of the proof of part 2 in three parts depending on whether $k < 0$, $k = 0$, or $k > 0$.

   If $\mathbf{k < 0}$ then by part 1, $n^k \to 0$ as $n \to \infty$. Also the sequence (a^n) is bounded since $|a^n| = |a|^n < 1$. So by Theorem 3.9, $n^k a^n \to 0$ as $n \to \infty$.

   If $\mathbf{k = 0}$ then $|n^k a^n| = |a|^n$ and this forms a strictly decreasing sequence because $|a|^{n+1} = |a|^n \times |a| < |a|^n$. (Strictly decreasing because we assume $a \neq 0$.) Clearly the sequence $(|a|^n)$ is bounded below by 0, so we deduce that this sequence converges to some limit l, i.e.

   $$|a|^n \to l \text{ as } n \to \infty.$$

   This implies that the sequence $(|a|^{n+1})$ also converges to l because it is the same sequence with the first term deleted, so

   $$|a|^{n+1} \to l \text{ as } n \to \infty.$$

But $|a|^{n+1} = |a|^n \times |a|$, and so by the multiple rule,

$$|a|^{n+1} \to |a| \times l \text{ as } n \to \infty.$$

It follows that $l = |a|l$, giving $l(1 - |a|) = 0$, and since $|a| \neq 1$, we have $l = 0$. Hence $|a|^n \to 0$ as $n \to \infty$, which gives $a^n \to 0$ as $n \to \infty$. (As remarked in a comment after Theorem 3.6, for any sequence (x_n), if $|x_n| \to 0$ as $n \to \infty$, then $x_n \to 0$ as $n \to \infty$.)

If k $>$ 0 then $k \geq 1$. Here the argument is similar to the case $k = 0$. We show that $|n^k a^n|$ is decreasing from some point onwards, and then the determination of the limit follows the same pattern.

From part 1 of this theorem, $\frac{1}{n} \to 0$ as $n \to \infty$. By the addition and product rules it follows that $(1 + \frac{1}{n})^k \to 1$ as $n \to \infty$. Put $\epsilon^* = \frac{1-|a|}{|a|}$ then $\exists N^*$ s.t. $\forall n > N^*$,

$$\begin{aligned}\left(1 + \frac{1}{n}\right)^k &< 1 + \epsilon^* \\ &= 1 + \frac{1 - |a|}{|a|} \\ &= \frac{1}{|a|} \\ &= \frac{|a|^n}{|a|^{n+1}}.\end{aligned}$$

Hence if $n > N^*$ then

$$|(n+1)^k a^{n+1}| < |n^k a^n|.$$

It follows that $|n^k a^n|$ forms a strictly decreasing sequence for $n > N^*$. (We might say it is *eventually strictly decreasing* by ignoring the terms numbered with $n \leq N^*$.) This sequence is clearly bounded below by 0, so it converges to some limit l, i.e.

$$|n^k a^n| \to l \text{ as } n \to \infty.$$

This implies that the sequence $(|(n+1)^k a^{n+1}|)$ also converges to l because it is the same sequence with the first term deleted, so

$$|(n+1)^k a^{n+1}| \to l \text{ as } n \to \infty.$$

But $|(n+1)^k a^{n+1}| = \left(1 + \frac{1}{n}\right)^k \times |a| \times |n^k a|^n$, and so by the multiple rule,

$$|(n+1)^k a^{n+1}| \to 1 \times |a| \times l \text{ as } n \to \infty.$$

It follows that $l = |a|l$, giving $l(1 - |a|) = 0$, and since $|a| \neq 1$, we have $l = 0$. Hence $|n^k a^n| \to 0$ as $n \to \infty$, and so $n^k a^n \to 0$ as $n \to \infty$.

3. Suppose that $x_n = \frac{a^n}{n!}$. If $a = 0$ the resulting sequence is the constant sequence (0) which converges to 0. We may therefore assume that $a \neq 0$. We have

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{a^{n+1}}{(n+1)!}\,\frac{n!}{a^n}\right| = \left|\frac{a}{n+1}\right| < 1 \text{ if } n > |a|.$$

So $|x_n| = |\frac{a^n}{n!}|$ forms a strictly decreasing sequence for $n > |a|$ (again it is eventually strictly decreasing). It is also bounded below by 0, and consequently it converges to some limit l. Also $|x_{n+1}| = |\frac{a}{n+1}| \times |x_n|$. But $|x_{n+1}|$ also converges to l and $\frac{1}{n+1}$ converges to 0, so $l = |a| \times 0 \times l$, giving $l = 0$. Hence $|x_n| \to 0$ as $n \to \infty$, and this gives $x_n \to 0$ as $n \to \infty$. Thus for any $a \in \mathbb{R}$, $x_n = \frac{a^n}{n!} \to 0$ as $n \to \infty$.

4. Suppose that $x_n = \frac{n^k}{n!}$. Then

$$\frac{x_{n+1}}{x_n} = \left(\frac{n+1}{n}\right)^k \frac{n!}{(n+1)!} = \left(1 + \frac{1}{n}\right)^k \times \frac{1}{n}.$$

For a fixed integer k, by the combination rules $(1 + \frac{1}{n})^k \to 1$ as $n \to \infty$, so $\frac{x_{n+1}}{x_n} \to 1 \times 0 = 0$ as $n \to \infty$. So once again the sequence x_n is eventually strictly decreasing (meaning that there exists some number N such that if $n > N$ then $x_{n+1} < x_n$). The sequence is clearly bounded below by zero, and consequently it converges to some limit l. But

$$x_{n+1} = \left(1 + \frac{1}{n}\right)^k \times \frac{1}{n} \times x_n.$$

So $l = 1 \times 0 \times l$, giving $l = 0$. Hence $x_n = \frac{n^k}{n!} \to 0$ as $n \to \infty$.

5. Suppose that $x_n = \left(\frac{1}{n!}\right)^{\frac{1}{n}}$. Choose $\epsilon > 0$. From part 3 of this theorem, $\dfrac{(\frac{1}{\epsilon})^n}{n!} \to 0$ as $n \to \infty$, so $\exists N$ s.t. $\forall n > N$, $\dfrac{(\frac{1}{\epsilon})^n}{n!} < 1$. Hence for $n > N$, $\frac{1}{n!} < \epsilon^n$, i.e. $x_n = \left(\frac{1}{n!}\right)^{\frac{1}{n}} < \epsilon$. Thus $\forall n > N, |x_n - 0| < \epsilon$, and it follows that $x_n = \left(\frac{1}{n!}\right)^{\frac{1}{n}} \to 0$ as $n \to \infty$.

$\square$

Comment. Here again you might ask how to devise proofs like the ones above. So take the last sequence as an example. We'd like to prove that we can get $\left(\frac{1}{n!}\right)^{\frac{1}{n}} < \epsilon$. But n^{th} roots are tricky, so raise both sides to the power n to get the equivalent requirement that $\left(\frac{1}{n!}\right) < \epsilon^n$. This can be rewritten as $\dfrac{(\frac{1}{\epsilon})^n}{n!} < 1$. And we know we can achieve this by the earlier part 3 of the theorem.

Generally, we'd start to devise a proof by writing down what we'd like to achieve and seeing if we can find conditions that ensure it. This can be difficult for people starting a first course of Real Analysis since it is the opposite process to what most people encounter at school where most examples start with facts and require you to make deductions from those facts. Sometimes people get the idea that you can prove almost anything by this "reverse" process, but of course this isn't the case. If you try to prove something that is incorrect, you won't be able to find conditions that ensure it.

Here are a couple more basic sequences involving n^{th} roots. They aren't null, but they do have nice limits.

Theorem 3.12. $n^{\frac{1}{n}} \to 1$ as $n \to \infty$, and consequently if $a > 0$, $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Proof. Choose $\epsilon > 0$, then $\frac{1}{1+\epsilon} < 1$ and so, using Theorem 3.11, we see that $n\left(\frac{1}{1+\epsilon}\right)^n \to 0$ as $n \to \infty$. Hence $\exists N$ s.t. $\forall n > N$, $n\left(\frac{1}{1+\epsilon}\right)^n < 1$, i.e. $n < (1+\epsilon)^n$. So if $n > N$

$$1 - \epsilon < 1 < n^{\frac{1}{n}} < 1 + \epsilon,$$

which gives $|n^{\frac{1}{n}} - 1| < \epsilon$, and it follows that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

If $a \geq 1$ then $1 \leq a^{\frac{1}{n}} < n^{\frac{1}{n}}$ for $n > a$. So by the sandwich rule $a^{\frac{1}{n}} \to 1$ as $n \to \infty$. If $0 < a < 1$ then $\frac{1}{a} > 1$ and so $\left(\frac{1}{a}\right)^{\frac{1}{n}} \to 1$ as $n \to \infty$. Then by the quotient rule applied to $\left(1/(\frac{1}{a})^{\frac{1}{n}}\right) = a^{\frac{1}{n}}$, we deduce that $a^{\frac{1}{n}} \to 1$ as $n \to \infty$. $\square$

Exercises for Section 3.7

1. For $x_n = \dfrac{4n^2 + 3^n}{5n^3 + 2(3^n)}$ determine whether or not the sequence (x_n) converges and, if it does converge, determine its limit.

2. For $y_n = \dfrac{5n^2 + (-2)^n}{6^n + 5(n!)}$ determine whether or not the sequence (y_n) converges and, if it does converge, determine its limit.

3.8 Recurrence relations

A sequence might be defined by means of a recurrence relation. In such cases it is often possible to determine if the sequence converges and even to find its limit. Here is an example.

Example 3.7. The sequence (x_n) is defined by $x_1 = 3$ and, for $n \geq 1$, $x_{n+1} = \sqrt{x_n + 5}$. Prove that (x_n) converges and determine its limit.

Solution. If $x_n \to l$ as $n \to \infty$ then $x_{n+1} \to l$ as $n \to \infty$. Since $(x_{n+1})^2 = x_n + 5$ we find that $l^2 = l + 5$ and solving the quadratic equation $l^2 - l - 5 = 0$ tells us that either $l = \dfrac{1-\sqrt{21}}{2}$ or $l = \dfrac{1+\sqrt{21}}{2}$. However, it is clear from the original recurrence relation that $x_n > 0$ for every n, so we cannot have $l < 0$. Hence if the sequence does converge, the limit must be $\dfrac{1+\sqrt{21}}{2} = 2.7913$ (to 4 decimal places).

To prove convergence, we start by considering the first few terms of the sequence to get a feel for what is happening. We have $x_1 = 3$, so $x_2 = \sqrt{8} = 2.8284$ (to 4 decimal places). Then $x_3 = 2.7979$ and $x_4 = 2.7925$ (to 4 decimal places). So it seems possible (on admittedly flimsy evidence) that the sequence is strictly decreasing. We will attempt to prove that this is the case, and then the limit l will be the greatest lower bound of the sequence. To eliminate problems arising from the square root we consider

$$\begin{aligned} x_n^2 - x_{n+1}^2 &= x_n^2 - x_n - 5 \\ &= (x_n - \alpha)(x_n - \beta) \end{aligned}$$

where $\alpha = \dfrac{1-\sqrt{21}}{2}$ and $\beta = \dfrac{1+\sqrt{21}}{2}$ are the roots of the equation $x^2 - x - 5 = 0$. The aim is to prove that $x_n^2 - x_{n+1}^2 > 0$ for all n as this will prove that (x_n) is strictly decreasing. The factored form $(x_n - \alpha)(x_n - \beta)$ will clearly be positive if $x_n > \beta = l$, and this looks likely from the decimal approximations to l, x_2, x_3, x_4 noted above.

So we now try to prove that $x_n > \beta$ for all n. We will do this by induction. First, $x_1 = 3 > \beta = 2.7913$ (to 4 decimal places). Next assume that $x_k > \beta$ for some positive integer k. Then $x_{k+1} = \sqrt{x_k + 5} > \sqrt{\beta + 5}$, and so $x_{k+1}^2 > \beta + 5 = \beta^2$. Taking the positive roots we get $x_{k+1} > \beta$. It follows by induction that $x_n > \beta$ for every $n \in \mathbb{N}$. Thus the sequence is strictly decreasing, clearly bounded below by 0, and so it converges. The limit l must be $\dfrac{1+\sqrt{21}}{2}$, as already established.

You might like to consider the above example with various initial values x_1. Plainly we can't allow $x_1 < -5$ because of the square root in the recurrence relation. But you could try values of x_1 between -5 and β and also other values greater than β. What happens if $x_1 = \beta$?

Exercises for Section 3.8

1. Suppose that the sequence (x_n) is given by $x_1 = m$ and $x_{n+1} = \frac{1}{2}(1 + x_n)$ for $n \geq 1$. Prove that if $m < 1$ then (x_n) is bounded above by 1 and is strictly increasing, while if $m > 1$ then (x_n) is bounded below by 1 and is strictly decreasing. Determine the limit in each case. What happens if $m = 1$?

2. Suppose that $m > 1$ and that the sequence (x_n) is given by $x_1 = m$ and $x_{n+1} = \frac{1}{2}(x_n + \frac{m}{x_n})$ for $n \geq 1$. Prove that the sequence (x_n) is bounded below by $\sqrt{m}$ and that it is monotonically decreasing. Hence show that (x_n) is convergent and determine its limit. [Hint: start by considering $x_{n+1}^2 - m$.]

3.9 Non-convergent sequences

Next we classify non-convergent sequences. Such sequences are sometimes called *divergent* but this can be slightly confusing as it covers a range of possible behaviours.

Definition 3.11. Suppose that $\forall A, \exists N$ s.t. $\forall n > N, x_n > A$. Then we say that (x_n) diverges to $+\infty$ (read as "plus infinity") and write $x_n \to +\infty$ as $n \to \infty$. Sometimes the $+$ sign is omitted from the $+\infty$. Note that there is no attempt to define individual symbols such as ∞ or the meaning of "infinity"; the sentence must be taken as a whole.

Figure 3.1 illustrates a sequence (x_n) that diverges to $+\infty$. Given any vertical height (specified by A), there is a point N on the horizontal axis beyond which all the members of the sequence lie above A in the shaded region. As A increases, we'd anticipate having to move N further to the right on the picture.

Note that we do *not* say that (x_n) *converges* to $+\infty$. The word "converges" is used exclusively for sequences that converge to some limit $l \in \mathbb{R}$. For a convergent sequence (x_n) with limit l we may write $\lim_{n\to\infty} x_n = l$, so it may be thought natural to write $\lim_{n\to\infty} x_n = +\infty$ for a sequence (x_n) that diverges to $+\infty$. In the convergent case this implies that l is a Real Number ($l \in \mathbb{R}$), but in the divergent case this must not be taken as implying that ∞ is a Real Number.

Infinity, whatever it is, is most certainly is not a Real Number ($\infty \notin \mathbb{R}$). An even more distasteful practice is adding the $\exists$ sign and writing $\exists \lim_{n\to\infty} x_n = +\infty$ which suggests that the sequence converges to a numerical limit $+\infty$ - something to avoid!

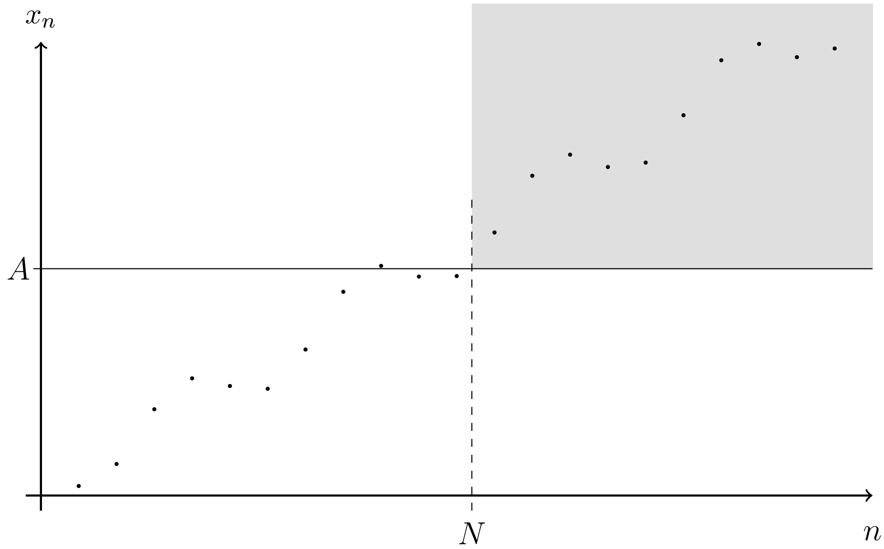


Figure 3.1: $x_n \to +\infty$ as $n \to \infty$.

Definition 3.12. If (x_n) is a sequence for which $(-x_n)$ diverges to $+\infty$, then we say that (x_n) diverges to $-\infty$ (read as "minus infinity") and write $x_n \to -\infty$ as $n \to \infty$. An equivalent definition is that $\forall B, \exists N$ s.t. $\forall n > N, x_n < B$. The $-$ sign is *never* omitted from the $-\infty$.

Definition 3.13. Sequences that diverge to $+\infty$ or $-\infty$ are sometimes said to be *properly divergent*. A sequence which is not convergent and not properly divergent is said to be *oscillatory*. An oscillatory sequence which is bounded is said to oscillate finitely, and an oscillatory sequence which is unbounded is said to oscillate infinitely.

Here are some examples, stated without proofs (which are easy).

1. $n \to +\infty$ as $n \to \infty$,
2. $-n^2 \to -\infty$ as $n \to \infty$,
3. $((-1)^n)$ oscillates finitely,
4. $((-1)^n n^3)$ oscillates infinitely.

Some of the results we saw for convergent sequences have analogies for properly divergent series, but some do not.

Theorem 3.13. Suppose that $x_n \to +\infty$ as $n \to \infty$ and that $y_n \to +\infty$ as $n \to \infty$. Then

1. (multiple rule) if $a > 0$ then $ax_n \to +\infty$ as $n \to \infty$,
2. (sum rule) $x_n + y_n \to +\infty$ as $n \to \infty$,
3. (product rule) $x_n y_n \to +\infty$ as $n \to \infty$.

We omit the proofs of these rules, but we observe that there can be no analogy of the quotient rule. To see this, suppose that $x_n > 0$ for every $n \in \mathbb{N}$ and that $x_n \to +\infty$ as $n \to \infty$. Put $y_n = 2x_n$ so that $y_n \to +\infty$ as $n \to \infty$. But then $\frac{x_n}{y_n} \to \frac{1}{2}$ as $n \to \infty$. Similarly, if we took $y_n = \sqrt{x_n}$ we'd get $\frac{x_n}{y_n} \to +\infty$ as $n \to \infty$, while $y_n = nx_n$ gives $\frac{x_n}{y_n} \to 0$ as $n \to \infty$.

In place of the quotient rule we have the following result.

Theorem 3.14. Suppose that the sequence (x_n) is eventually positive (meaning that $\exists N^*$ s.t. $\forall n > N^*, x_n > 0$). Then $x_n \to 0$ as $n \to \infty$ if and only if $\dfrac{1}{x_n} \to +\infty$ as $n \to \infty$.

Proof. Suppose first that $x_n \to 0$ as $n \to \infty$. Choose $A \in \mathbb{R}$. If $A \leq 0$ put $\epsilon = 1$ and if $A > 0$ put $\epsilon = \frac{1}{A}$. Then $\exists N$ s.t. $\forall n > N,\ 0 < x_n < \epsilon$. But then $\forall n > N,\ \dfrac{1}{x_n} > \dfrac{1}{\epsilon} \geq A$. Hence $\dfrac{1}{x_n} \to +\infty$ as $n \to \infty$.

Next suppose that $\dfrac{1}{x_n} \to +\infty$ as $n \to \infty$. Choose $\epsilon > 0$. Put $A = \frac{1}{\epsilon}$. Then $\exists N$ s.t. $\forall n > N,\ \dfrac{1}{x_n} > A$. But then $\forall n > N,\ 0 < x_n < \frac{1}{A} = \epsilon$. Hence $x_n \to 0$ as $n \to \infty$. $\square$

The sequence (x_n) is eventually positive if and only if the sequence $(\frac{1}{x_n})$ is eventually positive, and a sequence that diverges to $+\infty$ must be eventually positive. So if we know that $y_n \to +\infty$ as $n \to \infty$, then it follows (without having to specify "eventually positive") that $\frac{1}{y_n} \to 0$ as $n \to \infty$. It's also the case that if $y_n \to -\infty$ as $n \to \infty$ then $\frac{1}{y_n} \to 0$ as $n \to \infty$ (consider the sequence $(-y_n)$).

The reciprocals of the basic null sequences provide examples of sequences that diverge to $+\infty$, although we must insist that the constant a that appears in two of these null sequences is strictly positive.

Example 3.8. Suppose that x is not a multiple of π. Prove that the sequence $(\sin(nx))$ oscillates finitely. You may assume all the usual properties of the sine and cosine functions.

Solution. Since $|\sin(nx)| \leq 1$, the sequence is bounded, and so either it converges to some limit l or it oscillates finitely. Suppose that $\sin(nx) \to l$ as $n \to \infty$. Then

$$2\sin(x)\cos(nx) = \sin((n+1)x) - \sin((n-1)x) \to l - l = 0 \text{ as } n \to \infty.$$

But x is not a multiple of π, so $\sin(x) \neq 0$, and consequently we can deduce that $\cos(nx) \to 0$ as $n \to \infty$. But then

$$2\sin(x)\sin(nx) = \cos((n-1)x) - \cos((n+1)x) \to 0 - 0 = 0 \text{ as } n \to \infty,$$

from which we deduce that $\sin(nx) \to 0$ as $n \to \infty$. Then we can deduce that

$$\cos^2(nx) + \sin^2(nx) \to 0 + 0 = 0 \text{ as } n \to \infty.$$

But this contradicts the fact that $\cos^2(nx)+\sin^2(nx) = 1$. From this contradiction, we conclude that the sequence $(\sin(nx))$ does not converge to any limit, and so it oscillates finitely.

Exercises for Section 3.9

1. Classify the behaviour of each of the following sequences as convergent, divergent to $+\infty$ or $-\infty$, oscillating finitely or infinitely.
(i) $\left(n^2 + (-1)^n n\right)$ (ii) $\left(n + (-1)^n n^2\right)$ (iii) $\left(1 + (-1)^n\right)$
(iv) $\left(\sqrt{n+1} - \sqrt{n}\right)$ (v) $\left(\sqrt{n^2+n} - n\right)$ (vi) $\left(\sqrt{n^3+n^2} - \sqrt{n^3}\right)$

2. If $x_n \to +\infty$ as $n \to \infty$, prove that $\frac{1+x_n}{x_n} \to 1$ as $n \to \infty$.

3. If $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n}$, prove that $s_{n+1} - s_n > \frac{1}{2}$ and hence show that $s_n \to +\infty$ as $n \to \infty$.

4. If $t_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2}$, using induction or otherwise, prove that for all n, $t_n \leq 2 - \frac{1}{n}$. Deduce that (t_n) converges to some limit $l \leq 2$.

The last two questions are where I think Real Analysis starts to produce interesting and unexpected results. The terms $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ decrease steadily with limit 0, but their sum nevertheless grows without limit. Putting it crudely, these terms don't get smaller fast enough to ensure that their sum remains bounded above. If we square these individual terms and consider the resulting sum, the answer is very different. We will look at this again in more detail in the next chapter on series.

3.10 Subsequences

Next we look at subsequences of a given sequence and consider some associated results. First the definition.

Definition 3.14. Suppose that (n_r) is a strictly increasing sequence of positive integers and that (x_n) is a given sequence. Then we say that the sequence (x_{n_r}) is a *subsequence* of (x_n).

For example, if $n_r = 2r$ for $r = 1, 2, 3, \ldots$ then $x_{n_r} = x_{2r}$, giving the subsequence $(x_{n_r}) = (x_2, x_4, x_6, \ldots)$. In general, a subsequence of (x_n) comprises an infinite subset of the original sequence with the selected terms in their original ordering. Our first result about subsequences is fairly obvious.

Theorem 3.15. If $x_n \to l$ as $n \to \infty$ and if (x_{n_r}) is a subsequence of (x_n), then $(x_{n_r}) \to l$ as $r \to \infty$.

Proof. Choose $\epsilon > 0$. Since $x_n \to l$ as $n \to \infty$, there exists N such that for every $n > N$, $|x_n - l| < \epsilon$. But (n_r) is a strictly increasing sequence of positive integers, so $n_1 \geq 1, n_2 \geq 2, \ldots, n_r \geq r$. So for any $r > N$ we have $n_r > N$ and consequently $|x_{n_r} - l| < \epsilon$. It follows that $(x_{n_r}) \to l$ as $r \to \infty$. □

Corollary 3.15.1. if (x_n) has two subsequences converging to different limits, or has a non-convergent subsequence, then (x_n) cannot be convergent.

As an example, consider the sequence $((-1)^n)$. The odd numbered terms form the constant subsequence (-1) that converges to -1, while the even numbered terms form the constant subsequence (1) that converges to 1. So the sequence $((-1)^n)$ cannot be convergent and, since it is bounded, we deduce that it oscillates finitely.

One reason for looking at subsequences is that many non-convergent sequences do have convergent subsequences. In fact any bounded sequence must have a convergent subsequence. To get us started on a proof of this, consider a non-empty set of numbers $S \subseteq \mathbb{R}$ that is bounded above with least upper bound A (i.e. $A = \sup S$). If T is any non-empty subset of S, then it is also bounded above by A and so its least upper bound B must satisfy $B \leq A$. So $T \subseteq S$ implies that $\sup T \leq \sup S$. Now apply this principle to a sequence (x_n) that is bounded above with least upper bound M_1 (the reason for the subscript 1 will become apparent). So $M_1 = \sup(x_1, x_2, x_3, \ldots)$. Put $M_2 = \sup(x_2, x_3, x_4, \ldots)$. Then $M_2 \leq M_1$. Proceeding in this way with $M_r = \sup(x_r, x_{r+1}, x_{r+2}, \ldots)$, we have $M_{r+1} \leq M_r$ for every $r = 1, 2, 3 \ldots$. Consequently the sequence (M_r) is monotonically decreasing. If it is bounded below, it converges to some limit denoted by $\overline{\lim}_{n\to\infty} x_n$ or by $\limsup_{n\to\infty} x_n$. If we denote M_r as $\sup_{n\geq r}(x_n)$, then we can write the definition in the following form.

Definition 3.15. Suppose that the sequence $(\sup_{n\geq r}(x_n))$ (where r takes the values $1, 2, 3, \ldots$) is bounded below, so that it converges to some limit l. Then we define

$$\overline{\lim_{n\to\infty}}\, x_n = \limsup_{n\to\infty} x_n = \lim_{r\to\infty} \sup_{n\geq r}(x_n) = l.$$

The limiting value l is called the *upper limit* or *limit-superior* of the original sequence (x_n).

Using a similar argument, suppose now that $S \subseteq \mathbb{R}$ is non-empty and is bounded below with greatest lower bound A (i.e. $A = \inf S$). If T is any non-empty subset of S, then it is also bounded below by A and so its greatest lower bound B must satisfy $B \geq A$. So $T \subseteq S$ implies that $\inf T \geq \inf S$. Now apply this principle to a sequence (x_n) that is bounded below with greatest lower bound m_1, i.e. $m_1 = \inf(x_1, x_2, x_3, \ldots)$. Put $m_2 = \inf(x_2, x_3, x_4, \ldots)$. Then $m_2 \geq m_1$. Proceeding in this way with $m_r = \inf(x_r, x_{r+1}, x_{r+2}, \ldots)$, we have $m_{r+1} \geq m_r$ for every $r = 1, 2, 3 \ldots$. Consequently the sequence (m_r) is monotonically increasing. If it is bounded above, it converges to some limit denoted by $\underline{\lim}_{n\to\infty}\, x_n$ or by $\liminf_{n\to\infty} x_n$. If we denote m_r as $\inf_{n\geq r}(x_n)$, then we can write the definition in the following form.

Definition 3.16. Suppose that the sequence $(\inf_{n\geq r}(x_n))$ (where r takes the values $1, 2, 3, \ldots$) is bounded above, so that it converges to some limit l. Then we define

$$\underline{\lim_{n\to\infty}}\, x_n = \liminf_{n\to\infty} x_n = \lim_{r\to\infty} \inf_{n\geq r}(x_n) = l.$$

The limiting value l is called the *lower limit* or *limit-inferior* of the original sequence (x_n).

As an easy example, the sequence formed by $x_n = (-1)^n$ has

$$\limsup_{n\to\infty} x_n = 1, \text{ and } \liminf_{n\to\infty} x_n = -1.$$

Here is a more informative example.

Example 3.9. Consider the sequence $(x_n) = \left((-1)^n \dfrac{n+1}{n}\right)$. Figure 3.2 illustrates the situation. We will prove that $\limsup_{n\to\infty} x_n = 1$ and, by a similar argument, it follows that $\liminf_{n\to\infty} x_n = -1$.

Note first that $\frac{n+1}{n} = 1 + \frac{1}{n}$ forms a strictly decreasing sequence that converges to 1. If r is even, $\sup_{n \geq r}(x_n) = x_r = \dfrac{r+1}{r}$, while if r is odd, $\sup_{n \geq r}(x_n) = x_{r+1} = \dfrac{r+2}{r+1}$. Hence, whether r is even or odd, we have $1 \leq \sup_{n \geq r}(x_n) \leq \dfrac{r+1}{r}$. Hence, by the sandwich rule, $\lim_{r \to \infty} \sup_{n \geq r}(x_n) = 1$. The proof for $\liminf$ is similar.

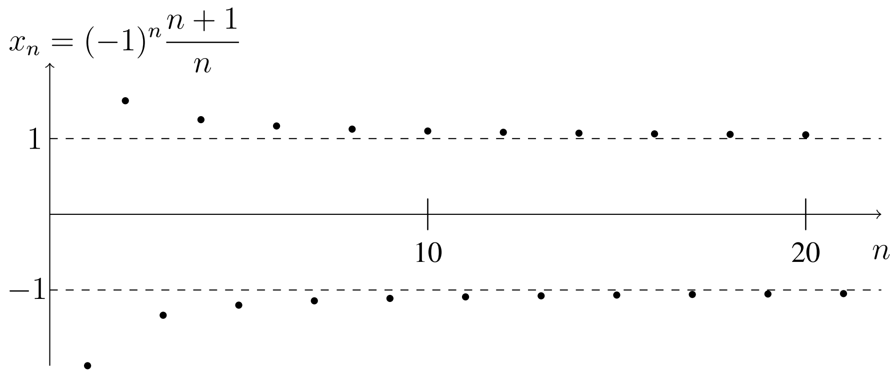


Figure 3.2: $\limsup$ and $\liminf$.

Here is another example.

Example 3.10. Consider the sequence $(x_n) = \left((-1)^n \dfrac{n}{n+1}\right)$. Figure 3.3 illustrates the situation. We state without proof that $\limsup_{n \to \infty} x_n = 1$ and $\liminf_{n \to \infty} x_n = -1$.

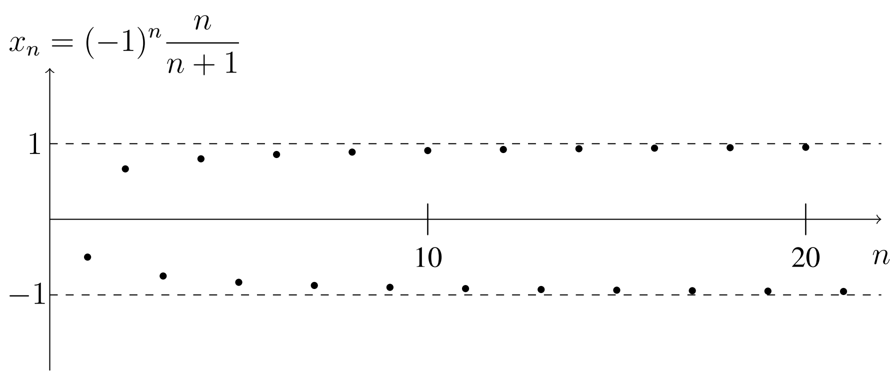


Figure 3.3: $\limsup$ and $\liminf$ again.

There is a subtlety in the definitions of $\limsup$ and $\liminf$ for a sequence (x_n) that may have escaped your notice. These are defined as limits of sequences of bounds of (x_n), not as limits of subsequences of (x_n). In Example 3.9, the bounds were actually members of the sequence, but this is not necessarily the case for all sequences, as shown by Example 3.10. However, we can prove that there is always a subsequence of (x_n) that converges to $\limsup\limits_{n\to\infty} x_n$ whenever the latter is well-defined (i.e. when (x_n) is bounded above and the sequence $(\sup\limits_{n\geq r}(x_n))$ is bounded below). We can do similarly for $\liminf$. A consequence (see Theorem 3.18 below) is that $x_n \to l$ as $n \to \infty$ if and only if $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n = l$.

Theorem 3.16. Suppose that (x_n) is bounded above and the sequence $(\sup\limits_{n\geq r}(x_n))$ is bounded below, so that there exists a Real Number $l = \limsup\limits_{n\to\infty} x_n$. Then there is a subsequence of (x_n) that converges to l.

Proof. Denote $(\sup\limits_{n\geq r}(x_n))$ as M_r for each $r = 1, 2, 3, \ldots$ so that $M_r \to l$ as $r \to \infty$. Take $r_1 = 1$. Then take $r_2 > r_1$ such that $M_{(r_1+1)} \geq x_{r_2} > M_{(r_1+1)} - \frac{1}{2}$. Proceeding in this way, take $r_{k+1} > r_k$ such that $M_{(r_k+1)} \geq x_{r_k} > M_{(r_k+1)} - \frac{1}{k}$. These choices are possible because each M_r is the ***least upper bound*** of the sequence obtained from (x_n) by deleting the first $r-1$ terms. [Take your time to convince yourself that this is correct.]

The sequence (r_k) $(k = 1, 2, 3, \ldots)$ is strictly increasing, so the subsequence (M_{r_k}) must converge to l. Consequently $(M_{(r_k+1)})$ converges to l as does $(M_{(r_k+1)} - \frac{1}{k})$. Then by the sandwich rule applied to the inequality $M_{(r_k+1)} \geq x_{r_k} > M_{(r_k+1)} - \frac{1}{k}$, we deduce that $x_{r_k} \to l$ as $k \to \infty$. □

We can prove the analogous result for $\liminf$ in a similar way. We state the result without proof.

Theorem 3.17. Suppose that (x_n) is bounded below and the sequence $(\inf\limits_{n\geq r}(x_n))$ is bounded above, so that there exists a Real Number $l = \liminf\limits_{n\to\infty} x_n$. Then there is a subsequence of (x_n) that converges to l.

From the previous two theorems we can deduce the following useful result known as the Bolzano-Weierstrass Theorem.

Corollary 3.17.1 (The Bolzano-Weierstrass Theorem)**.** If the sequence (x_n) is bounded then it has a convergent subsequence.

Proof. The boundedness of (x_n) ensures that both $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$ exist. By the previous two theorems, there is a subsequence of (x_n) that converges to the upper limit and a subsequence of (x_n) that converges to the lower limit. (These can be the same subsequence if the upper limit equals the lower limit - see Theorem 3.18 below.) □

Theorem 3.18. $x_n \to l$ as $n \to \infty$ if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l$.

Proof. Suppose first that $x_n \to l$ as $n \to \infty$ then (x_n) is bounded above and below (see Theorem 3.7), so both $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ exist. Every subsequence of (x_n) converges to l and, in particular, subsequences that converge to these upper and lower limits. So both the upper and lower limits must equal l.

Next suppose that $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l$. This implies that (x_n) is bounded above and below and that the sequences (M_r) and (m_r), where $M_r = (\sup_{n\geq r}(x_n))$ and $m_r = (\inf_{n\geq r}(x_n))$, converge to the common value l. But for each positive integer r, $m_r \leq x_r \leq M_r$, so by the sandwich rule we must have $x_r \to l$ as $n \to \infty$. $\square$

Using the previous result, we can easily prove the following (which may seem obvious).

Theorem 3.19. Suppose that $a_n > 0$ for every $n \in \mathbb{N}$ and that $a_n \to l$ as $n \to \infty$. Then $\sqrt{a_n} \to \sqrt{l}$ as $n \to \infty$.

Proof. Since (a_n) is a convergent sequence, it is bounded, i.e. $\exists A$ s.t. $\forall n \in \mathbb{N}, 0 < a_n < A$. But then $0 < \sqrt{a_n} < \sqrt{A}$, so the sequence $(\sqrt{a_n})$ is bounded. If this sequence were not convergent, it would have subsequences converging respectively to its upper limit $\overline{a}$ and to its lower limit $\underline{a}$, which are unequal. But then the corresponding subsequences of (a_n) would, by the product rule ($a_n = \sqrt{a_n} \times \sqrt{a_n}$), converge respectively to $\overline{a}^2$ and $\underline{a}^2$, which are unequal, a contradiction. Hence $(\sqrt{a_n})$ must be convergent, and if a is its limit, the product rule gives the limit of (a_n) as a^2. So $a^2 = l$, and consequently $a = \sqrt{l}$. $\square$

Of course the theorem and its proof are easily generalised to k^{th} roots for any positive integer k. We state it without giving the details.

Theorem 3.20. Suppose that $a_n > 0$ for every $n \in \mathbb{N}$ and that $a_n \to l$ as $n \to \infty$. Then if k is any positive integer, $\sqrt[k]{a_n} \to \sqrt[k]{l}$ as $n \to \infty$.

Exercises for Section 3.10

1. Prove that the sequence $\left(\left(\frac{1}{(2n)!}\right)^{\frac{1}{n}}\right)$ converges and find its limit.

2. If (x_n) is a monotonically increasing sequence which has a convergent subsequence with limit l, prove that $x_n \to l$ as $n \to \infty$.

3. Assuming it is possible to prove that $(1 + \frac{2}{n})^n$ forms a monotonically increasing sequence, prove that $(1 + \frac{2}{n})^n \to e^2$ as $n \to \infty$.

4. Prove that $(1 + \frac{2}{n})^n$ forms a monotonically increasing sequence.

5. Use the identity
$$1 - \frac{1}{n} = \frac{1}{1 + \frac{1}{n-1}}$$
to prove that $(1 - \frac{1}{n})^n \to e^{-1}$ as $n \to \infty$.

6. What do think might happen to $(1 + \frac{x}{n})^n$ as $n \to \infty$? (No proof required, just a guess.)

7. Suppose that the sequence (x_n) is bounded, but not convergent (i.e. it oscillates finitely). Prove that it has two subsequences that converge to different limits.

8. Prove that if (x_n) is unbounded above then it has a subsequence diverging to $+\infty$.

9. Suppose that (x_n) is a sequence that has a finite number of subsequences $S_1, S_2, \ldots S_k$ that cover $\{x_n : n \in \mathbb{N}\}$ (meaning that every x_n lies in at least one of these subsequences). If each of these subsequences converges to the same limit l, prove that $x_n \to l$ as $n \to \infty$.

3.11 Cauchy sequences

Cauchy's criterion provides a test for convergence of a sequence where the limit is not known. We make the following definition.

Definition 3.17. A sequence (x_n) is called a *Cauchy sequence* if and only if the following property is satisfied:
$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n > N, |x_m - x_n| < \epsilon.$$

Intuitively, this says that a Cauchy sequence is one where the terms of the sequence get arbitrarily close together as you move along the sequence. Rather surprisingly this is sufficient to ensure convergence.

Theorem 3.21. A sequence (x_n) is convergent if and only if it is a Cauchy sequence.

Proof. We start with the easier implication by showing that convergence implies that the Cauchy criterion is satisfied. Choose $\epsilon > 0$. If $x_n \to l$ as $n \to \infty$ then $\exists N$ s.t. $\forall n > N, |x_n - l| < \frac{\epsilon}{2}$. So if $m, n > N$ we have
$$|x_m - x_n| = |(x_m - l) + (l - x_n)| \leq |x_m - l| + |x_n - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence a convergent sequence is necessarily a Cauchy sequence.

Next assume that (x_n) is a Cauchy sequence. We first show that (x_n) is bounded by taking $\epsilon = 1$ in the definition and choosing M to be the first integer greater than N, so that if $n > N$ then $|x_M - x_n| < 1$. Put

$$A = \max(|x_1|, |x_2|, \ldots, |x_M|, |x_M| + 1).$$

If $n \leq M$ then $|x_n| \leq A$, and if $n > M$ then

$$|x_n| = |x_M + (x_n - x_M)| \leq |x_M| + |x_M - x_n| < |x_M| + 1 \leq A.$$

So for every $n \in \mathbb{N}$, $|x_n| \leq A$, meaning that the sequence (x_n) is bounded. It follows that (x_n) has a subsequence (x_{n_r}) converging to some limit l. We will prove that the entire sequence (x_n) converges to l. To do this, choose $\epsilon > 0$. Then $\exists N_1$ s.t. $\forall m, n > N_1, |x_m - x_n| < \epsilon/2$, and $\exists N_2$ s.t. $\forall r > N_2, |x_{n_r} - l| < \epsilon/2$. Put $N^* = \max(N_1, N_2)$. Take $n > N^*$ and $r > N^*$ (so that $n_r > N^*$ because $n_r \geq r$). We have

$$|x_n - l| = |(x_n - x_{n_r}) + (x_{n_r} - l)| \leq |x_n - x_{n_r}| + |x_{n_r} - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that $x_n \to l$ as $n \to \infty$. □

EXERCISES 3.11

1. If $|x_n - x_{n+1}| \leq \frac{1}{2^n}$ for all n, prove that (x_n) is a Cauchy sequence and therefore convergent.

Chapter 4

Series

4.1 Basic results

The first thing to say is that mathematicians use the words "sequence" and "series" to mean different things. A *sequence* is an ordered list, often (but not always) a list of numbers. On the other hand a *series* is formed from a sequence of numbers by adding them together. In this chapter we will look at infinite series formed from infinite sequences. We won't keep saying "infinite". Here is the definition.

Definition 4.1. Suppose that $(a_n) = (a_1, a_2, a_3, \ldots)$ is an (infinite) sequence of Real Numbers. We define the n^{th} *partial sum* of the corresponding series to be

$$S_n = a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i.$$

If the sequence (S_n) of partial sums converges to some limit S as n tends to infinity, then we say that S is the sum of the (infinite) series $a_1 + a_2 + a_3 + \ldots$. We may write

$$S = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots.$$

We can also express this by saying that the series $\sum_{i=1}^{\infty} a_i$ converges to S, more briefly by writing $\exists \sum_{i=1}^{\infty} a_i = S$, or even more briefly by writing $\sum_{i=1}^{\infty} a_i = S$.

If the sequence (S_n) of partial sums does not converge then we say that $\sum_{i=1}^{\infty} a_i$ diverges. A series is said to be *properly* divergent if the sequence of partial sums is properly divergent, i.e. it tends to either $+\infty$ or to $-\infty$. In such cases we may write $\sum_{i=1}^{\infty} a_i = +\infty$ or $\sum_{i=1}^{\infty} a_i = -\infty$, as appropriate. But remember this is shorthand and should not be used to treat things like $+\infty$ or $-\infty$ as if they were numbers.

The numbers a_i are generally called the *terms* of the series. Sometimes it is more convenient to start the summation at an index value i other than $i = 1$. A common convenient alternative is $i = 0$. So we may use notations like $\sum_{i=0}^{\infty} a_i = a_0 + a_1 + a_2 + \ldots$, and choose to define S_n as either the sum of the first n terms (in which case the sum ends at a_{n-1}), or as the sum of terms up to and including a_n (i.e. $n+1$ terms). Such minor changes are just notational and do not affect issues of convergence.

Something to observe about notation is that the "i" in $\sum_{i=1}^{\infty} a_i$ is a *dummy* variable. You can replace it by any symbol not used elsewhere. For example, $\sum_{i=1}^{\infty} a_i$ and $\sum_{n=1}^{\infty} a_n$ mean exactly the same thing. Of course if you define S_n as the n^{th} partial sum, you must not write $S_n = \sum_{n=1}^{n} a_n$ because this uses n for two different purposes: as a dummy variable and as a variable specifying the end of the summation.

Note. We allow ourselves to talk about $\sum_{i=1}^{\infty} a_i$ whether or not it converges. So the mere writing down of such a symbol should not be taken to imply that the series has a sum. Moreover, it should not be assumed that it is impossible to obtain meaningful results about divergent series - whole books have been written about them, see [G. H. Hardy,"Divergent Series"].

Obviously we would not talk about convergent series unless there were some examples. Here is an easy one.

Example 4.1. Prove that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1.$

Solution. We start by examining the first few terms:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots.$$

Observe that the n^{th} term is given by

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \qquad \text{(partial fractions).}$$

Hence the n^{th} partial sum S_n is given by

$$S_n = \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) + \ldots + \Big(\frac{1}{n-1} - \frac{1}{n}\Big) + \Big(\frac{1}{n} - \frac{1}{n+1}\Big).$$

Observe that the second term in each bracket cancels with the first term in the next bracket. We are left with the first term in the first bracket and the second term in the last bracket. So $S_n = 1 - \frac{1}{n+1} \to 1$ as $n \to \infty$, and the result follows.

There are some potentially upsetting results about infinite series. If you look back to the previous chapter, in the last two questions of Exercises 3.9 you will see that whether or not a series $\sum_{i=1}^{\infty} a_i$ converges can depend on how fast the terms get smaller. But even worse is to come. We will see that the sum can vary depending on the order of the terms, or even how we might bracket them together. Of course these two operations alter the underlying sequence (a_n), so maybe such results are not so surprising, even though finite sums are not affected by re-ordering or bracketing. We start with two fairly simple results that are extremely useful.

Theorem 4.1. Suppose that $a_i \geq 0$ for every $i \in \mathbb{N}$ and that the sequence formed by the partial sums $S_n = \sum_{i=1}^{n} a_i$ is bounded above, then $\sum_{i=1}^{\infty} a_i$ converges.

Proof. Since $a_i \geq 0$ for every $i \in \mathbb{N}$, the sequence (S_n) is monotonically increasing. If this sequence is bounded above, it must therefore converge. $\square$

Theorem 4.2. If $\sum_{i=1}^{\infty} a_i$ converges then $a_n \to 0$ as $n \to \infty$.

Proof. Put $S_n = \sum_{i=1}^{n} a_i$ and let S denote the limit of (S_n), so that $S_n \to S$ as $n \to \infty$. But then $S_{n-1} \to S$ as $n \to \infty$. Hence $S_n - S_{n-1} \to S - S = 0$ as $n \to \infty$. However $S_n - S_{n-1} = a_n$, so we have proved that $a_n \to 0$ as $n \to \infty$. $\square$

A consequence of this result is:

Corollary 4.2.1. If (a_n) does not converge to zero then $\sum_{i=1}^{\infty} a_i$ cannot converge.

The converse of Theorem 4.2 is FALSE. That is to say, a series may diverge even though the terms tend to zero. We will prove (Theorem 4.5) that $\sum_{i=1}^{\infty} \frac{1}{n}$ diverges, even though $\frac{1}{n} \to 0$ as $n \to \infty$. But before that we examine geometric series.

Theorem 4.3. The geometric series $\sum_{i=0}^{\infty} x^i$ converges with sum $\frac{1}{1-x}$ if $|x| < 1$, but it diverges if $|x| \geq 1$.

Proof. The first few terms of the series are: $1 + x + x^2 + x^3 + \ldots$. We put $S_n = \sum_{i=0}^{n-1} x^i$, the sum of the first n terms. We examined such a sum in the previous chapter, Section 3.1. We write S_n in full and xS_n below it, lining up similar powers of x.

$$\begin{aligned} S_n &= 1 + x + x^2 + x^3 + \ldots + x^{n-1} \\ xS_n &= \phantom{1 + {}} x + x^2 + x^3 + \ldots + x^{n-1} + x^n \\ \text{so } \ S_n - xS_n &= 1 - x^n. \end{aligned}$$

Hence $(1-x)S_n = 1 - x^n$, which gives $S_n = \dfrac{1-x^n}{1-x}$, provided that $x \neq 1$.

If $|x| < 1$ then $x^n \to 0$ as $n \to \infty$, and so $S_n \to \frac{1}{1-x}$ as $n \to \infty$.
If $|x| \geq 1$ then the terms of the series do not tend to zero, and so by Corollary 4.2.1 above, the series cannot converge. $\square$

Here is a verbal and pictorial illustration of Theorem 4.3 in the case when $x = \frac{1}{2}$. Take ruler of unit length (say 10cm). Add on half the length again (5cm), then add on half that length (2.5cm), and so on. If you keep going you will approach two units of length (20cm). Each successive term added to the partial sum halves the remaining distance to two units. So the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges to 2. Figure 4.1 illustrates what is happening.

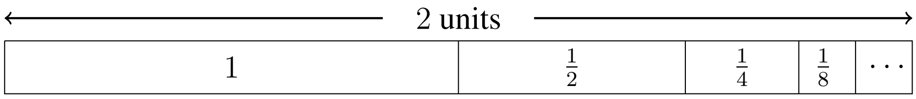


Figure 4.1: $\sum_{i=0}^{\infty}(\frac{1}{2})^i = 2$.

Theorem 4.3 enables us to deal with recurring decimals. You are probably aware that $0.\bar{9} = 0.999\cdots = 1$. However, many people retain a sense of unease about this, probably feeling that in some way $0.\bar{9}$ is ever so slightly less than 1. I hope the following explanation will resolve this unease. The resolution lies in the fact that $0.\bar{9}$ is really an (infinite) series in disguise:

$$0.\bar{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots.$$

The sum of the first n terms is

$$\begin{aligned} S_n &= \frac{9}{10}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \ldots + \frac{1}{10^{n-1}}\right] \\ &= \frac{9}{10}\left[\frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}}\right] \\ &= 1 - \frac{1}{10^n} \\ &\to 1 \text{ as } n \to \infty. \end{aligned}$$

So when we write $0.\bar{9} = 1$ the equals sign is really doing a bit more work than it usually does. It is asserting that the infinite series $0.\bar{9}$ converges as well as asserting that its limit is 1. Maybe we should really write $\exists 0.\bar{9} = 1$ to emphasise the convergence but somehow I don't see this ever gaining acceptance. Of course the partial sums are all less than 1, but the limit itself, represented by $0.\bar{9}$, is exactly 1.

Recognising recurrent decimals as infinite series justifies performing arithmetic operations as we have done since school days, for example $3 \times 0.\bar{3} = 0.\bar{9}$ ($3 \times \frac{1}{3} = 1$, right?). The length of the recurrent section doesn't matter, such decimals are always infinite series in disguise. For example,

$$0.\overline{12} = 0.121212\cdots = \frac{12}{100}\left[1 + \frac{1}{100} + \frac{1}{100^2} + \dots\right].$$

Next we give some simple results on series that are easy consequences of earlier results on sequences

Theorem 4.4 (Combination Rules for Series)**.**

1. If $\exists \sum_{n=1}^{\infty} x_n = S$ and α is any Real Number, then $\exists \sum_{n=1}^{\infty}(\alpha x_n) = \alpha S$.
2. If $\sum_{n=1}^{\infty} x_n$ is divergent and $\alpha \neq 0$, then $\sum_{n=1}^{\infty}(\alpha x_n)$ is divergent.
3. If $m \in \mathbb{N}$ and $\exists \sum_{n=1}^{\infty} x_n = S$, then $\exists \sum_{n=m+1}^{\infty} x_n = S - \sum_{n=1}^{m} x_n$.
4. If $m \in \mathbb{N}$ and $\exists \sum_{n=m+1}^{\infty} x_n = S$, then $\exists \sum_{n=1}^{\infty} x_n = S + \sum_{n=1}^{m} x_n$.
5. If $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ differ in only a finite number of terms then either both series converge, or they both diverge.
6. If $\exists \sum_{n=1}^{\infty} x_n = S$ and $\exists \sum_{n=1}^{\infty} y_n = T$, then $\exists \sum_{n=1}^{\infty}(x_n + y_n) = S + T$.

Proof.

1. Put $S_n = \sum_{i=1}^{n} x_i$ so that $S_n \to S$ as $n \to \infty$. Then the n^{th} partial sum of $\sum_{n=1}^{\infty}(\alpha x_n)$ is $\sum_{i=1}^{n}(\alpha x_i) = \alpha S_n \to \alpha S$ as $n \to \infty$.
2. If $\exists \sum_{n=1}^{\infty}(\alpha x_n) = S$, then by part (1) $\exists \sum_{n=1}^{\infty}(\frac{1}{\alpha}\alpha x_n) = \frac{S}{\alpha}$, i.e $\exists \sum_{n=1}^{\infty} x_n$, a contradiction.
3. The partial sums of the two series differ by the fixed amount $\sum_{n=1}^{m} x_n$.
4. As in part (3).
5. $\exists m \in \mathbb{N}$ s.t. $\forall n > m, x_n = y_n$. Hence $\sum_{n=m+1}^{\infty} x_n$ and $\sum_{n=m+1}^{\infty} y_n$ are identical, and so either they both converge of they both diverge. The result then follows from parts (3) and (4).
6. Denote the n^{th} partial sums of the two series as S_n and T_n respectively. Then the n^{th} partial sum of the combined series is $\sum_{i=1}^{n}(x_i + y_i) = S_n + T_n \to S + T$ as $n \to \infty$.

□

You might wonder why we haven't given a result about multiplying convergent series. If you multiply $(a+b)(c+d)$ you get 4 terms: $ac+ad+bc+bd$. If you multiply two sums each having 10 terms, you get a product with 100 terms. So you can see that multiplying two series, each with infinitely many terms, is likely to be a complicated business. We will address this later in this chapter. Meanwhile here is a non-obvious result.

Theorem 4.5. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent if $\alpha > 1$, but divergent if $\alpha \leq 1$.

The particular case $\alpha = 1$ corresponds to the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$, which is known as the *harmonic* series and is DIVERGENT.

Proof. We denote the sum of the first n terms by S_n.

Suppose first that $\alpha > 1$. Clearly (S_n) is a strictly increasing sequence because as n increases, more positive terms are added to the existing partial sum. If we can show that (S_n) is bounded above, it will follow that (S_n) is convergent. To do this, note first that because $n \leq 2^n - 1$, we have $S_n \leq S_{2^n-1}$ for every positive integer n. But

$$\begin{aligned} S_{2^n-1} = \frac{1}{1^\alpha} &+ \Big(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\Big) + \Big(\frac{1}{4^\alpha} + \frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha}\Big) + \ldots \\ &+ \Big(\frac{1}{2^{(n-1)\alpha}} + \ldots + \frac{1}{(2^n-1)^\alpha}\Big). \end{aligned}$$

Think carefully about the final bracket, it contains 2^{n-1} terms. It follows that

$$\begin{aligned} S_n \leq S_{2^n-1} &\leq 1 + \frac{2}{2^\alpha} + \frac{4}{4^\alpha} + \ldots + \frac{2^{n-1}}{2^{(n-1)\alpha}} \\ &= 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{4^{\alpha-1}} + \ldots + \frac{1}{(2^{n-1})^{\alpha-1}} \\ &= 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{(2^{\alpha-1})^2} + \ldots + \frac{1}{(2^{\alpha-1})^{n-1}} \quad \text{(a geometric series)} \\ &= \frac{1 - \frac{1}{(2^{\alpha-1})^n}}{1 - \frac{1}{2^{\alpha-1}}} \\ &< \frac{1}{1 - \frac{1}{2^{\alpha-1}}} \quad \Big(\text{since } 0 < \frac{1}{2^{\alpha-1}} < 1, \text{ because } \alpha > 1\Big). \end{aligned}$$

So if $\alpha > 1$ the sequence of (strictly increasing) partial sums is bounded above and is therefore convergent.

Now suppose that $\alpha \leq 1$. Note that in this case $n^\alpha \leq n$, so that $\frac{1}{n^\alpha} \geq \frac{1}{n}$. We will show that the subsequence (S_{2^n}) is unbounded above and therefore divergent. We have

$$\begin{aligned} S_{2^n} = &\frac{1}{1^\alpha} + \frac{1}{2^\alpha} + \Big(\frac{1}{3^\alpha} + \frac{1}{4^\alpha}\Big) + \Big(\frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha} + \frac{1}{8^\alpha}\Big) + \dots \\ &+ \Big(\frac{1}{(2^{n-1}+1)^\alpha} + \dots + \frac{1}{(2^n)^\alpha}\Big). \end{aligned}$$

Again, the final bracket contains 2^{n-1} terms. Since $\frac{1}{n^\alpha} \geq \frac{1}{n}$, this gives

$$\begin{aligned} S_{2^n} &\geq \frac{1}{1} + \frac{1}{2} + \Big(\frac{1}{3} + \frac{1}{4}\Big) + \Big(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\Big) + \dots \\ &\quad + \Big(\frac{1}{(2^{n-1}+1)} + \dots + \frac{1}{(2^n)}\Big) \\ &\geq 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \dots + \frac{2^{n-1}}{2^n} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} \qquad (n+1 \text{ terms}) \\ &= 1 + \frac{n}{2}. \end{aligned}$$

Hence if $\alpha \leq 1$ the sequence (S_n) has a divergent subsequence and so it must also be divergent. Indeed, since (S_n) is increasing and not bounded above we can say $S_n \to +\infty$ as $n \to \infty$. We may write this as $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = +\infty$ when $\alpha \leq 1$. □

It is perhaps surprising that the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is divergent. Although the terms tend to zero, they do not do so fast enough to counteract the effect of adding more and more of them together. The proof of divergence is quite instructive. For example, if we want an estimate for a value of N that will ensure that $\sum_{n=1}^{N} \frac{1}{n}$ exceeds 10^6, it tells us that we can take $N = 2^m$ where $1 + \frac{m}{2} > 10^6$. So $N = 2^{2\times 10^6}$ would give $\sum_{n=1}^{N} \frac{1}{n} > 10^6$.

It is probably worth making another point about the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ that really has nothing to do with the proof but concerns what is meant by n^α. In Appendix C we show that if q is a positive integer then any positive Real Number a has a unique positive q^{th} root $a^{\frac{1}{q}}$. This facilitates a definition of rational powers of a positive number a. If p is any integer and q is a positive integer then for a positive Real Number a we can define $a^{\frac{p}{q}}$ to be the q^{th} root of a^p, i.e $a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}}$. It is easy to prove that

$$(a^p)^{\frac{1}{q}} = (a^{\frac{1}{q}})^p.$$

To see this, consider

$$\big[(a^{\frac{1}{q}})^p\big]^q = (a^{\frac{1}{q}})^{pq} = \big[(a^{\frac{1}{q}})^q\big]^p = a^p.$$

Then taking q^{th} roots we get $(a^{\frac{1}{q}})^p = (a^p)^{\frac{1}{q}}$, as required.

So rational powers are relatively easy to define and behave as we expect. But what about a^α when α is irrational? The answer is that we *can* define this for positive $a \in \mathbb{R}$. The definition is best left to a much later stage when we have discussed the exponential and logarithm functions ($\exp$ and $\log_e$). As a temporary expedient you can think of something like $3^{\sqrt{2}}$ as being the limit of a sequence (3^{x_n}) where (x_n) is any sequence of Rational Numbers with (irrational) limit $\sqrt{2}$. We won't make any use of expressions with irrational exponents until after we have defined them properly. However, the proof of Theorem 4.5 works just as well for irrational α as for rational α.

We conclude this section with a result on the alternating harmonic series. In the next section on convergence tests we will generalize this result.

Theorem 4.6. The *alternating harmonic series* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.

Proof. Put $S_n = \displaystyle\sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}$ (the n^{th} partial sum). Then

$$S_{2n} = \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) + \ldots + \Big(\frac{1}{2n-1} - \frac{1}{2n}\Big).$$

Clearly the subsequence (S_{2n}) is strictly increasing since all the bracketed pairs are positive. If we bracket differently we have

$$\begin{aligned} S_{2n} &= 1 - \Big(\frac{1}{2} - \frac{1}{3}\Big) - \Big(\frac{1}{4} - \frac{1}{5}\Big) - \ldots - \Big(\frac{1}{2n-2} - \frac{1}{2n-1}\Big) - \frac{1}{2n} \\ &\le 1. \end{aligned}$$

So (S_{2n}) is bounded above by 1. It follows that (S_{2n}) converges to some limit $l \le 1$. But $S_{2n-1} = S_{2n} + \frac{1}{2n} \to l + 0 = l$ as $n \to \infty$. So (S_n) has two subsequences that cover $\{S_n : n \in \mathbb{N}\}$ and have the same limit l. It follows (see Exercises 3.10) that $S_n \to l$ as $n \to \infty$. [In fact $l = \log_e 2$ but we cannot prove that yet.] $\square$

The key features of this proof are (i) that the terms of the series alternate in sign, (ii) their absolute values (in this case $\frac{1}{n}$) monotonically decrease, and (iii) the terms converge to zero. So this is a sort of partial converse to Corollary 4.2.1, albeit with the stringent extra conditions (i) and (ii). Leibniz' Alternating Series test in the next section generalises this result with an almost exact copy of the proof.

Exercises for Section 4.1

1. Prove that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{n(n+2)}$ converges and determine its sum.

2. Determine the sum of the geometric series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$.

3. Determine the sum of the geometric series $\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \ldots$ (i.e. with common ratio $-\frac{1}{2}$).

4.2 Convergence Tests

How can we tell if something like $\displaystyle\sum_{n=1}^{\infty} \frac{1+2n-n^2}{2n^4+3n-1}$ converges? It would be horrible to have to treat every possible case from basic principles. What are needed are some simple tests that will deal with most (or at least many) cases. As we have already mentioned Leibniz' alternating series test, we present this first.

Theorem 4.7 (Leibniz' Alternating Series Test)**.** Suppose that (x_n) is a sequence with the following three properties:

(i) the members of the sequence alternate in sign,

(ii) $(|x_n|)$ is a monotonically decreasing sequence,

(iii) $x_n \to 0$ as $n \to \infty$.

Then the series $\displaystyle\sum_{n=1}^{\infty} x_n$ converges.

Proof. Put $S_n = \displaystyle\sum_{i=1}^{n} x_i$ (the n^{th} partial sum). Without loss of generality we may assume that $x_1 > 0$ so that $x_n > 0$ for n odd and $x_n < 0$ for n even. Then

$$S_{2n} = (x_1 + x_2) + (x_3 + x_4) + \ldots + (x_{2n-1} + x_{2n}).$$

By condition (ii) $|x_1| \geq |x_2|$, $|x_3| \geq |x_4|$, etc., so all the bracketed pairs are at least zero, and it follows that (S_{2n}) is a monotonically increasing subsequence. If we bracket differently we have

$$\begin{aligned} S_{2n} &= x_1 + (x_2 + x_3) + (x_4 + x_5) + \ldots + (x_{2n-2} + x_{2n-1}) + x_{2n} \\ &\leq x_1, \end{aligned}$$

because all the bracketed terms, and the final (unbracketed) term, are now at most zero. So (S_{2n}) is bounded above by x_1. It follows that (S_{2n}) converges to some limit $l \leq x_1$. But $S_{2n-1} = S_{2n} - x_{2n} \to l + 0 = l$ as $n \to \infty$. So (S_n) has two subsequences that cover $\{S_n : n \in \mathbb{N}\}$ and have the same limit l. Hence $S_n \to l$ as $n \to \infty$. □

The three conditions in this test are of equal importance. If you examine the proof carefully you will see that we have used conditions (i) and (ii) to prove that (S_{2n}) is increasing and bounded above, and we used condition (iii) to prove that (S_{2n}) and (S_{2n-1}) have the same limit. Of course convergence of a series is unaffected by altering a finite number of terms, so it suffices for conditions (i) and (ii) to hold for all sufficiently large n, i.e. all $n > N$ for some number N. So we have the following corollary.

Corollary 4.7.1. Suppose that (x_n) is a sequence with the following three properties:

(i) the members of the sequence alternate in sign for $n > N$,

(ii) $(|x_n|)$ is an eventually monotonically decreasing sequence (i.e. monotonically decreasing for $n > N$),

(iii) $x_n \to 0$ as $n \to \infty$.

Then the series $\sum_{n=1}^{\infty} x_n$ converges.

Example 4.2. Prove that if $-1 \leq x < 0$ then $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

Solution. Since $x < 0$, the terms $a_n = x^n/n$ alternate in sign, and because $|x| \leq 1$ we have $|a_n| \leq \frac{1}{n}$, and so $(|a_n|)$ is a null sequence. Moreover,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \left|\frac{xn}{n+1}\right| \leq \frac{n}{n+1} < 1,$$

so $(|a_n|)$ is a monotonically decreasing sequence. Thus the three conditions of Leibniz' alternating series test are met, and we deduce that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges when $-1 \leq x < 0$. [The case $x = -1$ is the alternating harmonic series.]

Example 4.3. Prove that if $x < 0$ then $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

Solution. Since $x < 0$, the terms $a_n = x^n/n!$ alternate in sign, and because $\left(\frac{x^n}{n!}\right)$ is a basic null sequence, $(|a_n|)$ is also a null sequence. Moreover,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| \\ &= \frac{|x|}{n+1} \\ &< 1 \text{ for } n > |x| \end{aligned}$$

[Note how factorials cancel: $n!/(n+1)! = 1/(n+1)$, e.g. $6!/7! = 1/7$.]
So $(|a_n|)$ is an eventually monotonically decreasing sequence (decreasing for $n > |x|$). Thus the three conditions of Leibniz' alternating series test are met, and we deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges when $x < 0$.

All the remaining convergence tests in this section deal with series whose terms are non-negative. In the next section (Absolute Convergence) we will see how these tests can be adapted to deal with series that contain a mixture of positive and negative terms. So our initial restriction to non-negative terms is not as bad as it may seem.

Theorem 4.8. [Comparison Test]
Suppose that $\sum_{n=1}^{\infty} c_n$ is a convergent series of non-negative terms, and that $\sum_{n=1}^{\infty} d_n$ is a divergent series of non-negative terms.

1. If $0 \leq x_n \leq c_n$ $(\forall n \in \mathbb{N})$, then $\sum_{n=1}^{\infty} x_n$ converges and $\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} c_n$.
2. If $x_n \geq d_n$ $(\forall n \in \mathbb{N})$, then $\sum_{n=1}^{\infty} d_n$ diverges.

Proof.

1. Suppose that $0 \leq x_n \leq c_n$ $(\forall n \in \mathbb{N})$. The partial sums of $\sum_{n=1}^{\infty} x_i$ form a monotonically increasing sequence because each term x_n is non-negative. Furthermore, $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} c_i \leq \sum_{i=1}^{\infty} c_i$. So the partial sums of $\sum_{i=1}^{\infty} x_n$ are bounded above. Hence these partial sums form a convergent sequence with limit at most $\sum_{i=1}^{\infty} c_i$, i.e. $\exists \sum_{i=1}^{\infty} x_i \leq \sum_{i=1}^{\infty} c_i$.
2. Suppose that $x_n \geq d_n$ $(\forall n \in \mathbb{N})$. Then $\sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} d_i \to +\infty$ as $n \to \infty$. Hence $\sum_{i=1}^{n} x_i$ is divergent.

$\square$

Since convergence or divergence of a series is unaffected by altering a finite number of terms, the comparison test can be used to determine convergence (or divergence) if $\exists N$ such that $0 \leq x_n \leq c_n$ (or $x_n \geq d_n$) for all $n > N$. Furthermore, positive multiples of convergent (divergent) series are convergent (divergent), we can relax the conditions of the test to give the following corollary

Corollary 4.8.1.
Suppose that $\sum_{n=1}^{\infty} c_n$ is a convergent series of non-negative terms, and that $\sum_{n=1}^{\infty} d_n$ is a divergent series of non-negative terms.

1. If $\alpha > 0$ and $0 \leq x_n \leq \alpha c_n$ ($\forall n > N$), then $\sum_{n=1}^{\infty} x_n$ converges.
2. If $\alpha > 0$ and $x_n \geq \alpha d_n$ ($\forall n > N$), then $\sum_{n=1}^{\infty} d_n$ diverges.

Now let us look at that horrible example we mentioned at the start of this section.

Example 4.4. Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+2n-n^2}{2n^4+3n-1}$ converges.

Solution. First examine $x_n = \dfrac{1+2n-n^2}{2n^4+3n-1}$ informally. The dominant term in the numerator is $-n^2$, and the dominant term in the denominator is $2n^4$. So, for large n we'd expect x_n to be close to $(-n^2)/(2n^4) = (-1)/2n^2$. Ignoring the numerical factor $-\frac{1}{2}$, this suggests that the series will behave like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is convergent (see Theorem 4.5). So we attempt to prove this by using (essentially) the comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. Because of the $-$ sign in front of the n^2 in the numerator, it's best to look at $-x_n = y_n$, say. Then

$$\begin{aligned} y_n &= \frac{-1-2n+n^2}{2n^4+3n-1} \\ &< \frac{n^2}{2n^4+3n-1} \\ &< \frac{n^2}{2n^4} \quad \text{since } 3n-1>0 \text{ for } n \geq 1, \\ &= \frac{1}{2n^2}. \end{aligned}$$

Also, $y_n \geq 0$ provided $n^2 \geq 1+2n$, which is certainly true if $n \geq 3$. (Note that $y_n \geq 0$ is an important condition in the comparison test for convergence.)

So we now have that for $n \geq 3$, $0 \leq y_n \leq \dfrac{1}{2n^2}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows from the comparison test that $\sum_{n=1}^{\infty} y_n$ converges, and hence $\sum_{n=1}^{\infty} x_n$ also converges because it is a multiple (by a factor -1) of that series.

Although we won't do this here, it would be possible to get some estimate of the value of $\sum_{n=1}^{\infty} x_n$, since the comparison test ensures that $\displaystyle\sum_{n=3}^{\infty} y_n \leq \frac{1}{2}\sum_{n=3}^{\infty} \frac{1}{n^2}$.

The following version of the comparison test can save a bit of arithmetic.

Corollary 4.8.2. [Limit Comparison Test]
Suppose that $\sum_{n=1}^{\infty} c_n$ is a convergent series of non-negative terms, and that $\sum_{n=1}^{\infty} d_n$ is a divergent series of non-negative terms.

1. If $\dfrac{x_n}{c_n} \to l$ as $n \to \infty$ and $l > 0$, then $\sum_{n=1}^{\infty} x_n$ converges.

2. If $\dfrac{x_n}{d_n} \to l$ as $n \to \infty$ and $l > 0$ (or if $l = +\infty$), then $\sum_{n=1}^{\infty} x_n$ diverges.

Proof. 1. If $\dfrac{x_n}{c_n} \to l$ as $n \to \infty$ and $l > 0$, then taking $\epsilon = \frac{l}{2}$ in the definition of convergence, $\exists N$ s.t. $\forall n > N$, $\left|\dfrac{x_n}{c_n} - l\right| < l/2$. But then for $n > N$,

$$0 < \frac{l}{2} < \frac{x_n}{c_n} < \frac{3l}{2},$$

which gives $0 < x_n < (3l/2)c_n$ for all $n > N$. It then follows from Corollary 4.8.1 that $\sum_{n=1}^{\infty} x_n$ converges.

2. Using a similar argument to part (1) when $l \neq +\infty$,

$$\exists N \text{ s.t. } \forall n > N,\ \frac{l}{2} < \frac{x_n}{d_n},$$

which gives $x_n > (l/2)d_n$ for all $n > N$. It then follows from Corollary 4.8.1 that $\sum_{n=1}^{\infty} x_n$ diverges. In the case when $l = +\infty$, $\exists N$ s.t. $\forall n > N, x_n > d_n$ and again we deduce from the Corollary that $\sum_{n=1}^{\infty} x_n$ diverges. $\square$

To show how this works, look again at Example 4.4. Again we compare $y_n = \dfrac{-1 - 2n + n^2}{2n^4 + 3n - 1}$ with $c_n = \dfrac{1}{n^2}$. We have

$$\begin{aligned}
\frac{y_n}{c_n} &= \frac{n^2(-1 - 2n + n^2)}{2n^4 + 3n - 1} \\
&= \frac{-\frac{1}{n^2} - 2\frac{1}{n} + 1}{2 + \frac{3}{n^3} - \frac{1}{n^4}} \quad \text{(dividing top and bottom by } n^4\text{)} \\
&\to \frac{1}{2} \text{ as } n \to \infty.
\end{aligned}$$

So by the limit comparison test $\sum_{n=1}^{\infty} y_n$ converges.

Here is another example.

Example 4.5. Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{2n^2\sqrt{n}+10}$ converges.

Solution. For large n we'd expect the term $x_n = \dfrac{n^2+3n}{2n^2\sqrt{n}+10}$ to behave like $\dfrac{n^2}{2n^2\sqrt{n}} = \dfrac{1}{2\sqrt{n}}$. So we compare x_n with $d_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$, noting that, by Theorem 4.5, $\sum_{n=1}^{\infty} d_n$ is a divergent series of positive terms. We have

$$\begin{aligned}
\frac{x_n}{d_n} &= \frac{\sqrt{n}(n^2+3n)}{2n^2\sqrt{n}+10} \\
&= \frac{1+\frac{3}{n\sqrt{n}}}{2+\frac{10}{n^2\sqrt{n}}} \quad \text{(dividing top and bottom by } n^2\sqrt{n}) \\
&\to \frac{1}{2} \text{ as } n \to \infty.
\end{aligned}$$

It then follows from the limit comparison test that $\sum_{n=1}^{\infty} x_n$ diverges.

The comparison test (in whatever form it is used) does have certain disadvantages.

1. It is necessary to have some idea in advance of whether or not the series in question converges or diverges.

2. Then you have to find a suitable series with which to make the comparison. At present we just have geometric series $\sum_{n=0}^{\infty} x^n$ and series of the form $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$.

The following test does not suffer from these disadvantages. But this is really because it has comparison with geometric series built into it. So the test is less subtle than the comparison test, although much easier to use.

Theorem 4.9. [D'Alembert's Ratio Test]
Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of positive terms.

1. If there exists a number $k < 1$ such that for every $n \in \mathbb{N}$, $\dfrac{x_{n+1}}{x_n} \le k$, then $\sum_{n=1}^{\infty} x_n$ converges.

2. If for every $n \in \mathbb{N}$, $\dfrac{x_{n+1}}{x_n} > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

Proof. 1. If $\dfrac{x_{n+1}}{x_n} \le k < 1$ for every $n \in \mathbb{N}$, then we have

$$x_2 \le kx_1, \quad x_3 \le kx_2 \le k^2x_1, \quad x_4 \le kx_3 \le k^3x_1, \quad \ldots.$$

In general, $x_n \le k^{n-1}x_1$. Since $0 < k < 1$, it follows by comparison with the geometric series $\sum_{n=1}^{\infty} k^{n-1}$ that the series $\sum_{n=1}^{\infty} x_n$ converges.

2. If $\dfrac{x_{n+1}}{x_n} > 1$ for every $n \in \mathbb{N}$, then we have $x_n > x_1$ for every $n \in \mathbb{N}$. By the assumption implicit in writing the ratio x_{n+1}/x_n we have $x_1 \ne 0$. Hence the sequence of terms (x_n) cannot converge to zero, and so $\sum_{n=1}^{\infty} x_n$ diverges.

□

Notes.
1. In part (1) of the test (convergence), it is important to find a constant $k < 1$. It isn't sufficient merely to show that $x_{n+1}/x_n < 1$ because that does not ensure convergence. It would only show that the sequence of terms is decreasing, it may not even be the case that $x_n \to 0$ as $n \to \infty$. And even if the terms do tend to zero, the series may not converge, as is shown by the DIVERGENT harmonic series where $x_n = \frac{1}{n}$, giving $x_{n+1}/x_n = n/(n+1) < 1$.
2. If you look at the proof of part (2) of the test you will see that the conditions ensure that the terms x_n do not tend to zero, and consequently the series cannot converge. In this part of the test we can drop the requirement that the terms x_n are positive provided we have $|x_{n+1}/x_n| > 1$, i.e. we introduce modulus signs. Then we deduce that $|x_n|$ does not tend to zero, and so x_n does not tend to zero. A similar observation applies to subsequent versions of the test given below.

As with the comparison test, we note that convergence or divergence of a series is unaffected by altering a finite number of terms, so it suffices if the conditions apply to all sufficiently large n, i.e. all $n > N$. So we can relax the conditions of the test to give the following corollary

Corollary 4.9.1.
Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of positive terms.

1. If there exists a number $k < 1$ such that $\forall n > N$, $\dfrac{x_{n+1}}{x_n} \le k$, then $\sum_{n=1}^{\infty} x_n$ converges.

2. If $\forall n > N$, $\dfrac{x_{n+1}}{x_n} > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

There is also a limit form of the test that often saves some arithmetic.

Corollary 4.9.2.
Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of positive terms and that $\dfrac{x_{n+1}}{x_n} \to l$ as $n \to \infty$. Then

1. if $l < 1$, $\sum_{n=1}^{\infty} x_n$ converges,
2. if $l > 1$ (including the case $l = +\infty$), $\sum_{n=1}^{\infty} x_n$ diverges,
3. if $l = 1$, the test is inconclusive - the series might converge or it might diverge.

Proof. For $l < 1$, the basic idea of the proof is that if we pick a number k between l and 1, such as $k = (1+l)/2$, then the ratio x_{n+1}/x_n will be less than k provided that n is sufficiently large. Similarly if $l > 1$ and we pick a number k between 1 and l, such as $k = (l+1)/2$, then the ratio x_{n+1}/x_n will be greater than k provided that n is sufficiently large. Here are the details.

1. If $l < 1$ take $\epsilon = \frac{1-l}{2}$ in the definition of sequence convergence. We have that $\exists N$ s.t. $\forall n > N, \left|\dfrac{x_{n+1}}{x_n} - l\right| < \dfrac{1-l}{2}$. But then if $n > N$, $\dfrac{x_{n+1}}{x_n} < \dfrac{1-l}{2} + l = \dfrac{1+l}{2} = k$, say, and $k < 1$. The result then follows from the previous corollary.

2. If $l > 1$ (but $l \neq +\infty$) take $\epsilon = \frac{l-1}{2}$ in the definition of sequence convergence. We have that $\exists N$ s.t. $\forall n > N, \left|\dfrac{x_{n+1}}{x_n} - l\right| < \dfrac{l-1}{2}$. But then if $n > N$, $\dfrac{x_{n+1}}{x_n} > l - \dfrac{l-1}{2} = \dfrac{l+1}{2} > 1$. Again, the result follows from the previous corollary.

   In the case $l = +\infty$, the terms are eventually increasing and so do not tend to zero, consequently the series diverges.

3. Both the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ give $l = 1$, but the former diverges, while the latter converges.

□

Example 4.6. Prove that if $0 \leq x < 1$ then the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges, while if $x \geq 1$ the series diverges.

Solution. If $x = 0$ then the series converges because all the terms (and hence all the partial sums) are zero. So suppose $x > 0$ and put $a_n = x^n/n$. Then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \\ &= x \cdot \frac{n}{n+1} \to x \text{ as } n \to \infty.\end{aligned}$$

By the limit version of D'Alembert's ratio test, the series converges if the limiting value, namely x, satisfies $x < 1$, and diverges if $x > 1$.

In the case $x = 1$, the series is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is DIVERGENT. If you look back to Example 4.2, you will see that we now know that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges when $-1 \leq x < 1$. It diverges if $|x| > 1$ because then the terms do not tend to zero. We'll indicate a more satisfactory proof of all this once we have discussed the concept of absolute convergence in the next section.

Example 4.7. Prove that if $x \geq 0$ then the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

Solution. If $x = 0$ then the series converges because all the terms apart from the first term (corresponding to $n = 0$) are zero. So suppose $x > 0$ and put $a_n = x^n/n!$. Then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \\ &= \frac{x}{n+1} \to 0 \text{ as } n \to \infty.\end{aligned}$$

By the limit version of D'Alembert's ratio test, the series converges because the limiting value, namely 0, is less than 1.

If you look back to Example 4.3, you will see that we now know that $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$. Again, we'll indicate a more satisfactory proof once we have discussed the concept of absolute convergence in the next section. It is perhaps a little surprising that the series converges no matter what value of x you choose. The truth is that size matters: $n!$ will eventually massively outrun x^n as n gets larger, whatever the value of x might be.

A further test, which can be useful is the following.

Theorem 4.10 (Cauchy's n^{th} Root Test)**.**
Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of non-negative terms.

1. If there exists a number $k < 1$ such that for every $n \in \mathbb{N}$, $\sqrt[n]{x_n} \leq k$, then $\sum_{n=1}^{\infty} x_n$ converges.

2. If for every $n \in \mathbb{N}$, $\sqrt[n]{x_n} > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

Proof. 1. If $\sqrt[n]{x_n} \leq k$ for every $n \in \mathbb{N}$, and $k < 1$, then we have $x_n \leq k^n$, and consequently $\sum_{n=1}^{\infty} x_n$ converges (by comparison with $\sum_{n=1}^{\infty} k^n$).

2. If $\sqrt[n]{x_n} \geq 1$ for every $n \in \mathbb{N}$, then $x_n \geq 1$, and consequently $\sum_{n=1}^{\infty} x_n$ diverges (the terms do not tend to zero).

□

As with previous tests, it suffices for the conditions to hold for all sufficiently large n, i.e $n > N$. So we have the corollary.

Corollary 4.10.1.
Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of non-negative terms.

1. If there exists a number $k < 1$ such that for every $n > N$, $\sqrt[n]{x_n} \leq k$, then $\sum_{n=1}^{\infty} x_n$ converges.
2. If for every $n > N$, $\sqrt[n]{x_n} > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

There is also a limit version of this test.

Corollary 4.10.2.
Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of non-negative terms and that $\sqrt[n]{x_n} \to l$ as $n \to \infty$. Then

1. if $l < 1$, $\sum_{n=1}^{\infty} x_n$ converges,
2. if $l > 1$ (including the case $l = +\infty$), $\sum_{n=1}^{\infty} x_n$ diverges,
3. if $l = 1$, the test is inconclusive - the series might converge or it might diverge.

Proof. As in the proof of the limit form of D'Alembert's ratio test, we have in case

1. $\exists N$ and $k < 1$ such that $\forall n > N$, $\sqrt[n]{x_n} \leq k$,
2. $\exists N$ such that $\forall n > N$, $\sqrt[n]{x_n} > 1$,
3. the same examples provide the justification.

□

Example 4.8. Use Cauchy's n^{th} root test to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$ converges for every real Number x.

Solution. The expression x^{2n} is non-negative for every $x \in \mathbb{R}$ and for every $n \in \mathbb{N}$, and hence all the terms of the series are non-negative. Moreover, $\sqrt[n]{\dfrac{x^{2n}}{(2n)!}} = \dfrac{x^2}{\sqrt[n]{(2n)!}}$. But

$$(2n)! \geq (2n)(2n-1)(2n-2)\cdots(n+1)$$

and the latter product has n terms, all greater than n, and so $(2n)! > n^n$, giving $\sqrt[n]{(2n)!} > n$. It follows that $0 \leq \sqrt[n]{\dfrac{x^{2n}}{(2n)!}} \leq \dfrac{x^2}{n} \to 0$ as $n \to \infty$. Consequently $\sqrt[n]{\dfrac{x^{2n}}{(2n)!}} \to 0$ as $n \to \infty$ and then by the limit version of Cauchy's n^{th} root test we deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$ converges for every Real Number x.

Exercises for Section 4.2

1. Investigate whether or not the following series converge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$. b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\left(\frac{n}{n^3+1}\right)}$. c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{4n^5+1}}$.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+\sqrt{n}}}$. e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$. f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$.

2. Prove that $\displaystyle\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})$ diverges, but $\displaystyle\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n})$ converges.

3. Define $a_n = \begin{cases} 0 & \text{if } n \text{ has a zero in its decimal representation,} \\ 1 & \text{otherwise.} \end{cases}$

For example, $a_3 = 1, a_{50} = 0, a_{103} = 0, a_{579} = 1$.

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

[Hint: m-digit integers are those from 10^{m-1} to $10^m - 1$; count how many of these have no zeros amongst their m digits. Hence estimate $\displaystyle\sum_{n=10^{m-1}}^{10^m-1} \frac{a_n}{n}$.]

4.3 Absolute Convergence

All the convergence tests in the previous section rely on the terms being well-behaved in the sense of either all being non-negative or alternating in sign. But there is no reason why series should always look like one of these alternatives. To deal with this issue, suppose informally that $\sum_{n=1}^{\infty} x_n$ is a series which has a mixture of positive and negative terms. With such a series we can associate a series containing only non-negative terms, namely $\sum_{n=1}^{\infty} |x_n|$. In this latter series there is no possibility of positive and negative terms cancelling out, so it is plausible that $\sum_{n=1}^{\infty} |x_n|$ is less likely to converge than $\sum_{n=1}^{\infty} x_n$. We will prove that if $\sum_{n=1}^{\infty} |x_n|$ converges then so necessarily does $\sum_{n=1}^{\infty} x_n$. This provides us with a way to investigate the convergence of $\sum_{n=1}^{\infty} x_n$, since $\sum_{n=1}^{\infty} |x_n|$ contains no negative terms and the earlier tests may be applied to it. We start with a definition.

Definition 4.2. Suppose that $\sum_{n=1}^{\infty} x_n$ is a series and that the associated series $\sum_{n=1}^{\infty} |x_n|$ converges. The we say that $\sum_{n=1}^{\infty} x_n$ *converges absolutely*.

This definition is only seen to be sensible once we have proved the result mentioned above, namely that if a series converges absolutely, then it converges.

Theorem 4.11. If $\sum_{n=1}^{\infty} x_n$ converges absolutely (i.e. if $\sum_{n=1}^{\infty} |x_n|$ converges), then $\sum_{n=1}^{\infty} x_n$ converges.

Proof. Roughly speaking, the strategy is to split the series into its positive and negative parts and prove by comparison with $\sum_{n=1}^{\infty} |x_n|$ that these separate parts converge. Accordingly we define

$$x_n^+ = \begin{cases} x_n & \text{if } x_n \geq 0, \\ 0 & \text{if } x_n < 0 \end{cases}$$

$$x_n^- = \begin{cases} 0 & \text{if } x_n \geq 0, \\ x_n & \text{if } x_n < 0 \end{cases}$$

Then $x_n = x_n^+ + x_n^-$ and $|x_n| = x_n^+ - x_n^-$. Moreover, $0 \leq x_n^+ \leq |x_n|$ and $0 \leq -x_n^- \leq |x_n|$. By the comparison test with the convergent series $\sum_{n=1}^{\infty} |x_n|$, it follows from the last two inequalities that both $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n^-$ converge. Hence $\sum_{n=1}^{\infty}(x_n^+ + x_n^-)$ converges, i.e. $\sum_{n=1}^{\infty} x_n$ converges. □

Motivated by the preceding theorem we make another definition.

Definition 4.3. If the series $\sum_{n=1}^{\infty} x_n$ converges, but is not absolutely convergent (i.e. $\sum_{n=1}^{\infty} |x_n|$ diverges), the we say that $\sum_{n=1}^{\infty} x_n$ is *conditionally convergent* or that it *converges conditionally*.

An example of a conditionally convergent series is the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$. This is convergent (Theorem 4.6), but the harmonic series itself $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent (Theorem 4.5).

Example 4.9. Determine whether or not the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges. (Assume that the sine function has all its usual properties.)

Solution. Here we cannot employ Leibniz' alternating series test because, although $\sin(n)$ is positive for some values of n and negative for other values, it certainly does not alternate (e.g. $\sin(n) > 0$ for $n = 1, 2, 3$, but $\sin(n) < 0$ for $n = 4, 5, 6$). And because the terms certainly do take positive and negative values, we cannot directly employ any of the other tests (comparison, ratio, n^{th}-root).

Therefore we consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{|\sin(n)|}{n^2}$. We have $0 \le \dfrac{|\sin(n)|}{n^2} \le \dfrac{1}{n^2}$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (Theorem 4.5). So, by the comparison test, $\displaystyle\sum_{n=1}^{\infty}\frac{|\sin(n)|}{n^2}$ converges, and hence the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$ converges (it is absolutely convergent).

A note on Complex sequences and series
So far we have assumed that we are working in the field of Real Numbers ($\mathbb{R}$). But many of the definitions and theorems also apply in the field of Complex Numbers ($\mathbb{C}$). In particular, the triangle inequality, the definition of a convergent sequence, the boundedness of a convergent sequence, the rules for sums, products and quotients of convergent sequences, the definition of a convergent series, the fact that terms of a convergent series must tend to 0, the behaviour of geometric series with complex common ratios, and the result that absolute convergence ensures convergence. For the latter, $|z|$ is interpreted as the complex modulus of z. If you subsequently study Complex Analysis you will encounter all these familiar friends(?) again, generally with almost identical proofs. Of course there are some differences, for example the alternating series test cannot apply to a series of Complex Numbers because there is no notion of a Complex Number being positive (or negative).

In this section on absolute convergence there are two important aspects. One aspect concerns the application of absolute convergence to power series. This generates lots of examples to test your knowledge of convergence tests and, more importantly, it is the first step in defining familiar functions such as $\sin x$. So this first aspect is important to you because you are highly likely to be tested on it in any examination or test on series. The other aspect is that we can show that conditionally convergent series can be *rearranged* to converge to whatever you like, or to oscillate finitely or infinitely, or to diverge to either $+\infty$ or to $-\infty$. Indeed, the words "conditionally convergent" refer to the condition that the terms should not be rearranged. So the basic message from this is that **you must not rearrange a conditionally convergent series**. This is quite different to all finite sums, which are unaltered by any rearrangement of the terms. It's good that we never form an infinite series of positive and negative financial transactions when we go shopping - the shopkeeper would undoubtedly argue that we should owe $+\infty$, while we might prefer $-\infty$. But first we will deal with power series.

Definition 4.4. An infinite series of the form $\displaystyle\sum_{n=0}^{\infty} a_n x^n$, (where each a_n is independent of x) is called a *power series* in the variable x. The number a_0 is called the constant term and, for $n \geq 1$, the number a_n is referred to as the coefficient of x^n.

For example, the geometric series $\sum_{n=0}^{\infty} x^n$ is a power series in which all the coefficients and the constant term are equal to 1. Another example is $\sum_{n=1}^{\infty} \frac{x^n}{n}$, where the constant term is 0 and the coefficient of x^n is $\frac{1}{n}$ for each $n \geq 1$.

A power series in x may converge only for $x = 0$ (when $a_n x^n = 0$ for $n \geq 1$), or it may converge for some non-zero values of x but diverge for other values of x, or it may converge for all values of x. We will see examples of all of these behaviours. The next two theorems show that every power series has a degree of regularity in its behaviour. For example, it can't diverge for $x = 2$, and converge for $x = 3$.

Theorem 4.12. Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = \xi$ and that $|\xi| > 0$ (i.e. $\xi \neq 0$). Then the power series converges absolutely for any value of x satisfying $|x| < |\xi|$.

Proof. Since $\sum_{n=0}^{\infty} a_n \xi^n$ converges, the terms tend to zero, i.e. $a_n \xi^n \to 0$ as $n \to \infty$. Hence there exists a bound A such that $|a_n \xi^n| < A$ for $n = 0, 1, 2, \ldots$. But then we have

$$\begin{aligned} |a_n x^n| &= |a_n \xi^n| \left| \frac{x}{\xi} \right|^n \\ &\leq A \left| \frac{x}{\xi} \right|^n. \end{aligned}$$

Hence, if $|x| < |\xi|$, the series $\sum_{n=0}^{\infty} |a_n x^n|$ converges (by the comparison test with the geometric series $\sum_{n=0}^{\infty} \left| \frac{x}{\xi} \right|^n$ whose common ratio is $\left| \frac{x}{\xi} \right| < 1$). Thus the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for any value of x satisfying $|x| < |\xi|$. $\square$

Corollary 4.12.1. If the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = \xi$, then it diverges for any value of x satisfying $|x| > |\xi|$.

Proof. Suppose that the series converges for $x = \eta$ where $|\eta| > |\xi|$. Then by Theorem 4.12, the series converges for all values of x satisfying $|x| < |\eta|$. Therefore the series must converge for $x = \xi$, a contradiction. Hence the series must diverge for any value of x satisfying $|x| > |\xi|$. □

Theorem 4.13. [The radius of convergence of a power series]
Given a power series $\sum_{n=0}^{\infty} a_n x^n$, precisely one of the following three alternatives must apply.

1. The series converges only for $x = 0$.
2. The series converges for all values of x.
3. There exists a number $R > 0$ such that the series converges if $|x| < R$ and diverges if $|x| > R$.

In case (3), the number R is called the *radius of convergence* of the series. In case (1) we say that the radius of convergence is zero and write $R = 0$. In case (2) we say that the radius of convergence is infinite and write $R = \infty$.

Proof. Suppose that (1) and (2) are not the case. Then there exists $\xi > 0$ such that the series converges for $x = \xi$, and there exists η such that the series diverges for $x = \eta$. From Theorem 4.12 it follows that $|\xi| < |\eta|$. Put

$$S = \left\{ |x| : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

The set S is bounded above by $|\eta|$ so S has a least upper bound $R \in \mathbb{R}$, and $R \leq |\eta|$. If $|x| < R$ it follows from the definition of S and Theorem 4.12 that the series converges. If $|x| > R$ it follows from Corollary 4.12.1 that the series diverges. □

Comments

1. The reason that R is called the *radius* of convergence is that Theorems 4.12 and 4.13 remain valid if we allow each a_n and x to be Complex Numbers. In the complex plane, the inequality $|x| < R$ determines a circular region centred on the origin with radius R.
2. Theorem 4.13 tells us nothing about convergence when $|x| = R$ (unless $R = 0$ or ∞). For any $R > 0$ ($R \neq \infty$) there are power series with that radius of convergence, which converge if $|x| = R$, others which diverge if $|x| = R$, and ones which converge for $x = R$ and diverge for $x = -R$ (and vice-versa).

3. In many cases, D'Alembert's ratio test or Cauchy's n^{th} root test can be used to determine the value of R.

Example 4.10. Consider the geometric series $\sum_{n=0}^{\infty} x^n$. If we ignore the $n = 0$ term, which is irrelevant to questions of convergence, the n^{th} term is x^n. The n^{th} root of $|x^n|$ is just $|x|$. So if $|x| < 1$ the series converges (absolutely), while if $|x| > 1$ it diverges, so the radius of convergence is $R = 1$. In the cases when $|x| = 1$ (i.e. $x = 1$ and $x = -1$) the series diverges because the terms do not tend to zero.

Example 4.11. Consider the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Here the n^{th} root of $|x^n/n|$ is $|x|/(n^{\frac{1}{n}}) \to |x|$ as $n \to \infty$. So if $|x| < 1$ the series converges (absolutely), while if $|x| > 1$ it diverges. Thus the radius of convergence is again $R = 1$. In the case when $x = 1$ the series diverges (it is the harmonic series), and in the case when $x = -1$ it converges (use Leibniz' alternating series test or observe that it is the negative of the alternating harmonic series).

Example 4.12. Consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Put $a_n = x^n/n!$ and, for $x \neq 0$, consider the ratio $|a_{n+1}/a_n| = |x|/(n+1) \to 0$ as $n \to \infty$, so the series converges absolutely by D'Alembert's ratio test. Hence the series converges for every $x \in \mathbb{R}$, including $x = 0$ of course, and the radius of convergence of this series is $R = \infty$.

Example 4.13. Consider the series $\sum_{n=0}^{\infty} n!x^n$. Put $a_n = n!x^n$ and, for $x \neq 0$, consider the ratio $|a_{n+1}/a_n| = (n + 1)|x| \to +\infty$ as $n \to \infty$, so the series diverges by D'Alembert's ratio test for every $x \in \mathbb{R}$, except for $x = 0$ of course. Hence the radius of convergence of this series is $R = 0$.

Power series can be used to define familiar functions. In Chapter 7 we will take the following power series as definitions of $\sin(x), \cos(x)$ and $\exp(x)$. In each of these three cases the series converges for all $x \in \mathbb{R}$ (the radius of convergence $R = \infty$).

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$
$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We will show how all the familiar properties of these functions may be deduced from the power series that define them. We can't do that here because important properties of functions, such as continuity and differentiability have not yet been defined. So I'm afraid we have to delay for the time being.

At this point you might like to try the exercises for this section, which lie a few pages further on. However, before we get to them in the text, we are going to look at the issue of rearranging the terms of a series and considering what effect, if any, it can have on convergence and divergence. First we need to define what we mean by a rearrangement.

Definition 4.5. Suppose that S is any set of numbers (here we have in mind $S = \mathbb{N}$). A permutation ϕ of S is a one-to-one mapping of S *onto* itself (i.e. ϕ is a bijection). This means that every $n \in S$ has an image $\phi(n)$ that also lies in S, and every $m \in S$ is the image of some $n \in S$, i.e. $m = \phi(n)$. Every permutation ϕ has an inverse permutation ϕ^{-1} such that $\phi^{-1}(\phi(n)) = n$ and $\phi(\phi^{-1}(m)) = m$.

A *rearrangement* of an infinite sequence (a_n) is a sequence of the form $(a_{\phi(n)})$, where ϕ is a permutation of the indexing set (generally $\mathbb{N}$ or the set $\mathbb{N} \cup \{0\}$). If (b_n) is a rearrangement of (a_n), then (a_n) is a rearrangement of (b_n). The infinite series formed from each of these sequences are also said to be rearrangements of one another.

Somewhat less formally, if (b_n) is a rearrangement of (a_n) then the two sequences contain exactly the same numbers with the same multiplicity, meaning that if a number appears k times in one sequence then it appears k times in the other. Rather trivially every infinite series is a rearrangement of itself. But the real interest lies with rearrangements that are not confined to any finite number of terms. As an example, consider the alternating harmonic series that we have already shown to be (conditionally) convergent. Let S denote the sum of this series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots.$$

Plainly $S \neq 0$ since all the partial sums are at least $\frac{1}{2}$. Let us rearrange this series by taking one positive term, followed by two negative terms, in blocks of three, giving the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots.$$

Let S_n and T_n (respectively) denote the partial sums of the first n terms in each of the two series. Then

$$\begin{aligned}
S_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{2n-1} - \frac{1}{2n}, \text{ and} \\
T_{3n} &= \Big(1 - \frac{1}{2} - \frac{1}{4}\Big) + \Big(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\Big) + \dots + \Big(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\Big) \\
&= \Big(\frac{1}{2} - \frac{1}{4}\Big) + \Big(\frac{1}{6} - \frac{1}{8}\Big) + \dots + \Big(\frac{1}{4n-2} - \frac{1}{4n}\Big) \\
&= \frac{1}{2} S_{2n}.
\end{aligned}$$

We have $S_{2n} \to S$ as $n \to \infty$ and so $T_{3n} \to \frac{1}{2}S$ as $n \to \infty$. But $T_{3n+1} - T_{3n} = \frac{1}{2n+1} \to 0$ as $n \to \infty$ and $T_{3n+2} - T_{3n} = \frac{1}{2n+1} - \frac{1}{4n+2} \to 0$ as $n \to \infty$, and so we deduce that $T_n \to \frac{1}{2}S$ as $n \to \infty$. Hence the rearranged series converges with sum $\frac{1}{2}S$, i.e.

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots = \frac{S}{2}.$$

Since $S \neq 0$ the rearranged series has a different sum $\frac{1}{2}S$ from the original series (the alternating harmonic series) whose sum is S.

While the bad news (as in this example) is that we must be cautious when rearranging a series, there is some good news. We now show that a rearrangement of a series that has only non-negative terms does not affect convergence/divergence, or the sum of such a series in the case of convergence. Moreover, this result extends to absolutely convergent series.

Theorem 4.14. Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of non-negative terms (i.e $a_n \geq 0$ for all $N \in \mathbb{N}$), and that $\sum_{n=1}^{\infty} b_n$ is a rearrangement of this series. Then either both series diverge to $+\infty$, or both converge to the same sum S.

Proof. The partial sums S_n of $\sum_{n=1}^{\infty} a_n$ are monotonically increasing. Let us assume first that this series converges with sum S. Then $S_n \leq S$ for every $n \in \mathbb{N}$. Now consider a partial sum T_n of $\sum_{n=1}^{\infty} b_n$, i.e. $T_n = \sum_{i=1}^{n} b_i$. Since (b_i) is a rearrangement of (a_i), there will be some integer m such that $T_n \leq S_m$. But then we have $T_n \leq S$, so the monotonically increasing sequence (T_n) is bounded above by S and must therefore converge to some limit $T \leq S$. We can now reverse the roles of the two series, noting that $\sum_{n=1}^{\infty} a_n$ is a rearrangement of $\sum_{n=1}^{\infty} b_n$, and deduce that $S \leq T$. It follows that $S = T$, so both series converge to the same sum.

To deal with the divergence case is now easy. If $\sum_{n=1}^{\infty} a_n$ diverges, could $\sum_{n=1}^{\infty} b_n$ converge? The answer must be no because, if $\sum_{n=1}^{\infty} b_n$ converges then by the previous argument $\sum_{n=1}^{\infty} a_n$ must also converge, a contradiction. $\square$

Corollary 4.14.1. Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of non-positive terms (i.e $a_n \leq 0$ for all $N \in \mathbb{N}$), and that $\sum_{n=1}^{\infty} b_n$ is a rearrangement of this series. Then either both series diverge to $-\infty$, or both converge to the same sum S.

Proof. This follows immediately from Theorem 4.14 by considering $\sum_{n=1}^{\infty}(-a_n)$ and $\sum_{n=1}^{\infty}(-b_n)$. $\square$

Theorem 4.15. Suppose that $\sum_{n=1}^{\infty} x_n$ is an absolutely convergent series, and that $\sum_{n=1}^{\infty} y_n$ is a rearrangement of this series. Then both series converge to the same sum.

Proof. The strategy of the proof is similar to that of Theorem 4.11, so it is a good idea to look back there and see how the series was split into its positive and negative parts. Here we define

$$x_n^+ = \begin{cases} x_n & \text{if } x_n \geq 0, \\ 0 & \text{if } x_n < 0 \end{cases} \qquad y_n^+ = \begin{cases} y_n & \text{if } y_n \geq 0, \\ 0 & \text{if } y_n < 0 \end{cases}$$

$$x_n^- = \begin{cases} 0 & \text{if } x_n \geq 0, \\ x_n & \text{if } x_n < 0 \end{cases} \qquad y_n^- = \begin{cases} 0 & \text{if } y_n \geq 0, \\ y_n & \text{if } y_n < 0 \end{cases}$$

Then (y_n^+) is a rearrangement of the non-negative sequence (x_n^+), and (y_n^-) is a rearrangement of the non-positive sequence (x_n^-). Moreover, both $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n^-$ converge (see the proof of Theorem 4.11), and so both $\sum_{n=1}^{\infty} y_n^+$ and $\sum_{n=1}^{\infty} y_n^-$ converge, and with the same limits (respectively). But $x_n = x_n^+ + x_n^-$ and $y_n = y_n^+ + y_n^-$. Hence

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} x_n^+ + \sum_{n=1}^{\infty} x_n^- = \sum_{n=1}^{\infty} y_n^+ + \sum_{n=1}^{\infty} y_n^- = \sum_{n=1}^{\infty} y_n.$$

$\square$

We saw above how the alternating harmonic series (which is conditionally convergent) can be rearranged to converge to a different sum. In fact we can rearrange any conditionally convergent series to do pretty much whatever we want. The first step is to show that in a conditionally convergent series the sum of the positive terms and the sum of the negative terms must both diverge (to $+\infty$ and $-\infty$ respectively). This follows from the proof of Theorem 4.11. Here are the details.

Theorem 4.16. Suppose that $\sum_{n=1}^{\infty} x_n$ is a conditionally convergent series. Define

$$x_n^+ = \begin{cases} x_n & \text{if } x_n \geq 0, \\ 0 & \text{if } x_n < 0 \end{cases}$$

$$x_n^- = \begin{cases} 0 & \text{if } x_n \geq 0, \\ x_n & \text{if } x_n < 0 \end{cases}$$

Then $\sum_{n=1}^{\infty} x_n^+$ diverges to $+\infty$ and $\sum_{n=1}^{\infty} x_n^-$ diverges to $-\infty$.

Proof. As in the proof of Theorem 4.11, we have $|x_n| = x_n^+ - x_n^-$. Since $\sum_{n=1}^{\infty} |x_n|$ diverges, at least one of $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n^-$ must diverge. Suppose that the former converges. Then since $x_n^- = x_n^+ - x_n$ and *both* $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n$ converge, we would have $\sum_{n=1}^{\infty} x_n^-$ convergent, a contradiction. If instead we suppose that $\sum_{n=1}^{\infty} x_n^-$ converges we get a contradiction by the same method. It follows that both $\sum_{n=1}^{\infty} x_n^+$ and $\sum_{n=1}^{\infty} x_n^-$ diverge. Since the former has monotonically increasing partial sums and the latter monotonically decreasing partial sums, the divergence must be (respectively) to $+\infty$ and to $-\infty$. □

Theorem 4.17. Suppose that $\sum_{n=1}^{\infty} x_n$ is a conditionally convergent series. Then it is possible to rearrange the series to converge to any desired sum $S \in \mathbb{R}$, to oscillate finitely between any desired upper and lower limits, to oscillate infinitely, to diverge to $+\infty$, or to diverge to $-\infty$.

Proof. We will deal with the case $S \in \mathbb{R}$. The other cases can be dealt with in the same way. We note that by the previous theorem (and using the same notation) $\sum_{n=1}^{\infty} x_n^+$ diverges to $+\infty$ and $\sum_{n=1}^{\infty} x_n^-$ diverges to $-\infty$.

Let (p_n) denote the subsequence of (x_n) formed by selecting precisely those terms for which $x_n \geq 0$, and let (q_n) denote the subsequence (x_n) formed by selecting precisely those terms for which $x_n < 0$. Then every x_n lies in one of these two subsequences. The subsequence (p_n) is not quite the same as (x_n^+), because the sequence (x_n^+) has zeros for those values of n for which $x_n < 0$, while (p_n) omits these zeros. Similarly (q_n) is not quite the same as (x_n^-). For example, if (x_n) forms the terms of the alternating harmonic series, then we have

$$\begin{aligned} (x_n) &= (1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{5}, -\tfrac{1}{6}, \ldots) \\ (x_n^+) &= (1, 0, \tfrac{1}{3}, 0, \tfrac{1}{5}, 0, \ldots) \\ (p_n) &= (1, \tfrac{1}{3}, \tfrac{1}{5} \ldots) \\ (x_n^-) &= (0, -\tfrac{1}{2}, 0, -\tfrac{1}{4}, 0, -\tfrac{1}{6} \ldots) \\ (q_n) &= (-\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{6}, \ldots) \end{aligned}$$

However, it is easy to see that $\sum_{i=1}^{\infty} p_i$ diverges to $+\infty$ and $\sum_{i=1}^{\infty} q_i$ diverges to $-\infty$, since deleting zeros from a series will not alter the fact that the partial sums are unbounded.

Now choose $S \in \mathbb{R}$. The strategy (roughly) is to take alternating subsets of positive and negative terms from (x_n) and so build up a suitable rearrangement. We start at x_1 and take only positive (or zero) terms until their sum first exceeds S, then we go back to x_1 and take negative terms until the new sum is first below S. Then we return to taking positive (or zero) terms until the total again exceeds S, and follow this with negative terms until the sum is below S. See-sawing in this way, we gradually home in on S while ensuring that we have included all the original terms of the series. The rearrangement converges to S because the terms x_n, and hence also p_n and q_n, tend to zero. Here are the details.

Take r_1 to be the least integer such that $\sum_{i=1}^{r_1} p_i > S$. This is possible because $\sum_{i=1}^{\infty} p_i$ diverges to $+\infty$. Put $P_1 = \sum_{i=1}^{r_1} p_i$. Next take s_1 to be the least integer such that $P_1 + \sum_{i=1}^{s_1} q_i < S$. This is possible because $\sum_{i=1}^{\infty} q_i$ diverges to $-\infty$. Put $Q_1 = \sum_{i=1}^{s_1} q_i$

Next take r_2 to be the least integer greater than r_1 such that $P_1 + Q_1 + \sum_{i=r_1+1}^{r_2} p_i > S$ and put $P_2 = \sum_{i=r_1+1}^{r_2} p_i$. Likewise take s_2 to be the least integer greater than s_1 such that $P_1+Q_1+P_2+\sum_{i=s_1+1}^{s_2} q_i < S$ and put $Q_2 = \sum_{i=s_1+1}^{s_2} q_i$. Again, these choices are possible because of the divergence of $\sum_{i=1}^{\infty} p_i$ and $\sum_{i=1}^{\infty} q_i$.

At this point pause and note that $r_2 > r_1 \geq 1$ and $s_2 > s_1 \geq 1$. We also have

$$\begin{aligned}
P_1 &= p_1 + \ldots + p_{r_1} > S \\
P_1 + Q_1 &= p_1 + \ldots + p_{r_1} + q_1 + \ldots + q_{s_1} < S \\
P_1 + Q_1 + P_2 &= p_1 + \ldots + p_{r_1} + q_1 + \ldots + q_{s_1} + p_{r_1+1} + \ldots + p_{r_2} > S \\
P_1 + Q_1 + P_2 + Q_2 &= p_1 + \ldots + p_{r_1} + q_1 + \ldots + q_{s_1} + p_{r_1+1} + \ldots + p_{r_2} + \\
&\qquad + q_{s_1+1} + \ldots + q_{s_2} < S \ \ \text{(sorry it doesn't fit on one line!)}
\end{aligned}$$

You can see that we are beginning to use all the terms of both (p_n) and (q_n), and hence building up a rearrangement of (x_n). If we look at the third of the four sums displayed above and omit the positive term p_{r_2}, then the total will be less than or equal to S. In other words $S + p_{r_2} \geq (P_1 + Q_1 + P_2) > S$. So the partial sum of the rearrangement given by that third sum differs by at most $|p_{r_2}|$ from S. These bounds on the sum will continue to hold as we add the negative terms q_{s_1+1} up to and including q_{s_2-1}, because up until and including that last addition, the sum remains greater than S (or just possibly equal to S). At any rate all these partial sums differ from S by at most $|p_{r_2}|$. If we then look at the last of the four sums displayed above and omit the negative term q_{s_2}, then that total will be greater than or equal to S. In other words $S > (P_1 + Q_1 + P_2 + Q_2) \geq S + q_{s_2}$. So the partial sum of the rearrangement given by that fourth sum differs by at most $|q_{s_2}|$ from S. Again that bound will continue to hold as we add further positive terms until we get to p_{r_3} which is defined by the requirement that it is the least integer greater than r_2 such that $P_1 + Q_1 + P_2 + Q_2 + \sum_{i=r_2+1}^{r_3} p_i > S$.

We continue in this fashion, alternating groups of entries from (p_n) and (q_n) to produce a rearrangement of (x_n) with the property that the partial sums of the associated series differ from S by a subsequence of values taken from (x_n), and which subsequence must therefore tend to zero. Hence the rearranged series converges to S.

[Strictly speaking we should use induction to define the sequences $(r_1, r_2, r_3, \ldots)$ and $(s_1, s_2, s_3, \ldots)$ but sometimes it is best to take a more relaxed attitude, and this is certainly one such occasion. How would you do it? First define r_1 and s_1 as we have done and then, on the assumption that $r_1, r_2, \ldots, r_n$ and $s_1, s_2, \ldots s_n$ are already defined, give a definition of r_{n+1} and s_{n+1} - good luck!!]

To deal with the other cases (divergence, oscillation) we follow a similar procedure, we just alter the targets. So for divergence to $+\infty$, we take blocks of positive and negative terms alternately above and below $1, 2, 3, \ldots$. $\square$

Exercises for Section 4.3

1. Which of the following series are alternating? You may assume that the trigonometric functions have their usual properties.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$, b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/2)}{n^2}$,

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin((2n+1)\pi/2)}{n}$, d) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan(n\pi/2)}{n}$.

2. Which, if any, of the series in Question 1 converge and is the convergence absolute or conditional?

3. Determine the values of x (if any) for which the following series converge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+x^2}$, b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+x^{2n}}$,

c) $\displaystyle\sum_{n=1}^{\infty} x^{n!}$, d) $\displaystyle\sum_{n=1}^{\infty} n^2 \frac{x^{3n}}{2^n}$.

4. Prove that the following series converges absolutely and determine its sum: $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \ldots$ (terms taken three at a time, one positive and two negative).

5. Prove that the series $\sum_{n=0}^{\infty}(n+1)x^n$ converges if and only if $|x| < 1$, and determine the sum when it is convergent. [Let S_n denote the n^{th} partial sum and calculate $S_n - 2xS_n + x^2S_n$.]

6. Prove that the series (defined earlier in this section) as $\sin(x), \cos(x)$ and $\exp(x)$ all have infinite radius of convergence.

7. Determine the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$.

8. Use the method given in the proof of Theorem 4.17 to write down in the correct order the first ten terms of a rearrangement of the alternating harmonic series that will converge with the sum $\frac{7}{8} = 0.875$.

4.4 Multiplication of Series

Multiplication of series poses a problem about the order of terms. If we examine a very simple finite product $(a_0 + a_1)(b_0 + b_1)$, you will have several choices about how to write down the answer. Probably the most obvious ones are $a_0b_0 + a_0b_1 + a_1b_0 + a_1b_1$ and $a_0b_0 + a_1b_0 + a_0b_1 + a_1b_1$. But there are four terms in the product and so $4! = 24$ ways to order the four terms. Clearly if we had 3 terms in each bracket, there would be a lot of choice ($9!$ in fact). So you can see that with an infinite number of terms in each bracket, we need to proceed with some caution. If you multiply two polynomials in some variable x you will, I hope, feel an urge to group like powers of x together. For example, $(a_0 + a_1x)(b_0 + b_1x) = a_0b_0 + (a_0b_1 + a_1b_0)x + a_1b_1x^2$. If we take two quadratic factors and record the terms up to x^2 in the product we get

$$\begin{aligned}(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^x) = a_0b_0 &+ (a_0b_1 + a_1b_0)x \\ &+ (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots .\end{aligned}$$

This gives us a clue about how we might proceed, and at least this is consistent with what we do for multiplying polynomials.

Definition 4.6. Given two infinite series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their *Cauchy product* is defined as the infinite series $\sum_{n=0}^{\infty} c_n$, where $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \dots + a_nb_0$.

This definition says absolutely nothing about issues of convergence. Investigating that aspect is what this Section is all about. We will show that if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both absolutely convergent with sums A and B respectively, then their Cauchy product is absolutely convergent with sum $C = AB$. But the condition of absolute convergence on the two ingredients is important. To start the proof, we will look at the possibility of summing an infinite 2-dimensional array.

Consider the following array of Real Numbers.

$$\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots & a_{0,j} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots & a_{1,j} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & a_{2,j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i,0} & a_{i,1} & a_{i,2} & \cdots & a_{i,j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

First we try to sum the numbers in each row, i.e. we form the series

$$\sum_{j=0}^{\infty} a_{0,j}, \quad \sum_{j=0}^{\infty} a_{1,j}, \quad \sum_{j=0}^{\infty} a_{2,j}, \quad \ldots, \quad \sum_{j=0}^{\infty} a_{i,j},$$

and see if each one is convergent. Now suppose that each one is convergent and denote the sum $\sum_{j=0}^{\infty} a_{i,j}$ by R_i. Then we can form the combined sum of all the rows:

$$\sum_{i=0}^{\infty} R_i = \sum_{i=0}^{\infty} \Big(\sum_{j=0}^{\infty} a_{i,j} \Big),$$

and see if this is also convergent. If this sum is convergent then we say that the array has the row sum S_R given by

$$S_R = \sum_{i=0}^{\infty} R_i = \sum_{i=0}^{\infty} \Big(\sum_{j=0}^{\infty} a_{i,j} \Big).$$

In a similar way, working first with columns, we may (when all the series involved are convergent) form the column sum of the array $S_C = \sum_{j=0}^{\infty} C_j$, where $C_j = \sum_{i=0}^{\infty} a_{i,j}$. Our interest then is whether the two sums

$$S_R = \sum_{i=0}^{\infty} \Big(\sum_{j=0}^{\infty} a_{i,j} \Big) \text{ and } S_C = \sum_{j=0}^{\infty} \Big(\sum_{i=0}^{\infty} a_{i,j} \Big)$$

are in fact equal. Sometimes this can happen.

Theorem 4.18. With the same notation as in the preceding discussion, suppose that $a_{i,j} \geq 0$ for all i and j. Then if the row sum converges, so does the column sum (and vice-versa), and the sums are equal.

Proof. Suppose that the row sum is convergent. This means that $\sum_{j=0}^{\infty} a_{i,j}$ is convergent for each i and, denoting this sum by R_i, the sum $\sum_{i=0}^{\infty} R_i$ is also convergent to some Real Number S_R.

We observe that $a_{i,j} \leq R_i$ for each j. Therefore, summing over i and using the comparison test it follows that $\sum_{i=0}^{\infty} a_{i,j}$ converges and the sum (denoted by C_j) satisfies

$$C_j = \sum_{i=0}^{\infty} a_{i,j} \leq \sum_{i=0}^{\infty} R_i = S_R.$$

In other words, each column of the array forms a convergent series.

In fact, we have a lot more that just $a_{i,j} \leq R_i$. If J is any non-negative integer, we have $\sum_{j=0}^{J} a_{i,j} \leq R_i$. So again summing over i and using the comparison test we obtain

$$\sum_{j=0}^{J} C_j \leq \sum_{i=0}^{\infty} R_i = S_R.$$

So the partial sums of the series $\sum_{j=0}^{\infty} C_j$ are bounded above by S_R and therefore this series of non-negative terms must converge to some value $S_C \leq S_R$. Thus the column sum is convergent and, by reversing the roles of rows and columns (i.e. taking the transpose of the array), we also get $S_R \leq S_C$. It follows that $S_R = S_C$, and that completes the proof. $\square$

The theorem above establishes a result when all the terms $a_{i,j}$ are non-negative. The corollary below establishes a similar result without this restriction provided that we have absolute convergence of the row or column sum. And, roughly speaking, the proof involves splitting the array into its positive and negative parts as we have done previously.

Corollary 4.18.1. Suppose that $(a_{i,j})$ is an (infinite) array of Real Numbers and that the row (or column) sum of the array $(|a_{i,j}|)$ converges. Then both the row and column sums of $(a_{i,j})$ converge to a common value.

Proof. Define

$$a_{i,j}^{+} = \begin{cases} a_{i,j} & \text{if } a_{i,j} \geq 0, \\ 0 & \text{if } a_{i,j} < 0 \end{cases} \qquad a_{i,j}^{-} = \begin{cases} 0 & \text{if } a_{i,j} \geq 0, \\ a_{i,j} & \text{if } a_{i,j} < 0 \end{cases}$$

If either the row or column sum of $(|a_{i,j}|)$ converges, then by Theorem 4.18 above, so does the other. We may therefore suppose that both converge.

Consider the array $(a_{i,j}^{+})$. Since $a_{i,j}^{+} \leq |a_{i,j}|$, it follows from the comparison test that $\sum_{j=0}^{\infty} a_{i,j}^{+}$ converges to some Real Number, R_i^{+}, say, and $R_i^{+} \leq \sum_{j=0}^{\infty} |a_{i,j}|$. Then, once again by the comparison test, $\sum_{i=0}^{\infty} R_i^{+}$ converges. Let S_R^{+} denote its sum.

[Although we don't need this, it may help you to observe that

$$S_R^{+} = \sum_{i=0}^{\infty} R_i^{+} = \sum_{i=0}^{\infty} \Big(\sum_{j=0}^{\infty} a_{i,j}^{+} \Big) \leq \sum_{i=0}^{\infty} \Big(\sum_{j=0}^{\infty} |a_{i,j}| \Big).]$$

In the same way it follows that $\sum_{i=0}^{\infty} a_{i,j}^{+}$ converges to some Real Number, C_j^{+}, say, and $C_j^{+} \leq \sum_{i=0}^{\infty} |a_{i,j}|$. And, once again by the comparison test, $\sum_{j=0}^{\infty} C_j^{+}$ converges. Let S_C^{+} denote its sum. By Theorem 4.18 we have $S_R^{+} = S_C^{+}$.

If we now consider the array $(a_{i,j}^{-})$, we may proceed in a similar fashion and find the row sum S_R^{-} and the column sum S_C^{-}, and note that $S_R^{-} = S_C^{-}$.

Then because $a_{i,j} = a_{i,j}^{+} + a_{i,j}^{-}$, it follows that the row sum S_R of $(a_{i,j})$ exists and is given by $S_R = S_R^{+} + S_R^{-}$, and also that the column sum S_C of $(a_{i,j})$ exists and is given by $S_C = S_C^{+} + S_C^{-}$. Finally, since $S_R^{+} + S_R^{-} = S_C^{+} + S_C^{-}$, it follows that $S_R = S_C$, i.e. the row and column sums of the array $(a_{i,j})$ are equal. □

We can now proceed to the main result of this section.

Theorem 4.19. Suppose that $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are both absolutely convergent series with sums A and B respectively. Then their Cauchy product is also absolutely convergent with sum AB.

Proof. Define $\bar{A} = \sum_{i=0}^{\infty} |a_i|$ and $\bar{B} = \sum_{i=0}^{\infty} |b_i|$. Then consider the two arrays:

$$\begin{pmatrix} a_0b_0 & 0 & 0 & 0 & \dots \\ a_0b_1 & a_1b_0 & 0 & 0 & \dots \\ a_0b_2 & a_1b_1 & a_2b_0 & 0 & \dots \\ a_0b_3 & a_1b_2 & a_2b_1 & a_3b_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \begin{pmatrix} |a_0b_0| & 0 & 0 & 0 & \dots \\ |a_0b_1| & |a_1b_0| & 0 & 0 & \dots \\ |a_0b_2| & |a_1b_1| & |a_2b_0| & 0 & \dots \\ |a_0b_3| & |a_1b_2| & |a_2b_1| & |a_3b_0| & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

In the second array, the $(j+1)^{\text{th}}$ column has sum $|a_j| \sum_{i=0}^{\infty} |b_i| = |a_j|\bar{B}$, so the overall column sum of this array is $(\sum_{j=0}^{\infty} |a_j|)\bar{B} = \bar{A}\bar{B}$. It follows from Corollary 4.18.1 that the row and column sums of the first array converge to a common value.

Looking at that first array, the $(j+1)^{\text{th}}$ column has sum $a_j \sum_{i=0}^{\infty} b_i = a_j B$, so the overall column sum of this array is $(\sum_{j=0}^{\infty} a_j)B = AB$. However, the row sum of this array is

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \dots,$$

which is precisely the Cauchy product of the series $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$. So the Cauchy product converges with sum AB. □

Theorem 4.19 is particularly useful when it comes to multiplying power series. Remember that a power series is absolutely convergent within its radius of convergence, so if we have two power series in a variable x with radii of convergence R_1 and R_2, and $R_1 \geq R_2$, then the Cauchy product will have radius of convergence at least R_2. In cases where the radii of convergence are both infinite, the Cauchy product will also have an infinite radius of convergence.

Exercises for Section 4.4

1. If $\exp(x)$ is defined as sum of the power series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$, prove that the Cauchy product of $\exp(x)$ with itself (i.e. $(\exp(x))^2$) is just $\exp(2x)$.

2. Prove that the Cauchy product of $\exp(x)$ and $\exp(y)$ is $\exp(x+y)$.
[If you have a very strong stomach for algebra you can prove in a similar way that $\sin(x)\cos(y) + \cos(x)\sin(y) = \sin(x+y)$, where $\sin(x)$ and $\cos(x)$ are given by the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ respectively. We will look at this in detail when we discuss these functions and investigate their properties in Chapter 7.]

3. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent. Show that the Cauchy product of this series with itself is divergent. [Hint: show that the terms of the Cauchy product do not form a null sequence.]

Chapter 5

Functions, Limits and Continuity

5.1 Functions

You will already have your own ideas about what we mean when we speak about functions. The purpose of this section is to give a formal definition and to describe some associated terminology.

Definition 5.1. A *mapping* ϕ from a set S to a set T is a *rule* that takes each $x \in S$ and associates with it one or more elements of T. The "rule" is specified by the set of all the ordered pairs (x, y) where $x \in S$ and y is any one of the elements of T associated with x by ϕ.

Example 5.1. Suppose that $S = \{1, 2, 3\}$ and $T = \{a, b, c, d, e, f\}$ and ϕ associates 1 with a, c, d, 2 with b, d, e, and 3 with a, b. Then ϕ is a mapping from S to T. As a collection of ordered pairs we can write

$$\phi = \{(1, a), (1, c), (1, d), (2, b), (2, d), (2, e), (3, a), (3, b).\}$$

Continuing with the definition, if ϕ associates $x \in S$ with $y \in T$, then we say that y is an image of x under ϕ. In the example above, a, c, d are the images of $1 \in S$ in the set T. In general, every point of S has at least one image in T, but there may be points in T that are not the image of any $x \in S$. In the example above, f is not the image of any of the points of S. The set S is generally called the *domain* of ϕ, and the set T is generally called the *co-domain*. The set of all the images, denoted by $\phi(S)$, may be called the *range* or the *image set* of ϕ. Clearly, $\phi(S) \subseteq T$. In the example above, $\phi(S) = \{a, b, c, d, e\}$. If $\phi(S) = T$, i.e. every element of the co-domain is an image of some point in the domain, then we say that ϕ is a *surjective* or *onto* mapping.

It may happen that each $x \in S$ has precisely one image in T under ϕ. In such a case, ϕ is called a *function*, and the unique image of $x \in S$ under ϕ is denoted by $\phi(x)$.

Summarising: a function has one image of each point in its domain.

It is possible that a function may map more than one element of S to the same element of T, i.e. we may have $\phi(x_1) = \phi(x_2)$ for $x_1 \neq x_2$. If that happens, the function is described as being a *many-one* function.

Example 5.2. Suppose that $S = T = \mathbb{R}$ and ϕ is defined as the mapping that takes $x \in \mathbb{R}$ to $x^2 \in \mathbb{R}$. Then ϕ is a function from S to T because every x gives rise to a unique value x^2. However this is a many-one function because, for example, $4 \in T$ is the image of both 2 and -2 in S.

In previous chapters we studied sequences. A sequence can be seen as a function. For example, the sequence $(\frac{1}{n^2})$ is the function defined by all the ordered pairs $(n, \frac{1}{n^2})$ for $n \in \mathbb{N}$. The domain of this function is $\mathbb{N}$, the co-domain can be taken as $\mathbb{R}$, and the image set is the set of all Real Numbers of the form $\frac{1}{n^2}$ for $n \in \mathbb{N}$.

Again continuing with the definition, in the case when ϕ is a function defined by a formula, as will often be the case, we may write $\phi : x \mapsto \phi(x)$. So we can describe the mapping of Example 5.2 as $\phi : x \mapsto x^2$ (read as "x maps to x^2 "). Indeed, we often abbreviate further, just referring to the formula itself, as in "the function x^2 ". However, giving a formula such as $\phi(x) = x^2$ does not specify the domain or the co-domain. In such cases, either the domain must be specified separately, or (the default position) taken as the largest set for which the formula makes sense. Once the domain is known, if the co-domain is not specified, it can be taken to be the image set. In terms of ordered pairs the mapping from the example is $\phi = \{(x, x^2) : x \in \mathbb{R}\}$, and this does specify the domain. If we wish to restrict the domain to a specific interval, for example the interval from 0 to 1, we might write something like "$\phi(x) = x^2$ for $0 \leq x \leq 1$". We often do restrict a domain in order to make deductions about the image set.

One way to specify the domain S and co-domain T of a mapping ϕ is to use the $\rightarrow$ symbol as in $\phi : S \rightarrow T$. So we could write $\phi : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$. The $\rightarrow$ symbol is also used to denote convergence, but that isn't likely to cause any confusion. So why use the $\mapsto$ symbol at all, couldn't we just stick to $\rightarrow$? That's a good question. It seems to have become an accepted convention to use $\mapsto$ when specifying the image of an individual point $x \in S$. Personally, I wouldn't get too hung up about this, particularly in handwriting; only a pedant would criticise $x \rightarrow x^2$.

Example 5.3. If $\psi(x) = \sqrt{x}$, the default domain is the set of non-negative Real Numbers: $\{x : x \geq 0\}$. This is because $\sqrt{x}$ only makes sense (if we expect a Real Number for $\sqrt{x}$) for $x \geq 0$. [Note that $\sqrt{x}$ *means* the non-negative root. For example, $\sqrt{4} = 2$, not -2. Of course this is a notational convention and it cannot alter the fact that there are two square roots of 4, namely $\sqrt{4}$ and $-\sqrt{4}$.] The image set for this function is also the set of all non-negative Real Numbers.

In some cases a function ϕ with domain S will have the property that each $y \in \phi(S)$ is the image of a unique point $x \in S$. This is the case with the function $\psi(x) = \sqrt{x}$ of the previous example: if $y \geq 0$ then y is the image of y^2 under ψ, i.e. $\psi(y^2) = y$. In such cases we say that the function is *injective* or *one-one* (sometimes written as "one-to-one" or "1:1"). Many functions are not injective. For example the function $\phi(x) = x^2$ with domain $\mathbb{R}$ is certainly not injective because 2 and -2 have the same image, namely 4. However, if we restrict the domain of $\phi(x) = x^2$ to non-negative Real Numbers ($\{x : x \geq 0\}$) then it (strictly speaking it's a new function) becomes injective.

If the function $\phi : S \to T$ is both surjective (onto) and injective (one-one), then is is said to be *bijective*. In some ways bijective functions are the nicest ones; in particular, each bijective function has an inverse function.

If $f : S \to T$ is a bijective function then the *inverse function*, denoted by f^{-1}, is defined to be the function with domain T and co-domain S given by the rule

$$f^{-1}(y) = x \text{ if and only if } f(x) = y.$$

It follows immediately from this definition that $f(f^{-1}(y)) = y$, $f^{-1}(f(x)) = x$, and $(f^{-1})^{-1} = f$. Moreover, each $x \in S$ is the image under f^{-1} of the unique value $y = f(x) \in T$, so that f^{-1} is also bijective.

Example 5.4. Determine the inverse of the function given by $f(x) = 3x + 2$ with domain $\mathbb{R}$.

Solution. Put $y = 3x + 2$ then $x = (y - 2)/3$. So $f^{-1}(y) = (y - 2)/3$. Of course this is what you've always done when solving equations. The only (sometimes) tricky bit is finding the domain of f^{-1}, which is the original image set of f. That's fairly obvious here since every Real Number y will give a corresponding Real Number x. So the domain of f^{-1} in this example is $\mathbb{R}$.

We conclude this Section with description of notation for composite functions and a description of what is meant by an implicit function.

Definition 5.2 (Composite functions)**.**
Suppose that $g : S \to T$ and $f : T \to U$ are two functions. Then, given $s \in S$, we have $g(s) \in T$ and so we can compute $f(g(s)) \in U$. The resulting function

with domain S and co-domain U, given by $s \mapsto f(g(s))$ is called the *composite* of f and g and may be denoted as $f \circ g$. Figure 5.1 illustrates the formation of $f \circ g$ from f and g.

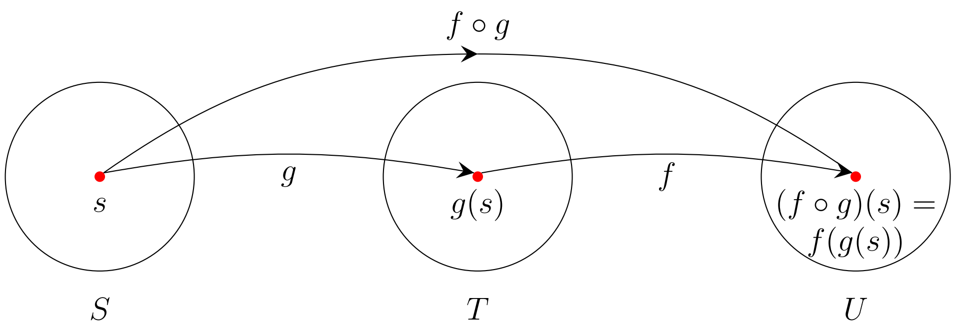


Figure 5.1: Formation of $f \circ g$

Note that in $f \circ g$, g is applied first and f second, so that $f \circ g$ is not the same thing as $g \circ f$. In fact the existence of $f \circ g$ does not imply the existence of $g \circ f$ (or vice-versa), and even when both exist, they may be different functions. One exception is with a pair of inverse functions that happen to have the same domain, because then $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ and $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$, so in this case $f^{-1} \circ f = f \circ f^{-1}$.

Example 5.5. Take $f : \{x : x \geq 0\} \to \mathbb{R}$ given by $x \mapsto \sqrt{x}$ and $g : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto -x$. Then $g \circ f : \{x : x \geq 0\} \to \mathbb{R}$ is given by $x \mapsto -\sqrt{x}$. However, if you try to form $f \circ g$ you will find that you are attempting to calculate things like $\sqrt{-1}$ because the image set of g is not contained within the domain of f. So $f \circ g$ does not exist in this case.

Definition 5.3 (Implicit functions)**.**
Suppose that $f : S \to \mathbb{R}$ is a function, and that S is the set of ordered pairs (x, y) where $x \in S_1$ and $y \in S_2$ (S is then called the *Cartesian product* of S_1 and S_2, and we write $S = S_1 \times S_2$).

Consider the equation $f(x, y) = 0$. For a given $x \in S_1$ this equation may have (a) no solutions for y, or (b) one solution for y, or (c) more than one solution for y. Now suppose that (b) is the case for *every* $x \in S_1$. So for each value of x there is just one corresponding value of y which we will denote by $y(x)$ and which satisfies $f(x, y(x)) = 0$. Then $y(x)$ is a function with domain S_1 and co-domain S_2 that is known as the *implicit function* of x determined by f.

Example 5.6.

(a) If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $f(x,y) = x^2 + y$, then the implicit function is $y(x) = -x^2$.

(b) If $f : \{x : -1 \le x \le 1\} \times \{y : 0 \le y \le 1\} \to \mathbb{R}$ is given by $f(x,y) = x^2 + y^2 - 1$, then the implicit function is $y(x) = \sqrt{1 - x^2}$.

(c) If $f : \{x : -1 \le x \le 1\} \times \{y : -1 \le y \le 1\} \to \mathbb{R}$ is given by $f(x,y) = x^2 + y^2 - 1$, then there is no implicit function $y(x)$ because for at least one value of $x \in \{x : -1 \le x \le 1\}$ (e.g. $x = 0$) there are two solutions for y to the equation $x^2 + y^2 - 1 = 0$.

(d) If $f : \mathbb{R} \times \{y : 0 \le y \le 1\} \to \mathbb{R}$ is given by $f(x,y) = x^2 + y^2 - 1$, then there is no implicit function $y(x)$ because for at least one value of $x \in \mathbb{R}$ (e.g. $x = 2$) there is no solution for y to the equation $x^2 + y^2 - 1 = 0$.

It may be difficult or even impossible to "solve" an equation $f(x,y) = 0$ to get y in terms of x, even though f may define an implicit function. There is an example of this in the Exercises.

Exercises for Section 5.1

1. Determine which of the following mappings are surjective, and which are functions. For those that are surjective functions determine if they are many-one or one-one (injective). For those that are bijective determine the inverse function.

a) $\phi = \{(1,b),(2,c),(2,e),(3,b),(4,a)\}$, domain $\{1,2,3,4\}$, co-domain $\{a,b,c,d,e\}$,

b) $\phi = \{(1,b),(2,b),(3,b),(4,a)\}$, domain $\{1,2,3,4\}$, co-domain $\{a,b,c\}$,

c) $\phi = \{(1,b),(2,b),(3,b),(4,a)\}$, domain $\{1,2,3,4\}$, co-domain $\{a,b\}$,

d) $\phi = \{(1,b),(2,a),(3,d),(4,c)\}$, domain $\{1,2,3,4\}$, co-domain $\{a,b,c,d\}$,

e) $\phi(x) = x^3$, domain $\mathbb{R}$, co-domain $\mathbb{R}$,

f) $\phi(x) = x^4$, domain $\mathbb{R}$, co-domain $\mathbb{R}$,

g) $\phi(x) = \pm\sqrt{x}$, domain $\{x : x \ge 0\}$, co-domain $\mathbb{R}$,

h) $\phi(x) = x^2$, domain $\mathbb{R}$, co-domain $\{x : x \ge 0\}$,

i) $\phi(x) = x^2$, domain $\{x : x \le 0\}$, co-domain $\{x : x \ge 0\}$.

2. The function $f(x)$ is given by $f(x) = x^2 - 3x + 2$ with domain $\{x : x \geq \frac{3}{2}\}$ and co-domain equal to the image set. Determine the inverse function f^{-1} and state its domain and image set.

3. If $f(x) = 2x$ and $g(x) = x + 1$, both with domain and co-domain $\mathbb{R}$, find $f \circ g$ and $g \circ f$.

4. Put $f(x, y) = y^3 - y - x$, where x, y are any Real Numbers. Show that for a given $x \geq 6$ there is at most one y satisfying $f(x, y) = 0$. [In fact there is exactly one y for each such value of x, so f does define an implicit function $y = y(x)$ for $x \geq 6$.]

5.2 Cartesian Graphs

If we return to the ordered pair specification of a function when the domain and co-domain are $\mathbb{R}$ or subsets thereof, we can represent the function on a 2-dimensional Cartesian graph. For example if $f(x) = x + 1$ with domain and co-domain $\mathbb{R}$, the graph consists of all the points (x, y) for which $y = x + 1$. This can then be *illustrated* with a sketch. I emphasise the word "illustrated" because sketches can be misleading. That does not deny their utility in giving insight into how a function behaves. So, by all means, use sketches to suggest results and proofs, but do **not** use them *as proofs*. Having given this restricted licence to sketch graphs, we will take a look at some functions and graphs. Note that the domain is represented on the horizontal axis, and the co-domain on the vertical axis.

Example 5.7. If $\phi(x) = x^2$ with domain and co-domain $\mathbb{R}$, the Cartesian graph of ϕ is the set of ordered pairs $\{(x, x^2) : x \in \mathbb{R}\}$. This is sketched below in Figure 5.2. The dotted line shows that this is a many-one function. There are two values on the horizontal x-axis that give rise to the same value of $\phi(x)$.

Example 5.8. If $\psi(x) = \sqrt{x}$ with domain and co-domain $\{x : x \geq 0\}$, the Cartesian graph of ψ is the set of ordered pairs $\{(x, \sqrt{x}) : x \geq 0\}$. This is sketched below in Figure 5.3. This is an injective (one-one) function. As illustrated by the dotted line, any value on the non-negative part of the vertical axis arises from a unique value of x on the non-negative part of the horizontal axis.

The next example shows why sketching a graph can be misleading.

Example 5.9. Suppose that $f(x) = \begin{cases} 2 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational,} \end{cases}$ with domain $\mathbb{R}$ (and co-domain $\{1, 2\}$). Since every interval on the x-axis (no matter how small) contains both rational and irrational numbers, the best we can do to illustrate this

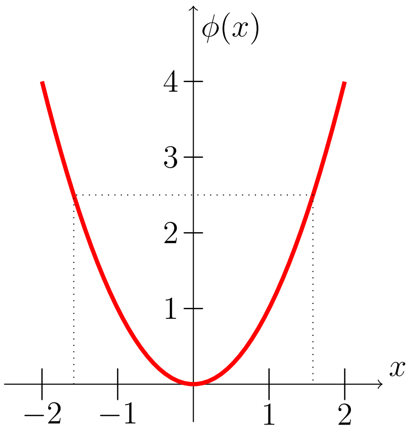


Figure 5.2: $\phi(x) = x^2$

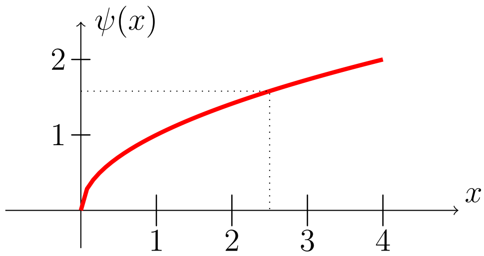


Figure 5.3: $\psi(x) = \sqrt{x}$

function is the following sketch (Figure 5.4). The blue line corresponds to rational values of x and the red line to irrational values of x. It *looks* like the dotted line cuts the graph twice, once on the blue line and once on the red line, but this is not the case. Any such vertical dotted line (corresponding to a value of x) will only cut one of the two lines (the blue line if x is rational, the red line if x is irrational). So the sketch makes it look like this function is not a function at all, even though it is. It's very misleading and, if you use it, you should give a health warning.

Sometimes we use filled or empty circles to denote points included or excluded from a sketch.

Example 5.10. Figure 5.5 shows a sketch of the floor function $f(x) = \lfloor x \rfloor$ with domain $\mathbb{R}$. [Reminder: $\lfloor x \rfloor$ is the integer satisfying the inequalities $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.] The filled circles indicate included points and the empty circles indicate excluded points. For example, there is a point on the graph at $(1, 1)$, but not at $(2, 1)$. Obviously such indications of included and excluded points have their limitations - it's the best that can be done in a simple sketch.

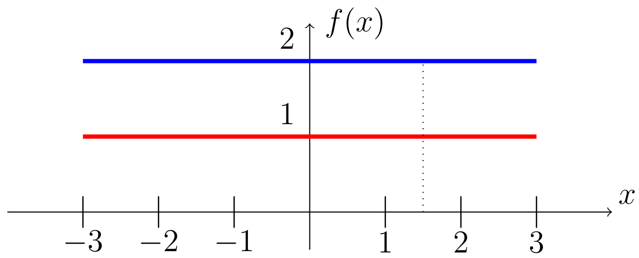


Figure 5.4: A misleading graph

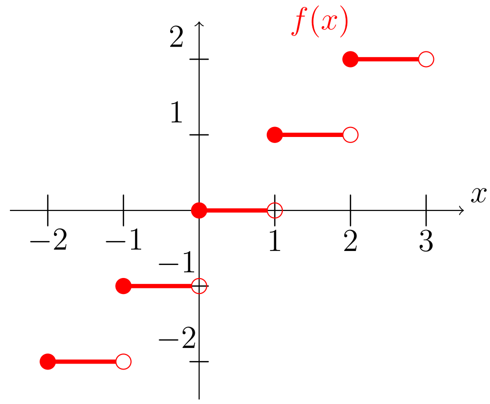


Figure 5.5: The floor function $f(x) = \lfloor x \rfloor$

Next we turn our attention to inverse functions. Suppose that f is a bijective function with domain $S \subseteq \mathbb{R}$ and co-domain $T \subseteq \mathbb{R}$. The Cartesian graph of f is obtained by plotting the points $(s, f(s))$ for $s \in S$. The Cartesian graph of f^{-1} is obtained by plotting the points $(f(s), s)$, effectively swapping the domain and co-domain of f, which means swapping the horizontal and vertical axes. Geometrically, this is achieved by reflecting the graph of f in the line with gradient 1 that passes through the origin, as shown in Figure 5.6 below.

Example 5.11. If $f(x) = x^2$ with domain and co-domain $\{x : x \geq 0\}$, then f is bijective because

(i) each y in the co-domain is the image of $x = \sqrt{y}$ in the domain, so the function is surjective (onto), and

(ii) it is only $x = \sqrt{y}$ in the domain that has image y in the co-domain, so the function is injective (one-one)

It follows that f has an inverse function ($f^{-1}(x) = \sqrt{x}$ of course) with domain and co-domain $\{x : x \geq 0\}$. These are sketched below in Figure 5.7.

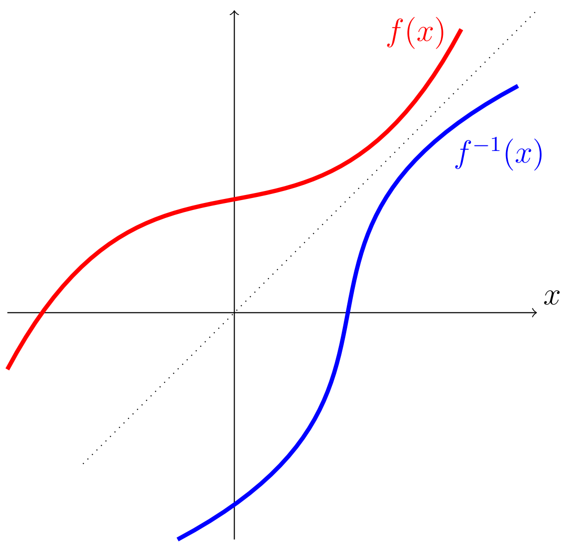


Figure 5.6: A function and its inverse

Two particular types of function are easily described by graphical illustrations. These are *even* and *odd* functions. First, here are the definitions.

Definition 5.4 (Even and odd functions)**.**
Suppose that $f : S \to \mathbb{R}$ is a function where S is a suitable subset of $\mathbb{R}$ (what we mean by "suitable" is clarified below).

If $f(x) = f(-x)$ for each $x \in S$, then we say that f is an *even* function.

If $f(x) = -f(-x)$ for each $x \in S$, then we say that f is an *odd* function.

Clearly this entails having $-x \in S$ whenever $x \in S$; that's what we mean by "suitable".

The reason for the terms "even" and "odd" is that even powers of x define even functions, and odd powers of x define odd functions. For example, $f(x) = x^2$ is even and $f(x) = x^3$ is odd.

The Cartesian graph of an even function is symmetrical about the vertical axis, and the Cartesian graph of an odd function has 180^o rotational symmetry about the origin. If f is odd and if $0 \in S$, this implies that $f(0) = 0$. The graphs below illustrate the two cases (Figure 5.8). Of course most functions are neither even nor odd (e.g. $f(x) = 1 + x$). But if the domain of f is "suitable", we can define even and odd parts of f, such that f is the sum of its even and odd parts.

Definition 5.5 (Even and odd parts)**.**
Suppose that $f : S \to \mathbb{R}$ is a function with domain $S \subseteq \mathbb{R}$ such that whenever $x \in S$, it is also the case that $-x \in S$. Then we define the even part of f as the

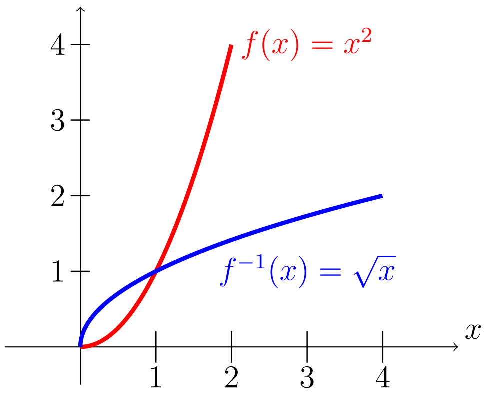


Figure 5.7: $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$

function

$$f_{even}(x) = \frac{f(x) + f(-x)}{2},$$

and the odd part of f as the function

$$f_{odd}(x) = \frac{f(x) - f(-x)}{2},$$

both with domain S and co-domain $\mathbb{R}$.

It is easy to check that f_{even} is an even function, that f_{odd} is an odd function, and that $f_{even}(x) + f_{odd}(x) = f(x)$ for every $x \in S$.

We conclude this Section with some notation for intervals. It is very tedious to have to express a domain such as $\{x : -1 \leq x \leq 1\}$ in this form. It is an interval in $\mathbb{R}$ and the important information comprises the end points and whether or not they are included.

Definition 5.6. Suppose that $a, b \in \mathbb{R}$ and that $a < b$. Then

$$\begin{aligned}
(a, b) &\text{ denotes the interval } \{x : a < x < b\},\\
[a, b] &\text{ denotes the interval } \{x : a \leq x \leq b\},\\
[a, b) &\text{ denotes the interval } \{x : a \leq x < b\},\\
(a, b] &\text{ denotes the interval } \{x : a < x \leq b\},\\
(a, \infty) &\text{ denotes the interval } \{x : x > a\},\\
[a, \infty) &\text{ denotes the interval } \{x : x \geq a\},\\
(-\infty, a) &\text{ denotes the interval } \{x : x < a\},\\
(-\infty, a] &\text{ denotes the interval } \{x : x \leq a\},\\
(-\infty, \infty) &= \mathbb{R}.
\end{aligned}$$

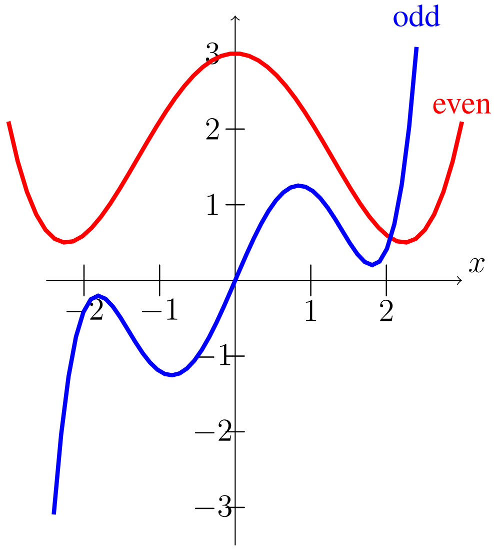


Figure 5.8: Even and odd functions

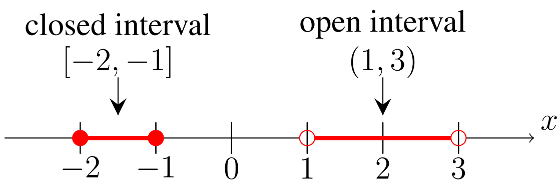


Figure 5.9: Open and closed intervals

You will observe that excluded end points are denoted by round brackets, and included end points by square brackets. So, for example, $a \notin (a, b)$. The interval (a, b) is described as the *open interval* from a to b, and $[a, b]$ is described as the *closed interval* from a to b. Figure 5.9 illustrates (a, b) and $[a, b]$ using empty circles and filled circles as described earlier.

Exercises for Section 5.2

1. Sketch the graphs of the following functions with the default domain in each case, and state whether the function is odd, even or neither. You can assume the usual properties of the sine function.

a) $f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$

b) $f(x) = \sqrt{x^2}$.

c) $f(x) = x - \lfloor x \rfloor$.

d) $f(x) = \sin(\frac{1}{x})$.

e) $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x\sin(\frac{1}{x}) & \text{if } x \neq 0. \end{cases}$

f) $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{q} & \text{if } x \text{ is rational } (x = \frac{p}{q} \text{ in lowest terms, with } q > 0). \end{cases}$

5.3 Limits of Functions

Suppose that $f(x)$ has domain $\mathbb{R}$. Then we might enquire about what happens to $f(x)$ for large positive x, i.e. as x tends to $+\infty$. This is similar to what we did for sequences. Equally well we might ask about the behaviour of $f(x)$ as x tends to $-\infty$. But with a continuous variable x, in place of the discrete variable n of a sequence, we can also enquire about the behaviour of $f(x)$ as x approaches any particular Real Number a from the left ($x < a$), or from the right ($x > a$), or unrestricted. We will look at each of these in turn, starting with the ones that are closest to what we did for sequences.

5.3.1 Limits at $\pm\infty$

We start very informally by considering the Cartesian graph of the function $f(x) = (x-1)/x$ with domain $[1, \infty]$. This is sketched in Figure 5.10. It looks fairly clear

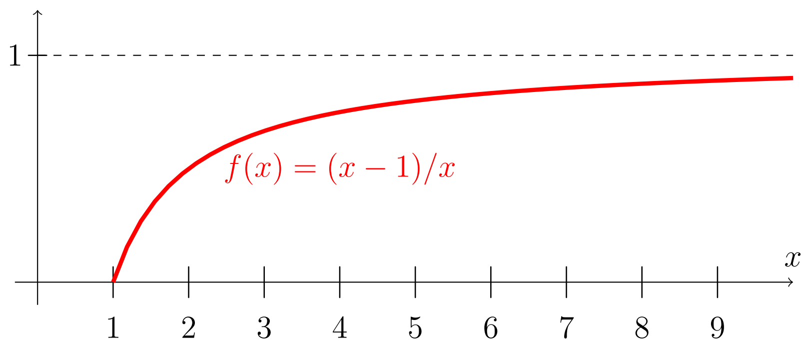


Figure 5.10: $f(x) = (x-1)/x$

from the graph that the value of $f(x)$ is approaching 1 as x increases. This is not surprising, particularly if you write $f(x) = (x-1)/x = 1 - \frac{1}{x}$ and think what is likely to happen to $\frac{1}{x}$ as x gets larger. We'd like to write "$f(x) \to 1$ as $x \to \infty$".

And to do enable us to do this we make a definition very similar to what we did for sequences.

Any function $f(x)$ with a domain (a, ∞) can be said to tend to a limit l as x tends to ∞ if, for any horizontal strip centred on l, there is some point X beyond which (i.e. for all $x > X$), all the values of $f(x)$ lie in the strip. See the illustration below (Figure 5.11).

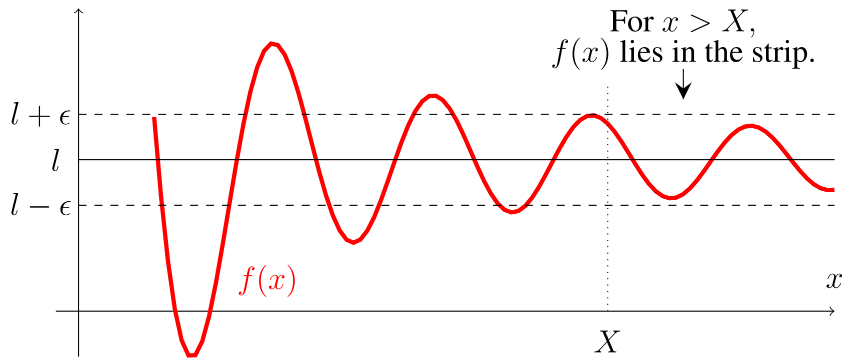


Figure 5.11: $f(x) \to l$ as $x \to \infty$

Our definition of what we mean by saying that "$f(x) \to l$ as $x \to \infty$" is that for any sized strip (characterized by $\epsilon > 0$) there exists a number X such that for every $x > X$, $|f(x) - l| < \epsilon$. In formal terminology:

Definition 5.7. We say that "$f(x) \to l$ as $x \to \infty$" if and only if

$$\forall \epsilon > 0, \exists X \text{ s.t. } \forall x > X, |f(x) - l| < \epsilon.$$

This is almost the same as what we had as the definition of "$x_n \to l$ as $n \to \infty$" for sequences. I invite you to look again at the discussion in Chapter 1. Almost every technique and result we had about sequences in Chapter 3 has its counterpart here for functions with limits at ∞. Note that the definition implies that f is defined on some domain (a, ∞).

Example 5.12. Prove that if $f(x) = (x - 1)/x$, then $f(x) \to 1$ as $x \to \infty$.

Solution. Choose $\epsilon > 0$. Put $X = \frac{1}{\epsilon}$. Take any $x > X$ and consider $|f(x) - 1| = \frac{1}{x} < \frac{1}{X} = \epsilon$. Hence $f(x) \to 1$ as $x \to \infty$.

The similarities with the results for sequences are so strong that we won't spend time looking at the details.

Sometimes we may replace "$x \to \infty$" with "$x \to +\infty$" in the definition above in order to emphasise that we are concerned with large positive values of x.

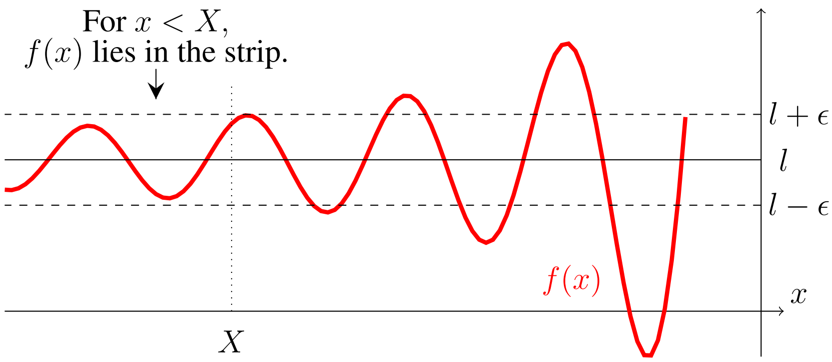


Figure 5.12: $f(x) \to l$ as $x \to -\infty$

This is because, unlike sequences where n heads off in the positive direction, for functions we may also consider what happens as x moves in the negative direction, as illustrated in Figure 5.12.

With the diagram in mind, we make the following definition.

Definition 5.8. We say that "$f(x) \to l$ as $x \to -\infty$" if and only if

$$\forall \epsilon > 0, \exists X \text{ s.t. } \forall x < X, |f(x) - l| < \epsilon.$$

This is almost the same as the previous definition. Previously we had $x > X$, now we have $x < X$. But all the arguments for dealing with limits at $-\infty$ are much the same as those for limits at ∞. And this definition implies that f is defined on some domain $(-\infty, a)$.

Example 5.13. Prove that if $f(x) = (x - 1)/x$, then $f(x) \to 1$ as $x \to -\infty$.

Solution. Choose $\epsilon > 0$. Put $X = -\frac{1}{\epsilon}$ (note X is negative). Take any $x < X$ (so $|x| > |X|$), and consider $|f(x) - 1| = |\frac{1}{x}| < |\frac{1}{X}| = \epsilon$. Hence $f(x) \to 1$ as $x \to -\infty$.

This is almost the same as the previous solution - we just had to introduce a few modulus signs because our choice for X was negative.

If you look back to what we did for sequences, you will see that we defined carefully what was meant by saying "$x_n \to +\infty$ as $n \to \infty$". Here are the equivalent definitions for a function $f(x)$, both for x moving in the positive direction, and for x moving in the negative direction.

Definition 5.9.

1. We say that "$f(x) \to +\infty$ as $x \to +\infty$" if and only if

$$\forall A, \exists X \text{ s.t. } \forall x > X, f(x) > A.$$

2. We say that "$f(x) \to -\infty$ as $x \to +\infty$" if and only if
$$\forall A, \exists X \text{ s.t. } \forall x > X, f(x) < A.$$
3. We say that "$f(x) \to +\infty$ as $x \to -\infty$" if and only if
$$\forall A, \exists X \text{ s.t. } \forall x < X, f(x) > A.$$
4. We say that "$f(x) \to -\infty$ as $x \to -\infty$" if and only if
$$\forall A, \exists X \text{ s.t. } \forall x < X, f(x) < A.$$

Don't try to memorise each of these. Learn the first one and how to change directions to get the other three. All four are illustrated in the following diagram (Figure 5.13) by four different functions exhibiting the four behaviours. And in all four cases the definition implies that f is defined on some appropriate domain.

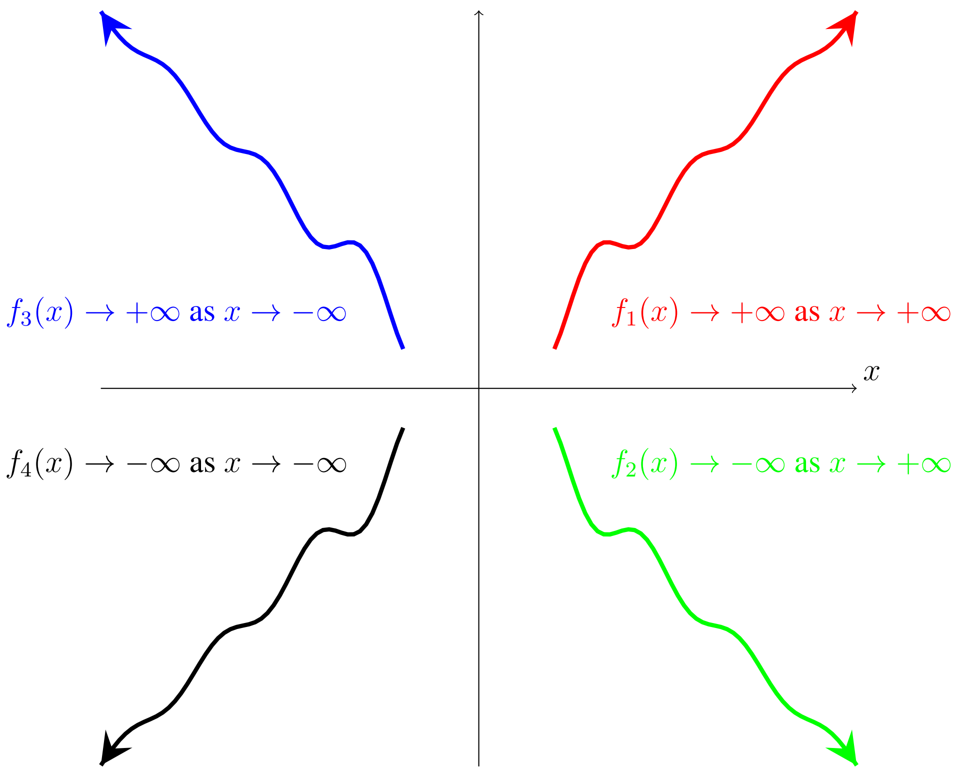


Figure 5.13: Various behaviours

Example 5.14. Prove that $x^3 \to -\infty$ as $x \to -\infty$.

Solution. Choose $A \in \mathbb{R}$. Put $X = -(|A| + 1)$, so that $X \leq -1$ and $X < -|A|$ Then take any $x < X$ and consider $x^3 < X^3 \leq X < -|A| \leq A$. Hence $x^3 \to -\infty$ as $x \to -\infty$.

There are a few examples for you to practice with in the Exercises at the end of this Section. But now we will leave this type of behaviour concerned with limits at $\pm\infty$ in favour of considering the behaviour of a function $f(x)$ in the vicinity of a point $a \in \mathbb{R}$.

5.3.2 Limits at a point

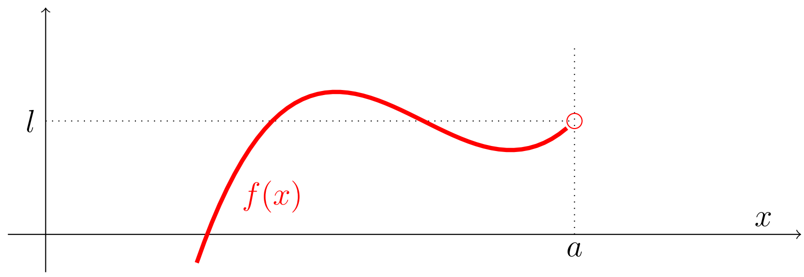


Figure 5.14: $f(x) \to l$ as $x \to a-$

Looking at the graph in Figure 5.14, you will see that $f(x)$ approaches the value l as x approaches a from the left hand side of a. Another way of expressing this is to say that f maps points near a (but below a) to points near l. We need to encapsulate this in a formal definition. The first thing to recognise is that it isn't necessary for $f(x)$ to be defined at $x = a$. In fact, even if $f(x)$ is defined at $x = a$ it will be convenient to avoid the assumption that $f(a) = l$. So in trying to find a definition of "$f(x) \to l$ as $x \to a-$", (to be read as "$f(x)$ tends to l as x tends to a *from below*"), we regard the value of $f(x)$ at $x = a$ as **totally irrelevant**!

From our work on sequences and on limits of functions at $\pm\infty$, we know how to enclose the limiting value in a strip. So take the previous diagram and form a strip centred on the horizontal dotted line. The strip is characterised (as before) by the positive number ϵ. This is shown in Figure 5.15. You will see that corresponding to the horizontal strip there is a one-sided vertical strip (to the left of a), that can be characterised by some positive number δ, and which has the property that if x lies in this vertical strip, then $f(x)$ lies in the horizontal strip

We would expect to find that for any horizontal strip (i.e. for any $\epsilon > 0$) there is a corresponding vertical strip (i.e. $\delta > 0$) such that for any x between $a - \delta$ and a, the value of $f(x)$ is within the horizontal strip (i.e. $|f(x) - l| < \epsilon$). This gives us the following definition.

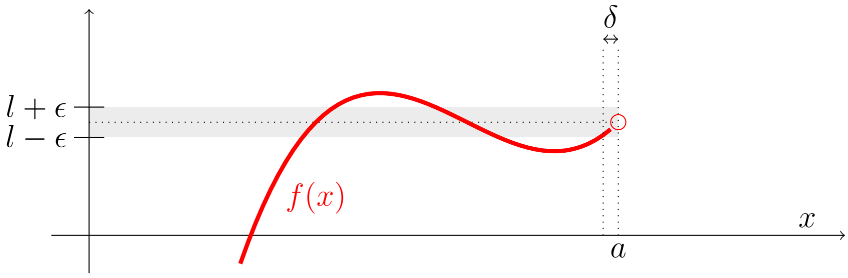


Figure 5.15: Towards a definition.

Definition 5.10. We say that "$f(x) \to l$ as $x \to a-$" if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in (a-\delta, a),\ |f(x) - l| < \epsilon.$$

The expression $x \in (a-\delta, a)$ is equivalent to $a - \delta < x < a$ so, if you prefer it, you can write the definition as

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \text{ satisfying } a-\delta < x < a,\ |f(x) - l| < \epsilon.$$

Of course the interest lies in small positive values of ϵ, but we don't say that explicitly (what is "small"?) because "all $\epsilon > 0$" certainly captures small $\epsilon > 0$, however you choose to think of "small". And normally if ϵ is made smaller, we'd anticipate that δ would have to be made smaller. Just like the definition of a sequence converging, this definition is an operational one, it tells us that to prove convergence we have to pick an arbitrary positive ϵ and determine a corresponding value of δ.

Example 5.15. Prove that $x^2 \to 4$ as $x \to 2-$.

Solution. Choose $\epsilon > 0$. Put $\delta = \min(1, \epsilon/4)$. Choose any x satisfying $2 - \delta < x < 2$, so that $x \in (1, 2)$ and $2 - x < \delta$. Then consider

$$|x^2 - 4| = 4 - x^2 = (2 + x)(2 - x) < 4(2 - x) < 4\delta \le \epsilon.$$

Hence $x^2 \to 4$ as $x \to 2-$.

Remarks.
1. In the solution, δ is essentially taken as $\epsilon/4$ but we make it the minimum of 1 and $\epsilon/4$ to ensure (via the 1) that x can't be too far away from 2. After all, there's nothing in the definition to stop you taking $\epsilon = 1000$ however crazy that might seem. The definition has to be verified *for all* $\epsilon > 0$.
2. You don't have to guess the $\epsilon/4$ at the start of the solution, just leave $\delta =$

$\min(1, \ldots)$ incomplete until you get to near the end. It's the same technique that was used to establish convergence of sequences.

Having seen the definition of convergence to a limit as x approaches a value a *from below*, it should be relatively easy to see how to alter that definition to deal with x approaching a *from above*. The graph below in Figure 5.16 illustrates this. Again we have a horizontal strip, characterised by $\epsilon > 0$, with a corresponding vertical strip characterised by $\delta > 0$, this time to the right of a. Once again, the value (if any) of $f(x)$ for $x = a$ is irrelevant.

Definition 5.11. We say that "$f(x) \to l$ as $x \to a+$" if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in (a, a+\delta),\ |f(x) - l| < \epsilon.$$

The expression $x \in (a, a+\delta)$ is equivalent to $a < x < a + \delta$ so, if you prefer it, you can write the definition as

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \text{ satisfying } a < x < a + \delta,\ |f(x) - l| < \epsilon.$$

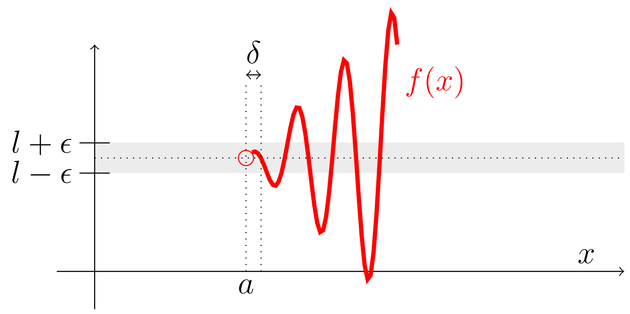


Figure 5.16: $f(x) \to l$ as $x \to a+$

Example 5.16. Prove that $x^2 \to 4$ as $x \to 2+$.

Solution. Choose $\epsilon > 0$. Put $\delta = \min(1, \epsilon/5)$. Choose any x satisfying $2 < x < 2 + \delta$, so that $x \in (2, 3)$ and $x - 2 < \delta$. Then consider

$$|x^2 - 4| = x^2 - 4 = (x+2)(x-2) < 5(x-2) < 5\delta \le \epsilon.$$

Hence $x^2 \to 4$ as $x \to 2+$.

This solution is almost identical to the one we had previously for $x \to 2-$. We just needed to alter the fraction from $\epsilon/4$ to $\epsilon/5$ because now x is (slightly) larger than 2.

Often we need to consider what happens to $f(x)$ as x approaches a, *without restricting x to be below or above a.* In other words, we want a definition of $f(x) \to l$ as $x \to a$ (read as "$f(x)$ tends to l as x tends to a") and we define this to mean that *both*

- $f(x) \to l$ as $x \to a-$, and
- $f(x) \to l$ as $x \to a+$

For a given $\epsilon > 0$ there may be two different δ values, one for $x \to a-$ and one for $x \to a+$. But by taking the smaller of the two, we can formulate this definition as follows.

Definition 5.12. We say that "$f(x) \to l$ as $x \to a$" if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \text{ satisfying } 0 < |x - a| < \delta,\ |f(x) - l| < \epsilon.$$

This provides a formal definition to capture the informal idea that f maps points near a (excluding a itself) to points near l. Note that the condition $0 < |x - a|$ ensures that the one value of x that is totally irrelevant is $x = a$, so we don't even need $f(x)$ to be defined at $x = a$. Why is this exclusion useful? It can be shown that $\dfrac{\sin(x)}{x} \to 1$ as $x \to 0$, even though $\dfrac{\sin(x)}{x}$ is (obviously) undefined at $x = 0$, and there are many similar examples.

Example 5.17. Prove that $x^2 \to 4$ as $x \to 2$.

Solution.
We already did this by showing that $x^2 \to 4$ as $x \to 2-$ and that $x^2 \to 4$ as $x \to 2+$. But it can be done, all in one go, as follows.

Choose $\epsilon > 0$. Put $\delta = \min(1, \epsilon/5)$, take any x satisfying $0 < |x - 2| < \delta$ (so that $|x| < 3$) and consider

$$|x^2 - 4| = |x - 2||x + 2| < \delta|x + 2| < 5\delta \leq \epsilon,$$

which is what we sought to prove.

The following example is more difficult and it does involve a function $f(x)$ that is undefined for $x = a$.

Example 5.18. Prove that if $a > 0$ then $f(x) = \dfrac{\sqrt{x} - \sqrt{a}}{x - a} \to \dfrac{1}{2\sqrt{a}}$ as $x \to a$.

Solution.
Choose $\epsilon > 0$. Put $\delta = \min(\frac{a}{2}, a^{3/2}\epsilon)$. [The $\frac{a}{2}$ is to ensure that x isn't too far from a, and the $a^{3/2}\epsilon$ comes out in the working below - you don't have to guess it in advance!]

Take any x satisfying $0 < |x - a| < \delta$. This implies that $x > \frac{a}{2}$, so that $\sqrt{x} > \sqrt{\frac{a}{2}}$. The first step is to get $\frac{\sqrt{x}-\sqrt{a}}{x-a}$ into a form that suggests it might be close to $\frac{1}{2\sqrt{a}}$. We have

$$\begin{aligned} f(x) = \frac{\sqrt{x}-\sqrt{a}}{x-a} &= \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})} \\ &= \frac{1}{(\sqrt{x}+\sqrt{a})} \end{aligned}$$

This looks a bit like $\frac{1}{2\sqrt{a}}$ and it follows that

$$\begin{aligned} \Big|f(x) - \frac{1}{2\sqrt{a}}\Big| &= \Big|\frac{1}{(\sqrt{x}+\sqrt{a})} - \frac{1}{2\sqrt{a}}\Big| \\ &= \Big|\frac{\sqrt{a}-\sqrt{x}}{2\sqrt{a}(\sqrt{x}+\sqrt{a})}\Big| \\ &= \Big|\frac{(\sqrt{a}-\sqrt{x})(\sqrt{a}+\sqrt{x})}{2\sqrt{a}(\sqrt{x}+\sqrt{a})^2}\Big| \\ &= \frac{|a-x|}{2\sqrt{a}(\sqrt{x}+\sqrt{a})^2} \\ &< \frac{\delta}{2\sqrt{a}(\sqrt{x}+\sqrt{a})^2} \\ &< \frac{\delta}{2\sqrt{a}(\sqrt{\frac{a}{2}}+\sqrt{a})^2} \quad \text{(using } \sqrt{x} > \sqrt{\tfrac{a}{2}}\text{)} \\ &= \frac{\delta}{(3+2\sqrt{2})a^{3/2}} < \frac{\delta}{a^{3/2}} \le \epsilon \end{aligned}$$

Remarks.
1. To prove that $|f(x) - l|$ is less than ϵ, the main tool at your disposal is the inequality $|x - a| < \delta$. In establishing a limit from first principles, it is almost always the case that you need to relate $|f(x) - l|$ to some multiple of $|x - a|$. That is precisely what all the nasty algebra was about in the previous example.
2. We obviously don't want to have to wade through lots of horrible algebra every time. So we need some general results (like we had for sequences) that tell us what happens if we add, multiply or divide functions. We also need some results about "standard" easy functions, such as $f(x) = x$. We move on to such results below.

3. If you can recall results about differentiation from previous experiences, you might notice a connection between the previous example and the result that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$. Differentiation is the subject of the next chapter where we will make use of results about limits obtained in this chapter.

The definition of convergence for functions at a point can be used to prove non-convergence, just like we did for sequences.

Example 5.19. Assuming the usual properties of $\sin(x)$, prove that $\sin(\frac{1}{x})$ has no limit as x tends to zero.

Solution. Suppose that $\sin(\frac{1}{x}) \to l$ as $x \to 0$. Take $\epsilon = 0.5$ in the definition. Then there exists $\delta > 0$ such that for any x satisfying $0 < |x| < \delta$, $|\sin(\frac{1}{x}) - l| < 0.5$. Now take $x_1 = \dfrac{1}{2n\pi - \frac{\pi}{2}}$ and $x_2 = \dfrac{1}{2n\pi + \frac{\pi}{2}}$, where the positive integer $n \geq 1$ is chosen so large that $x_1, x_2 < \delta$. But then $\sin\left(\frac{1}{x_1}\right) = -1$ and $\sin\left(\frac{1}{x_2}\right) = 1$. So we have both $|-1-l| < 0.5$ and $|1-l| < 0.5$, which is clearly impossible (l cannot be within 0.5 of both -1 and 1). We therefore conclude that $\sin(\frac{1}{x}) \not\to l$ as $x \to 0$ for any $l \in \mathbb{R}$.

Now let us move on to some simple results about limits of functions at a point. We won't give the proofs here because they are so similar to the proofs of the corresponding results for sequences. But you are asked to provide your own proofs in the exercises at the end of this section. The results remain true (with minor adjustments to the intervals) for one-sided convergence at a point a, and also for convergence at $\pm\infty$. Here then is a list summarised as a Theorem.

Theorem 5.1.

(a) The limit of a function at a point is unique: if $f(x) \to l_1$ as $x \to a$ and $f(x) \to l_2$ as $x \to a$ then $l_1 = l_2$.

(b) If f is the constant function with value l, then $f(x) \to l$ as $x \to a$.

(c) (Local boundedness) If $f(x) \to l$ as $x \to a$, then there exists $\delta > 0$ such that the set $S = \{y : y = f(x) \text{ for some } x \text{ satisfying } 0 < |x-a| < \delta\}$ is a bounded set of Real Numbers.

(d) (Combination rules) Suppose that $f(x) \to l$ as $x \to a$ and $g(x) \to m$ as $x \to a$. Then

  (i) (multiple rule) if k is any constant, $kf(x) \to kl$ as $x \to a$,

  (ii) (sum rule) $f(x) + g(x) \to l + m$ as $x \to a$,

  (iii) (product rule) $f(x)g(x) \to lm$ as $x \to a$,

(iv) (quotient rule) if $m \neq 0$ then $f(x)/g(x) \to l/m$ as $x \to a$.

The sum and product rules can of course be extended to cover a finite sum or product of n functions for any positive integer n. However you should not use these rules on infinite series or products. Here is an example of what can go wrong.

Example 5.20. Let $f(x)$ be defined with domain $\mathbb{R}$ by the formula

$$f(x) = x^2 + \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \ldots = \sum_{i=0}^{\infty} \frac{x^2}{(1+x^2)^i}.$$

Note that if $x \neq 0$ then $\frac{1}{1+x^2} < 1$ and so the (geometric) series converges with sum $f(x) = x^2 \dfrac{1}{1 - \frac{1}{1+x^2}} = 1 + x^2$. However, if $x = 0$ all the terms of the series are zero and so $f(0) = 0$. So we have a situation where each term of the sum tends to zero as x tends to zero, but $f(x) \to 1$ as $x \to 0$, even though $f(0) = 0$. [The moral is that any result connecting the limiting value of an infinite series to the limits of the separate terms will require further conditions on the nature of the series. A similar comment applies to infinite products.]

You could well be tempted to conjecture a result for composite functions, perhaps along the following lines: "If $f(x) \to l$ as $x \to a$ and $g(x) \to a$ as $x \to b$ then $(f \circ g)(x) = f(g(x)) \to l$ as $x \to b$." Unfortunately this is not necessarily true. The problem is that $g(x)$ might actually *equal* a. Indeed, it could even be the constant function with value a and, even though $f(x) \to l$ as $x \to a$, there is no necessity for $f(x)$ to be defined at a and, even if it is, we may have $f(a) \neq l$. The resolution of this difficulty will be found when we move on to discuss continuity in the next section.

Our next theorem tells us that polynomials and rational functions (i.e. quotients of polynomials) behave as we would expect.

Theorem 5.2. If $f(x)$ is a polynomial then for every $a \in \mathbb{R}$, $f(x) \to f(a)$ as $x \to a$. If $g(x)$ is also a polynomial then the rational function $r(x) = f(x)/g(x) \to f(a)/g(a)$ as $x \to a$ for every $a \in \mathbb{R}$ for which $g(a) \neq 0$.

Proof. We have already mentioned that the constant function $f_0(x) = l \to l = f_0(a)$ as $x \to a$.

Next consider the function $f_1(x) = x$. Choose $\epsilon > 0$. Put $\delta = \epsilon$ and take any x satisfying $0 < |x - a| < \delta$. Then consider $|f_1(x) - f_1(a)| = |x - a| < \delta = \epsilon$. Hence $f_1(x) = x \to f_1(a) = a$ as $x \to a$.

Now apply the product rule $k-1$ times to f_1 to deduce that for $k > 1$, $f_k(x) = x^k \to a^k$ as $x \to a$. By applying the multiple rule and the sum rule repeatedly, we deduce that if $f(x)$ is a polynomial then for every $a \in \mathbb{R}$, $f(x) \to f(a)$ as $x \to a$.

Finally, the quotient rule tells us that if f and g are polynomials, then $f(x)/g(x) \to f(a)/g(a)$ as $x \to a$ for every $a \in \mathbb{R}$ for which $g(a) \neq 0$. $\square$

As with sequences, there are "sandwich" rules that are sometimes helpful. Here's a version for convergence at a point.

Theorem 5.3 (Sandwich rule)**.** Suppose that $\gamma > 0$ and that $g(x) \leq f(x) \leq h(x)$ for all values of x satisfying $0 < |x-a| < \gamma$. Then if $g(x) \to l$ as $x \to a$ and if $h(x) \to l$ as $x \to a$, we have $f(x) \to l$ as $x \to a$.

Proof. Choose $\epsilon > 0$. There exists $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then $|g(x) - l| < \epsilon$, and there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $|h(x) - l| < \epsilon$. Put $\delta = \min(\gamma, \delta_1, \delta_2)$. Then if $0 < |x-a| < \delta$ we have

$$-\epsilon < g(x) - l \leq f(x) - l \leq h(x) - l < \epsilon,$$

and so $|f(x) - l| < \epsilon$. Hence $f(x) \to l$ as $x \to a$. $\square$

There are versions of this result for convergence on the left and on the right. The necessary modifications should be fairly obvious.

Example 5.21. Assuming that $\sin(x)$ has its usual properties, prove that $x^2 \sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0$.

Solution. Since $|\sin(x)| \leq 1$ for all values of x, we have $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$ for all $x \neq 0$. Both $g(x) = -x^2$ and $h(x) = x^2$ are polynomials and so $g(x) \to g(0) = 0$ as $x \to 0$ and $h(x) \to h(0) = 0$ as $x \to 0$. It follows from the sandwich rule that $x^2 \sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0$.

We will conclude this section by mentioning divergence to $\pm\infty$ at a point a, giving a few (hopefully) easy definitions that closely resemble their counterparts for sequences.

Definition 5.13.

- We say that $f(x) \to +\infty$ as $x \to a+$ if and only if $\forall A, \exists \delta > 0$ s.t. $\forall x$ satisfying $a < x < a + \delta,\ f(x) > A$.
- We say that $f(x) \to +\infty$ as $x \to a-$ if and only if $\forall A, \exists \delta > 0$ s.t. $\forall x$ satisfying $a - \delta < x < a,\ f(x) > A$.
- We say that $f(x) \to +\infty$ as $x \to a$ if and only if $\forall A, \exists \delta > 0$ s.t. $\forall x$ satisfying $0 < |x-a| < a + \delta,\ f(x) > A$.

- We say that $f(x) \to -\infty$ as $x \to a+$ if and only if
  $\forall A, \exists \delta > 0$ s.t. $\forall x$ satisfying $a < x < a + \delta,\ f(x) < A$.

- We say that $f(x) \to -\infty$ as $x \to a-$ if and only if
  $\forall A, \exists \delta > 0$ s.t. $\forall x$ satisfying $a - \delta < x < a,\ f(x) < A$.

- We say that $f(x) \to -\infty$ as $x \to a$ if and only if
  $\forall A, \exists \delta > 0$ s.t. $\forall x$ satisfying $0 < |x - a| < a + \delta,\ f(x) < A$.

As a single illustration of these definitions, see the graph below. It shows a horizontal line at height A and a corresponding vertical strip of width δ to the right of a having the property that, if x lies in the strip (i.e. if $a < x < a + \delta$), then $f(x)$ lies above the horizontal line (i.e. $f(x) > A$). So it illustrates the case $f(x) \to +\infty$ as $x \to a+$.

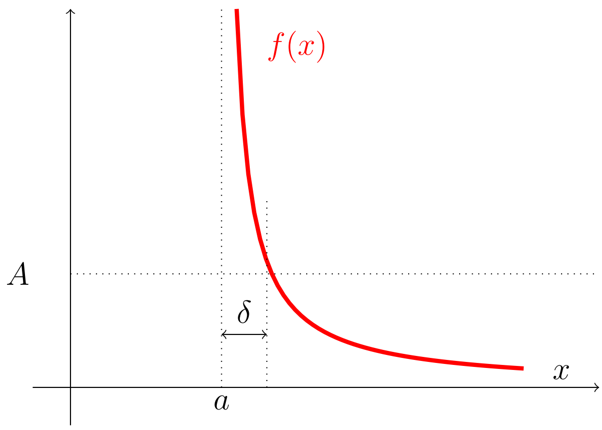


Figure 5.17: $f(x) \to +\infty$ as $x \to a+$

A corresponding example is the following.

Example 5.22. Prove that $f(x) = \frac{1}{x-2} \to +\infty$ as $x \to 2+$.

Solution. Choose A. Put $\delta = \frac{1}{|A|+1}$. Take any x satisfying $2 < x < 2 + \delta$ so that $0 < x - 2 < \delta$. Then consider

$$\frac{1}{x-2} > \frac{1}{\delta} = |A| + 1 > A.$$

Hence $f(x) = \frac{1}{x-2} \to +\infty$ as $x \to 2+$.
[Note that although we might wish to simplify the choice of δ to $\frac{1}{A}$, we must be careful because A could be negative or zero.]

Exercises for Section 5.3

1. If $f(x) = \dfrac{1}{1+x^2}$, prove that $f(x) \to 0$ as $x \to +\infty$.
2. If $f(x) = x^2$, prove that $f(x) \to +\infty$ as $x \to +\infty$.
3. Assuming the usual properties of $\sin(x)$, prove that $x \sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0+$.
4. If $f(x) = (x^2 - a^2)/(x - a)$ (for $x \neq a$), prove that $f(x) \to 2a$ as $x \to a$.
5. Provide proofs for the results stated in Theorem 5.1.
6. Prove that $f(x) = \frac{1}{x-2} \to -\infty$ as $x \to 2-$.
7. Suppose that $f(x)$ is monotonically increasing on an interval $[a, b]$, i.e whenever $x, y \in [a, b]$ with $x < y$ then $f(x) \leq f(y)$. Take any point $\xi \in [a, b)$. Prove that $f(x)$ has a limit from above at ξ, i.e $\exists \lim_{x\to\xi+} f(x)$. Similarly, prove that if $\xi \in (a, b]$ then $f(x)$ has a limit from below at ξ, i.e $\exists \lim_{x\to\xi-} f(x)$.

5.4 Continuity

In everyday English, "continuity" refers to something that persists without abrupt change. In terms of Cartesian graphs of functions, an informal description lies in the ability to draw a graph without taking the pencil off the page. Taking the pencil off the page results in a break in the graph. More mathematically, we might say that there are no points of discontinuity on a continuous graph. Of course this is a sort of tautology, since how can we speak of discontinuity before we speak of continuity? But it does point the way to proceed with a definition of continuity. Discontinuities happen at individual points, so if we can describe a discontinuity at an individual point, the absence of discontinuities will define continuity.

In the previous section we defined "$f(x) \to l$ as $x \to \infty$" and we repeatedly made the point that there was no necessity for l to be the same as $f(a)$. Indeed there was no need for $f(x)$ to be defined for $x = a$. However, it may happen that $l = f(a)$ in some circumstances. If $l \neq f(a)$ we would expect to see a discontinuity at a. If $l = f(a)$ we would expect an absence of discontinuity at a, i.e. continuity at the point a. Bearing this in mind we are led to a definition that describes discontinuity and continuity *at an individual point*.

Definition 5.14 (Continuity at a point)**.**
Suppose that $f(x) \to f(a)$ as $x \to a$. [This implies that $f(x)$ is defined on some interval (c, d) that contains the point a.] Then we say that f is *continuous* at the point a. If $f(x)$ is defined on some interval (c, d) that contains the point a, but $f(x) \not\to f(a)$ as $x \to a$, then we say that f is *discontinuous* at the point a.

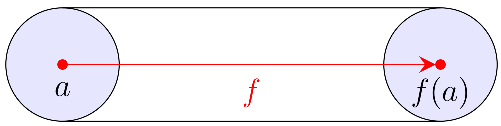
In a similar way, if $f(x) \to f(a)$ as $x \to a-$ then we say that f is *continuous on the left* at the point a. If $f(x)$ is defined on an interval $(c, a]$, but $f(x) \not\to f(a)$ as $x \to a-$, then we say that f is *discontinuous on the left* at the point a.

Likewise, if $f(x) \to f(a)$ as $x \to a+$ then we say that f is *continuous on the right* at the point a. If $f(x)$ is defined on an interval $[a, d)$ but $f(x) \not\to f(a)$ as $x \to a+$, then we say that f is *discontinuous on the right* at the point a.

In ϵ, δ terms, the definition of continuity of the function f at the point a can be expressed as:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \text{ satisfying } |x - a| < \delta, \ |f(x) - f(a)| < \epsilon.$$

This comes immediately from the earlier definition of $f(x) \to l$ as $x \to a$, with l replaced by $f(a)$. But note that there is now no need to exclude $x = a$ by the device "$0 < |x - a|$" because, if $x = a$, then it is automatically true that $|f(x) - f(a)| = 0 < \epsilon$. Informally speaking, continuity of f at the point a means that f maps points near a to points near $f(a)$. Some people find the type of illustration in Figure 5.18 helpful.



f maps points near a to points near $f(a)$.

Figure 5.18: Continuity of f at the point a

Example 5.23.
Put $f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$ Prove that f is not continuous at any point $a \in \mathbb{R}$.

Solution. Suppose that f is continuous at some point a. Take $\epsilon = \frac{1}{2}$. Then there exists $\delta > 0$ such that for any x satisfying $|x - a| < \delta$ we have $|f(x) - f(a)| < \frac{1}{2}$.

If a is irrational then $f(a) = 1$ and, if we choose a rational number x satisfying $|x - a| < \delta$, then we have $|f(x) - f(a)| = |0 - 1| = 1$, which contradicts $|f(x) - f(a)| < \frac{1}{2}$.

If a is rational then $f(a) = 0$ and, if we choose an irrational number x satisfying $|x - a| < \delta$, then we have $|f(x) - f(a)| = |1 - 0| = 1$, which again contradicts $|f(x) - f(a)| < \frac{1}{2}$.

So we conclude that f cannot be continuous at any point $a \in \mathbb{R}$.

Here is another example that looks a bit similar but with a very different result.

Example 5.24.
Put $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{q} & \text{if } x \text{ is rational } (x = \frac{p}{q} \text{ in lowest term with } q > 0). \end{cases}$
Prove that f is continuous at each irrational point, but discontinuous at each rational point.

Solution. We deal first with rational points (it's easier). Suppose that $a = \frac{p}{q}$ is rational in its lowest terms ($q > 0$), so that $f(a) = \frac{1}{q}$. If f were continuous at a, then by taking $\epsilon = \frac{1}{q}$, there would be $\delta > 0$ such that for any $x \in (a - \delta, a + \delta)$ (i.e. $|x - a| < \delta$) we have $|f(x) - \frac{1}{q}| < \frac{1}{q}$. But if we now take x to be irrational and in $(a - \delta, a + \delta)$ we have $f(x) = 0$, giving $|0 - \frac{1}{q}| < \frac{1}{q}$, an obvious contradiction. So f is discontinuous at each rational point a.

The more difficult part is when a is irrational. In this case choose $\epsilon > 0$. Take a positive integer $N > \frac{1}{\epsilon}$. There are only finitely many rational numbers $\frac{p}{q}$ (in lowest terms) with denominator $0 < q \leq N$ and lying in the finite interval $(a - 1, a + 1)$. So put

$$\delta = \min\left(\left|\frac{p}{q} - a\right| : \frac{p}{q} \in (a - 1, a + 1) \text{ is in lowest terms, and } 0 < q \leq N.\right)$$

Then $\delta > 0$. If $x = \frac{p}{q}$ (in lowest terms, $q > 0$) is rational and $|x - a| < \delta$, then $q > N$ and so $|f(x) - f(a)| = \frac{1}{q} - 0 < \frac{1}{N} < \epsilon$. If x is irrational then $|f(x) - f(a)| = 0 < \epsilon$. So whether or not x is rational or irrational, if $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Hence f is continuous at each irrational point a.

Once we have settled on the definition of continuity at a point, it is easy to define continuity on an interval. Of course the acid test of whether or not this is a good definition of continuity on an interval is by checking that it leads to the sorts of conclusions we would expect. Let us see ... But first the definition.

Definition 5.15 (Continuity on an interval)**.**
Suppose that $f(x)$ is defined on the open interval (c, d) (i.e. that is its domain), and that $f(x)$ is continuous at every point $a \in (c, d)$. Then we say that f is *continuous* on (c, d).

If $f(x)$ is defined on the closed interval $[c, d]$, and is continuous on (c, d), continuous on the right at c, and continuous on the left at d, then we say that f is continuous on $[c, d]$.

In the case of intervals of the form $[c, d)$ or $(c, d]$ we define continuity on the interval by requiring continuity on (c, d) together with one-sided continuity at the appropriate end point.

If f is continuous everywhere (i.e. at every point $a \in \mathbb{R}$) we may just say that "f is continuous" without explicitly mentioning the interval $\mathbb{R} = (-\infty, +\infty)$.

Example 5.25. Prove that $f(x) = \frac{1}{x}$ is continuous on any interval (c, d) that does not contain the point $x = 0$.

Solution. Suppose that $a \neq 0$. We prove that $f(x)$ is continuous at a. This follows from Theorem 5.2. Since $f(x) = \frac{1}{x}$ is a (particularly simple) rational function, we have $f(x) \to f(a) = \frac{1}{a}$ as $x \to a$, provided that $a \neq 0$. Consequently, $f(x) = \frac{1}{x}$ is continuous on any interval (c, d) that does not contain the point $x = 0$.

Of course $f(x) = \frac{1}{x}$ is not continuous at $x = 0$ since it isn't even defined at $x = 0$. It is easy to show that $f(x) \to +\infty$ as $x \to 0+$. Therefore, even if we were to define $f(0)$ arbitrarily at 0, for example by setting $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, then f is not continuous at 0 because $f(x) \not\to f(0)$ as $x \to 0$.

By applying the combination rules of Theorem 5.1 from the previous section to continuous functions, we have the following result.

Theorem 5.4 (Combination rules for continuous functions)**.** Suppose that f and g are continuous functions at the point a. Then

(i) (multiple rule) if k is any constant, then kf is continuous at a,

(ii) (sum rule) $f + g$ is continuous at a,

(iii) (product rule) fg is continuous at a,

(iv) (quotient rule) if $g(a) \neq 0$ then f/g is continuous at a.

Theorem 5.2 from the previous section can be reworded as follows.

Theorem 5.5 (Continuity of rational functions)**.** If f is a polynomial then f is continuous on $\mathbb{R}$. If g is also a polynomial then the rational function $r = f/g$ is continuous at every point a for which $g(a) \neq 0$.

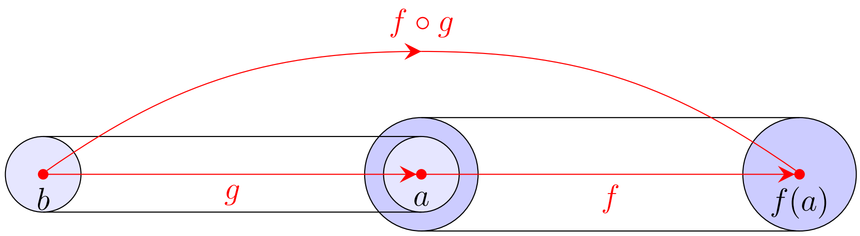
Theorem 5.3, also from the previous section, gives the following sandwich rule for continuity.

Theorem 5.6 (Sandwich rule)**.** Suppose that the functions g and h are continuous at the point a, that $g(a) = h(a)$, and that there exists $\gamma > 0$ such that $g(x) \leq f(x) \leq h(x)$ for all values of x satisfying $|x - a| < \gamma$. Then f is continuous at a. [Of course the conditions of the theorem ensure that $f(a) = g(a) = h(a)$.]

In the previous section, it was explained why there was no theorem to predict the limit of a composite function $f \circ g$ at a point b from knowledge of (i) the limit a of g at b and (ii) the limit of f at a. The good news is that, with the additional constraint of continuity, such a result can be established.

Theorem 5.7 (Composite rule for continuous functions)**.** Suppose that g is continuous at b, that $g(b) = a$, and that f is continuous at a. Then the composite function $f \circ g$ is continuous at b.
[This is often called the *function of a function* rule because $(f \circ g)(x) = f(g(x))$.]

Proof. Figure 5.19 illustrates the result with an informal description of what is happening. The precise argument is as follows.



g maps points near b to points near $g(b) = a$.
f maps points near a to points near $f(a) = f(g(b))$.
So $f \circ g$ maps points near b to points near $f(g(b)) = (f \circ g)(b)$.

Figure 5.19: Continuity of $f \circ g$

Choose $\epsilon > 0$. Since f is continuous at a, there exists $\delta > 0$ such that for any x satisfying $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. Since g is continuous at b, for this value of $\delta > 0$, there exists $\gamma > 0$ such that if $|y - b| < \gamma$ then $|g(y) - g(b)| = |g(y) - a| < \delta$ and, consequently, $|(f \circ g)(y) - f \circ g)(b)| = |f(g(y)) - f(g(b))| = |f(g(y)) - f(a)| < \epsilon$. Hence the composite function $f \circ g$ is continuous at b. $\square$

Once we have a basic repertoire of continuous functions, Theorem 5.7 is extremely useful for extending the repertoire. For example, if we knew that $\sin(x)$ was continuous on $\mathbb{R}$ then it would follow that $\sin\left(\frac{1}{1+x^2}\right)$ is also continuous on $\mathbb{R}$. [Of course $\sin(x)$ is continuous on $\mathbb{R}$, but we haven't proved that yet.]

Example 5.26. Assuming that the function $\cos(x)$ has its usual properties and is continuous on $\mathbb{R}$, prove that the function $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ is continuous on $\mathbb{R}$.

Solution. First we prove that f is continuous at any point $a \neq 0$. This follows because the rational function $g(x) = \frac{1}{x}$ is continuous at $a \neq 0$, so the composite function $h(x) = \cos\left(\frac{1}{x}\right)$ is continuous at $a \neq 0$, and by the product rule, the function $x\cos\left(\frac{1}{x}\right)$ is continuous at $a \neq 0$. It remains to prove continuity at 0.

Method (a) First Principles. Choose $\epsilon > 0$. Put $\delta = \epsilon$ and take any x satisfying $|x - 0| = |x| < \delta$. We then have $|f(x) - f(0)| = |f(x)|$. If $x = 0$ then $|f(x)| = 0 < \epsilon$, while if $x \neq 0$, $|f(x)| = |x||\cos\left(\frac{1}{x}\right)| \leq |x| < \delta = \epsilon$. So, whether or not $x = 0$, in each case $|f(x) - f(0)| < \epsilon$. Hence f is continuous at 0.

Method (b) One-sided Sandwich Rule. We have $-x \leq x\cos\left(\frac{1}{x}\right) \leq x$ for $x > 0$. Both x and $-x$ are (very simple) polynomials and so as x tends to zero (from above) both tend to their value (namely 0) at $x = 0$. It follows that $x\cos\left(\frac{1}{x}\right) \to 0$ as $x \to 0+$. A similar argument applies if $x < 0$ when $x \leq x\cos\left(\frac{1}{x}\right) \leq -x$, and this gives $x\cos\left(\frac{1}{x}\right) \to 0$ as $x \to 0-$. Consequently $f(x) \to 0 = f(0)$ as $x \to 0$, so f is continuous at 0.

We now state and prove some important results concerning continuous functions. The first of these (the Intermediate Value Theorem) roughly corresponds to the informal idea that a graph is continuous if it can be drawn without taking the pencil off the paper.

Theorem 5.8 (The Intermediate Value Theorem)**.** Suppose that f is continuous on the closed interval $[a, b]$ and that $f(a) \leq \eta \leq f(b)$. Then there exists $\xi \in [a, b]$ such that $f(\xi) = \eta$.

Proof. If $\eta = f(a)$, we can take take $\xi = a$, and if $\eta = f(b)$, we can take take $\xi = b$. By excluding these cases we can assume that $f(a) < \eta < f(b)$. On that assumption define the set S as follows.

$$S = \{x : x \in [a, b] \text{ and } f(x) < \eta\}.$$

Then S is a subset of $[a, b]$, $a \in S$ (so that $S \neq \emptyset$) and S is bounded above by b. It follows that S has a least upper bound (supremum) $\xi \leq b$. We will prove that $f(\xi) = \eta$. Figure 5.20 illustrates a function f with its associated set S, which in this case is the union of two intervals. You will see that it *looks like* $f(\xi) = \eta$, but of course we have to prove that is the case for *any* function f that satisfies the conditions of the theorem.

There are four steps in the proof: (i) proving that $\xi \neq a$, (ii) proving that $\xi \neq b$, (iii) proving that $f(\xi) \geq \eta$, and (iv) proving that $f(\xi) \leq \eta$. The argument is very similar in each step.

(i) Because f is continuous on the right at a, taking $\epsilon = \eta - f(a)$, there exists $\delta > 0$ such that if $x \in [a, a + \delta)$ then $|f(x) - f(a)| < \epsilon$. It follows that if $x \in [a, a + \delta)$ then $f(x) < f(a) + \epsilon = \eta$, and consequently $a + \delta/2 \in S$ and therefore the least upper bound of S (i.e. ξ) cannot be a.

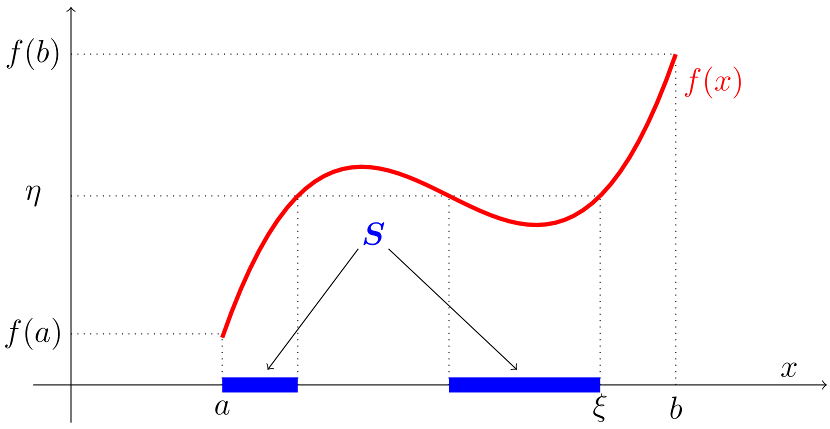


Figure 5.20: The set S

(ii) Because f is continuous on the left at b, taking $\epsilon = f(b) - \eta$, there exists $\delta > 0$ such that if $x \in (b-\delta, b]$ then $|f(x) - f(b)| < \epsilon$. It follows that if $x \in (b-\delta, b]$ then $f(x) > f(b) - \epsilon = \eta$, and consequently the least upper bound of S (i.e. ξ) cannot be greater than $b - \delta$.

It follows from (i) and (ii) that $\xi \neq a$ or b, so f is continuous at ξ.

(iii) Suppose that $f(\xi) < \eta$. Take $\epsilon = \eta - f(\xi)$. Then there exists $\delta > 0$ such that for any $x \in (\xi - \delta, \xi + \delta)$, $|f(x) - f(\xi)| < \epsilon$. It follows that if $x \in (\xi - \delta, \xi + \delta)$ then $f(x) < f(\xi) + \epsilon = \eta$, and consequently $\xi + \delta/2 \in S$ and therefore the least upper bound of S (i.e. ξ) cannot be ξ, a contradiction. Therefore we must have $f(\xi) \geq \eta$.

(iv) Finally suppose that $f(\xi) > \eta$. Take $\epsilon = f(\xi) - \eta$. Then there exists $\delta > 0$ such that for any $x \in (\xi - \delta, \xi + \delta)$, $|f(x) - f(\xi)| < \epsilon$. It follows that if $x \in (\xi - \delta, \xi]$ then $f(x) > f(\xi) - \epsilon = \eta$. But $\xi = \sup S$, so there exists $x^* \in S$ such that $x^* \in (\xi - \delta, \xi]$, and because $x^* \in S$ we have $f(x^*) < \eta$, a contradiction. Therefore we must have $f(\xi) \leq \eta$.

It follows from (iii) and (iv) that $f(\xi) = \eta$. $\square$

There is nothing special about the condition $f(a) \leq f(b)$. The result is still true if $f(a) \geq f(b)$.

Corollary 5.8.1. Suppose that f is continuous on the closed interval $[a, b]$ and that $f(a) \geq \eta \geq f(b)$. Then there exists $\xi \in [a, b]$ such that $f(\xi) = \eta$.

Proof. Just apply Theorem 5.8 to the function $-f$. $\square$

Note that the Intermediate Value Theorem relates to a *closed* interval $[a, b]$. It doesn't make sense to apply it to an open interval (a, b) because then f may not even be defined at a and at b and, even if it is, there may be discontinuities at these end points.

It is hard to overstate the importance of the Intermediate Value Theorem. It is what underpins numerical methods for solving equations. Suppose that we have an equation $f(x) = 0$, where f is a continuous function, and that we can show that $f(a) < 0$ and $f(b) > 0$, so that 0 is an intermediate value between $f(a)$ and$f(b)$. Then we can be sure that there is at least one solution for x between a and b.

Example 5.27. Prove that $f(x) = 4x^4-8x^2-x+2$ has zeros in $(-2,-1)$, $(-1, 0)$, $(0, 1)$ and $(1, 2)$.

Solution. The function f is a polynomial and so continuous on every interval. We draw up a table of values.

x	-2	-1	0	1	2
$f(x)$	36	-1	2	-3	32

Since $f(-2) = 36 > 0$ and $f(-1) = -1 < 0$, by the Intermediate Value Theorem, there must exist $x_1 \in [-2,-1]$ such that $f(x_1) = 0$ and, since $x_1 \neq -2$ or -1, we have $x_1 \in (-2,-1)$. In a similar way we deduce that there exists $x_2 \in (-1, 0)$ such that $f(x_2) = 0$, $x_3 \in (0, 1)$ such that $f(x_3) = 0$, and $x_4 \in (1, 2)$ such that $f(x_4) = 0$. [If we wish, we can get these zeros more accurately by repeatedly bisecting each of the four intervals.]

In Appendix C we show that the Completeness Axiom ensures that every positive Real Number has a positive n^{th} root for each positive integer n. The proof given there is from first principles. Here is a much easier proof using the Intermediate Value Theorem

Theorem 5.9. If n is a positive integer and $a > 0$ then there exists a number $b > 0$ such that $b^n = a$. In other words, a has a n^{th} root $b = a^{\frac{1}{n}}$.

Proof. Define $f(x) = x^n$ then f is a polynomial and so is continuous on $[0, X]$ for any $X > 0$. If $X = 1 + a$ then $f(X) = (1 + a)^n \geq 1 + na$ by the binomial theorem, so $f(X) > a$. We also have $f(0) = 0 < a$. It follows that there exists $b \in [0, 1 + a]$ such that $f(b) = a$, i.e. $b^n = a$. □

The next two results relate continuity and boundedness. We start with a definition.

Definition 5.16. Suppose that f has domain $S \subseteq \mathbb{R}$ and co-domain $\mathbb{R}$. Take the image set $f(S) = \{y : y = f(x) \text{ for some } x \in S\}$. If $f(S)$ is bounded above then we say that f is bounded above on S. If $f(S)$ is bounded below then we say that f is bounded below on S. If f is bounded above and below on S, then we say that f is bounded on S. We use the following notation.

$$\sup_{x \in S} f = \sup\{y : y = f(x) \text{ for some } x \in S\},$$

$$\inf_{x \in S} f = \inf\{y : y = f(x) \text{ for some } x \in S\}.$$

In an earlier chapter we proved that a convergent sequence is bounded. Here we prove that a continuous function *on a closed interval* is bounded.

Theorem 5.10 (Boundedness of a continuous function on a closed interval)**.** Suppose that f is continuous on the closed interval $[a, b]$. Then f is bounded on $[a, b]$.

Proof. The idea of this proof (there are other proofs) is to creep across the interval from a to b. With this in mind we define the set S.

$$S = \{x : x \in [a, b] \text{ and } f \text{ is bounded on } [a, x]\}.$$

Then $S \subseteq [a, b]$ and S is non-empty since $a \in S$. So S has a least upper bound (supremum) $\xi \in [a, b]$. The aim is to prove that $\xi = b$. First we prove that $\xi \neq a$.

Since f is continuous on the right at a, taking $\epsilon = 1$, we find that there exists $\delta > 0$ such that if $x \in [a, a + \delta)$ then $|f(x) - f(a)| < 1$, which implies that $|f(x)| < |f(a)| + 1$. So f is certainly bounded (by $|f(a)| + 1$) on the interval $[a, a + \delta/2]$, which implies that $(a + \delta/2) \in S$, and so $\xi \geq a + \delta/2$. Hence $\xi \neq a$.

Now suppose that $\xi < b$. Then f is continuous at ξ. Again taking $\epsilon = 1$, we find that there exists $\delta > 0$ such that if $x \in (\xi - \delta, \xi + \delta)$ then $|f(x) - f(\xi)| < 1$, which implies that $|f(x)| < |f(\xi)| + 1$. So f is certainly bounded on the interval $[\xi - \delta/2, \xi + \delta/2]$. But $\xi = \sup S$, so f is bounded on the interval $[a, \xi - \delta/2]$. Putting the two intervals together, we find that f is bounded on $[a, \xi + \delta/2]$, so $\xi + \delta/2 \in S$, which contradicts the fact that $\xi = \sup S$. It follows that $\xi = b$, and we are almost finished (but not quite).

Since f is continuous on the left at b, taking $\epsilon = 1$, we find that there exists $\delta > 0$ such that if $x \in (b - \delta, b]$ then $|f(x) - f(b)| < 1$, which implies that $|f(x)| < |f(b)| + 1$. So f is certainly bounded on the interval $[b - \delta/2, b]$. But we now know that $\sup S = b$, so f is bounded on the interval $[a, b - \delta/2]$. Putting the two intervals together, we find that f is bounded on $[a, b]$, which completes the proof.

□

Theorem 5.10 applies to a *closed* interval $[a, b]$. The function $\frac{1}{x}$ is continuous on the *open* interval $(0, 1)$ but it certainly is not bounded on this interval since $\frac{1}{x} \to +\infty$ as $x \to 0+$. So we cannot replace the closed interval $[a, b]$ in the theorem by the open interval (a, b).

Not only is a continuous function on a closed interval necessarily bounded, but it actually achieves its bounds, i.e. its supremum is actually a maximum value, and its infimum is actually a minimum value.

Theorem 5.11 (Attainment of bounds)**.** Suppose that f is continuous on $[a, b]$. Put $M = \sup_{x\in[a,b]} f$ and $m = \inf_{x\in[a,b]} f$. Then there exist ξ and ζ in $[a, b]$ such that $f(\xi) = M$ and $f(\zeta) = m$.

Proof. We prove the existence of ξ, the proof for ζ is similar. The proof depends on a cunning trick involving the previous theorem. We know that $M = \sup_{x\in[a,b]} f$, so $f(x) \le M$ for all $x \in [a, b]$. But suppose that $f(x) < M$ for all $x \in [a, b]$. Then $M - f(x) > 0$ for all $x \in [a, b]$ and the function

$$g(x) = \frac{1}{M - f(x)}$$

is continuous on $[a, b]$ by Theorem 5.4. Hence, by the previous theorem (Theorem 5.10), g is bounded on $[a, b]$ and consequently there exists some $K \in \mathbb{R}$ such that $g(x) < K$ for all $x \in [a, b]$. This gives

$$\begin{aligned} \frac{1}{M - f(x)} &< K, \quad \text{so} \\ \frac{1}{K} &< M - f(x), \quad \text{hence} \\ f(x) &< M - \frac{1}{K} \quad \text{for all } x \in [a, b]. \end{aligned}$$

But this contradicts the fact that $M = \sup_{x\in[a,b]} f$. It follows that there must exist some point $\xi \in [a, b]$ such that $f(\xi) = M$. □

Once again the theorem relates to a *closed* interval $[a, b]$. It does not hold for an open interval. For example, if $f(x) = x$ then $\sup_{x\in(0,1)} f = 1$ and $\inf_{x\in(0,1)} f = 0$. But for $x \in (0, 1)$ we have $0 < f(x) < 1$, so f does not attain its bounds on the open interval $(0, 1)$.

The following is a simple consequence of Theorem 5.11.

Corollary 5.11.1. Suppose that f is continuous on $[a, b]$. Then the image set is the closed interval $[m.M]$, where m, M are the greatest lower bound (infimum) and least upper bound (supremum) of f on $[a, b]$.

Our final theorem in this chapter is the Inverse Function Theorem (part 1). This gives conditions under which the inverse of a continuous function is itself continuous. Part 2 of the theorem relates to differentiability and we will cover this in the next chapter. First we need an easy definition.

Definition 5.17. Suppose that f is defined on an interval I (which may be open or closed) and has co-domain $\mathbb{R}$. If, for every $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) \leq f(x_2)$, then we say that f is monotonically increasing on the interval I. If $\leq$ can be replaced by $<$, then we say that f is strictly increasing on I. In the same way we define monotonically (or strictly) decreasing by the requirement that $f(x_1) \geq f(x_2)$ (or $f(x_1) > f(x_2)$) for every $x_1, x_2 \in I$ with $x_1 < x_2$.

Theorem 5.12 (The Inverse Function Theorem for continuous functions)**.**
Suppose that f is continuous and strictly increasing on the closed interval $[a, b]$. Put $\alpha = f(a),\ \beta = f(b)$. Then f has an inverse function g that is continuous and strictly increasing on $[\alpha, \beta]$ with the properties that $g(f(x)) = x$ for every $x \in [a, b]$ and $f(g(y)) = y$ for every $y \in [\alpha, \beta]$.
[Of course the inverse function is usually denoted as f^{-1}, but g is used here in the proof to simplify the notation. Figure 5.6 provides an illustration of a continuous and strictly increasing function along with its inverse.]

Proof. The function $f : [a, b] \to [\alpha, \beta]$ is injective (one-one) because f is strictly increasing. By the Intermediate Value Theorem (Theorem 5.8) if $\gamma \in [\alpha, \beta]$ then there exists $c \in [a, b]$ such that $f(c) = \gamma$, so f is surjective (onto). Hence f is a bijective function and so possesses an inverse function g such that $g(f(x)) = x$ for every $x \in [a, b]$ and $f(g(y)) = y$ for every $y \in [\alpha, \beta]$.

Next we show that g is strictly increasing on $[\alpha, \beta]$. To do this, take $y_1, y_2 \in [\alpha, \beta]$ with $y_1 < y_2$. Then there exist $x_1, x_2 \in [a, b]$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. We cannot have $x_1 = x_2$ since $y_1 \neq y_2$, so either $x_1 < x_2$ or vice-versa. But if $x_2 < x_1$ then $y_2 = f(x_2) < f(x_1) = y_1$, a contradiction. Hence if $y_1 < y_2$ we must have $x_1 < x_2$, i.e. $g(y_1) < g(y_2)$. So g is strictly increasing on $[\alpha, \beta]$.

Finally we show that g is continuous on $[\alpha, \beta]$. To do this take any point $\gamma \in [\alpha, \beta]$ and choose $\epsilon > 0$. Initially we will assume that $\gamma \neq \alpha$ or β, so that $c = g(\gamma) \neq a$ or b.

Take $\epsilon^* \leq \epsilon$ such that $(c - \epsilon^*, c + \epsilon^*) \subseteq (a, b)$. Then put $\gamma_1 = f(c - \epsilon^*)$ and $\gamma_2 = f(c + \epsilon^*)$. Because f is strictly increasing and $f(c) = \gamma$, we have $\gamma_1 < \gamma < \gamma_2$. Put $\delta = \min(\gamma - \gamma_1, \gamma_2 - \gamma)$. Then if $|y - \gamma| < \delta$, we have $\gamma_1 < y < \gamma_2$ and consequently $g(\gamma_1) < g(y) < g(\gamma_2)$, i.e. $c - \epsilon^* < g(y) < c + \epsilon^*$, which gives $|g(y) - c| < \epsilon^* \leq \epsilon$. But $c = g(\gamma)$, so we have $|g(y) - g(\gamma)| < \epsilon$ if $|y - \gamma| < \delta$. Hence g is continuous at γ.

To complete the proof we need to show that g is continuous on the right at α and on the left at β. To deal with continuity on the right at α, take $\epsilon^* \leq \epsilon$ such that $a + \epsilon^* < b$. Then put $\alpha^* = f(a + \epsilon^*)$. Because f is strictly increasing and $f(a) = \alpha$, we have $\alpha < \alpha^*$. Put $\delta = \alpha^* - \alpha$. Then if $\alpha \leq y < \alpha + \delta$, we have $\alpha \leq y < \alpha^*$ and consequently $g(\alpha) \leq g(y) < g(\alpha^*)$. This gives $0 \leq g(y) - g(\alpha) < g(\alpha^*) - g(\alpha) = (a + \epsilon^*) - a = \epsilon^* < \epsilon$. Hence $|g(y) - g(\alpha)| < \epsilon$ if $\alpha \leq y < \alpha + \delta$. Therefore g is continuous on the right at α.

Continuity of g on the left at β may be established in a similar fashion.

□

Although we have stated the result for a strictly *increasing* function f, it remains true if we replace "increasing" by "decreasing" throughout.

Corollary 5.12.1. Suppose that f is continuous and strictly decreasing on the closed interval $[a, b]$. Put $\alpha = f(a)$, $\beta = f(b)$. Then f has an inverse function g that is continuous and strictly decreasing on $[\beta, \alpha]$ with the properties that $g(f(x)) = x$ for every $x \in [a, b]$ and $f(g(y)) = y$ for every $y \in [\beta, \alpha]$.

Proof. Note that $\beta < \alpha$ in this case. Apply Theorem 5.12 to the function $h = -f$ with image set $[-\alpha, -\beta]$. If k is the inverse of h as guaranteed by that theorem, then g (the inverse of f) is given by $g(y) = k(-y)$. □

Corollary 5.12.2. Suppose that n is a positive integer. Then the function $g(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is continuous on $[0, \infty)$.

Proof. The function $f(x) = x^n$ is continuous and strictly increasing on $\mathbb{R}$ and so certainly continuous and strictly increasing on any closed interval $[0, b]$. The function g is the inverse of f and so continuous on $[f(0), f(b)] = [0, b^n]$. Since $b > 0$ is arbitrary, we deduce that g is continuous on $[0, \infty)$. □

It's hardly worth stating separately that, in consequence of the previous corollary, if r is a rational number then the function x^r is continuous at every point $c > 0$ (and also continuous on the right at 0 provided that $r > 0$). If $r = m/n$ with m, n integers and $n > 0$ then $x^r = (x^{\frac{1}{n}})^m$ to which we may apply the previous corollary and either the combination rules or the composite rule.

EXERCISES 5.4

1. Suppose f is continuous at a and that (x_n) is a sequence such that $x_n \to a$ as $n \to \infty$. Prove that the sequence $(f(x_n))$ converges to $f(a)$ (i.e. $f(x_n) \to f(a)$ as $n \to \infty$).

2. Suppose that the function f has the property that for *every* sequence (x_n) that converges to a, the sequence $(f(x_n))$ converges to $f(a)$. Prove that f must be continuous at a. [Hint: Assume that f is not continuous at a and use this assumption to construct a sequence (x_n) that converges to a but for which the sequence $(f(x_n))$ does not converge to $f(a)$.]

Remark. Questions 1 and 2 provide an alternative definition of continuity at a point:

f is continuous at a if and only if, for every sequence (x_n) that converges to a, the corresponding sequence $(f(x_n))$ converges to $f(a)$.

Something similar is possible for the limit of a function f at a point a. However one has then to specify that the sequences do not contain the value a because $f(a)$ may not be defined and, even if it is, $f(a)$ may not be the limiting value.

3. Prove that if $f(x) = x^4 - 4x^3 - 2x^2 + 10x + 3$ then the equation $f(x) = 0$ has solutions in each of the intervals $(-2, -1), (-1, 0), (1, 2)$ and $(3, 4)$. By repeatedly bisecting the interval $(-1, 0)$ find a subinterval of length $\frac{1}{64}$ that contains a solution.

4. For each of the following functions determine where it is continuous and the image set. Justify your answers. [There is no shame in sketching the graph to find the answer before proving it is the answer.]

a) $2x^2 + 5x - 3$, b) $\frac{x^2+1}{x^2-1}$,

c) $\sqrt{x^2 - 1}$, d) $\frac{1}{\sqrt{x^2-1}}$.

5. Suppose that $f(x)$ is a *monic* polynomial of degree n, i.e. it has leading coefficient 1, so that it has the form $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$. If n is odd prove that $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Deduce that the equation $f(x) = 0$ has at least one solution.
[Hint: Put $M = \max(|a_i|, i = 0, 1, \ldots, n-1)$ so that if $|x| \geq 1$,

$$|a_{n-1}x^{n-1} + \ldots + a_1x + a_0| \leq nM|x|^{n-1},$$

which gives $x^n - nM|x|^{n-1} \leq f(x) \leq x^n + nM|x|^{n-1}$.]

Remark. The *Fundamental Theorem of Algebra* asserts that a monic polynomial $f(z)$ of degree n with coefficients in the field of Complex Numbers $\mathbb{C}$ can be factored into n monic linear factors of the form $(z - \alpha_i)$ for $i = 1, 2, \ldots, n$, where each $\alpha_i \in \mathbb{C}$. Hence the equation $f(z) = 0$ has n roots (although some may be equal to others). The proof requires a course in Complex Analysis.

6. The function f has domain $\{x : x > 0\}$ and it has the following properties: (a) $f(x_1) + f(x_2) = f(x_1x_2)$ for all $x_1, x_2 > 0$, (b) $f(2) = 1$, and (c) $f(x)$ is continuous at $x = 1$.
Prove that $f(1) = 0$, $f(2^r) = r$ for each rational number r, and that f is continuous on $(0, \infty)$.

Chapter 6

Differentiability

6.1 Background and definition

In this chapter we consider what it means to say that a function is differentiable, rules for calculating derivatives, and theoretical consequences. I assume that you have already learned something about differentiation and that you have seen the connection with gradients of graphs and rates of change. Looking at a graph and discussing its gradient at a given point is usually how differentiation is introduced to students. We start therefore with a brief look at this informal approach.

Consider the graph shown in Figure 6.1. The aim is to find the gradient of the graph $y = f(x)$ at the point P, where $x = a$. To do this we first consider a neighbouring value of x, namely $x = a + h$. The quantity h is considered small, but it can be positive or negative. I can't show both on one diagram, so I show the case when h is positive. The triangle shown PQR is right-angled and the slope or gradient of the hypotenuse PQ is the vertical height $f(a+h)-f(a)$ divided by the length of the base, namely h. So PQ has gradient $\dfrac{f(a+h)-f(a)}{h}$. This remains true whether or not h is positive or negative, and whether or not P is below, level with, or above Q. The key idea is that as the size of h is reduced towards zero. the gradient of PQ approaches the gradient of the graph (i.e. the tangent to the graph) at the point P. You can envisage Q sliding along the curve towards P. We are therefore considering the limiting value (if any) of $\dfrac{f(a+h)-f(a)}{h}$ as h tends to zero. Of course you must not actually put $h = 0$ into this formula because division by zero is undefined. If the limit exists, we denote it by $f'(a)$ and call this the derivative of f at a.

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.}$$

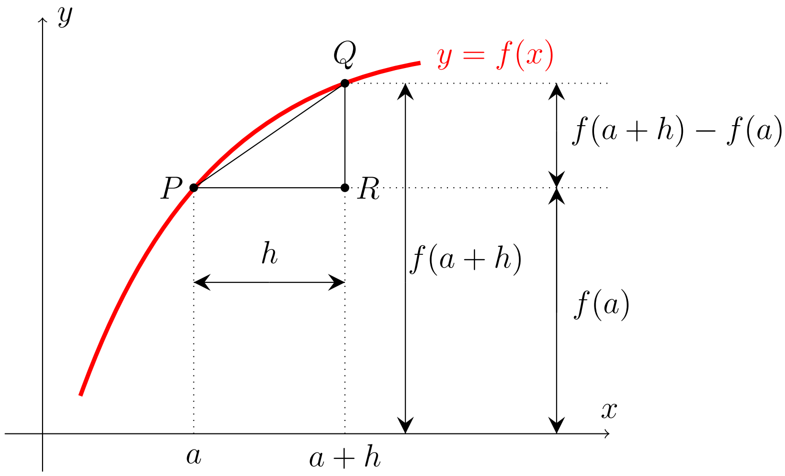


Figure 6.1: Calculating a gradient

Terminology may be different. If we replace h by δx (read this as a composite symbol meaning the change in the x-coordinate) and replace $f(a+h) - f(a)$ by δy (read this as a composite symbol meaning the change in the y-coordinate), then we can express the gradient of the line PQ as the ratio $\dfrac{\delta y}{\delta x}$. Of course δx and δy are composite symbols and there is no question of "cancelling" the δs. In this terminology we would hope that there is a limiting value of $\dfrac{\delta y}{\delta x}$ as δx tends to zero. Of course you must not actually put $\delta x = 0$. If the limit does exist it can be denoted by $\dfrac{dy}{dx}$ (or strictly speaking by $\left.\dfrac{dy}{dx}\right|_{x=a}$ since we are calculating the gradient at the point $x = a$). It is very important to note that $\dfrac{dy}{dx}$ is a single composite symbol; you must not treat it as a fraction, dy divided by dx, nor can you "cancel" the ds. Despite this warning, the notation has several practical advantages. In certain situations it does behave like a fraction, as we will see.

The two alternative notations for a derivative trace their origins back to Isaac Newton ($f'(a)$) and Gottfried Leibniz $\left(\dfrac{dy}{dx}\right)$, the principal inventors of Calculus. We will use either notation as and when convenient. Each has its own advantages. Time for a formal definition.

Definition 6.1. Suppose that f is a function defined on an open interval containing the point a. If there exists $l \in \mathbb{R}$ such that

$$\frac{f(a+h) - f(a)}{h} \to l \text{ as } h \to 0,$$

then we say that f is *differentiable* at the point a, and we denote the limiting value

l by $f'(a)$. This value is called the *derivative* of f at a. If we allow ourselves to write $y = f(x)$ then an alternative notation for the derivative is $\left.\dfrac{dy}{dx}\right|_{x=a}$.

The definition may be expressed in an alternative form. If we write x for $a+h$, so that $h = x - a$, then the limit l satisfies

$$\frac{f(x) - f(a)}{x - a} \to l \text{ as } x \to a.$$

Example 6.1. Prove that the function f defined by $f(x) = x^2$ is differentiable at $x = 3$ with derivative 6.

Solution. For $h \neq 0$ we have

$$\frac{f(3+h) - f(3)}{h} = \frac{(3+h)^2 - 3^2}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h} = 6 + h.$$

But $6 + h \to 6$ as $h \to 0$, so

$$\frac{f(3+h) - f(3)}{h} \to 6 \text{ as } h \to 0.$$

Hence the function $f(x) = x^2$ is differentiable at 3 with derivative 6.

As with continuity, we may look at one-sided definitions.

Definition 6.2. Suppose that f is a function defined on an interval $[a, b)$. If there exists $l \in \mathbb{R}$ such that

$$\frac{f(a+h) - f(a)}{h} \to l \text{ as } h \to 0+,$$

then we say that f is differentiable *on the right* at a and we denote the limiting value l by $f'(a+)$. This value is called the *right-derivative* of f at a.

Suppose that g is a function defined on an interval $(c, a]$. If there exists $l \in \mathbb{R}$ such that

$$\frac{g(a+h) - g(a)}{h} \to l \text{ as } h \to 0-,$$

then we say that g is differentiable *on the left* at a and we denote the limiting value l by $g'(a-)$. This value is called the *left-derivative* of g at a.

The function f is differentiable at a if and only if it is differentiable on both the left and the right at a, and the left and right derivatives are equal. The common value of these derivatives is then $f'(a)$.

Example 6.2. Show that $f(x) = |x|$ has left and right derivatives at $x = 0$, but these are unequal and so f is not differentiable at $x = 0$.

Solution. For $h > 0$ we have

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = 1 \to 1 \text{ as } h \to 0+ .$$

Hence the function has a right-derivative $f'(0+) = 1$. For $h < 0$ we have

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = -1 \to -1 \text{ as } h \to 0- .$$

Hence the function has a left-derivative $f'(0-) = -1$. Since the left and right derivatives are unequal, f is not differentiable at $x = 0$.

In geometric terms. the gradient on the right at 0 is $+1$ (45^o), while that on the left is -1 (-45^o). There is no well-defined tangent to the graph at $x = 0$. See Figure 6.2.

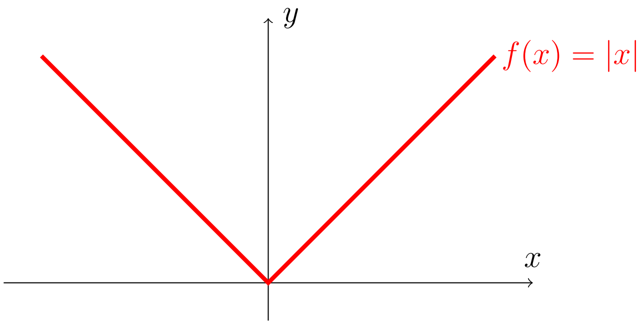


Figure 6.2: $f(x) = |x|$

As with continuity, interest tends to focus on intervals, rather than just the property at individual points. So we make the following definitions.

Definition 6.3. If the function f is differentiable at each point $x \in (a, b)$ then we say that f is differentiable on (the open interval) (a, b). The function defined by the values of the derivative is denoted by f', or by $\frac{df}{dx}$, and is called the *derived function*. If we allow ourselves to write $y = f(x)$ then f' may also be denoted by $\frac{dy}{dx}$.

Differentiability of f on a closed interval $[a, b]$ is taken to mean differentiability on the open interval (a, b), plus the existence of one-sided derivatives at the end points. Intervals $[a, b)$ and $(a, b]$ are dealt with in the same way.

If f is differentiable on $\mathbb{R}$, then we may simply say that f is differentiable.

Example 6.3. Prove that the function f defined by $f(x) = x^2$ is differentiable at each point $x \in \mathbb{R}$ with derivative $2x$.

Solution. For $h \neq 0$ we have

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \frac{2xh + h^2}{h} = 2x + h.$$

But $2x + h \to 2x$ as $h \to 0$, so

$$\frac{f(x+h)-f(x)}{h} \to 2x \text{ as } h \to 0.$$

Hence the function $f(x) = x^2$ is differentiable at x with derivative $2x$.

Exercises for Section 6.1

1.
   a) For $f(x) = \frac{1}{x}$ calculate the values of $(f(2+h) - f(2))/h$ for $h = \pm 0.1, \pm 0.01, \pm 0.001$. Compare your answers to $-\frac{1}{4}$.

   b) Repeat part a) for $f(x) = \sin(x)$ and compare the resulting answers with $\cos(2)$. [Use a calculator with angles set to radians.]

2. Prove that $f(x) = \frac{1}{x}$ is differentiable at every point x, except for $x = 0$, and give the value of the derivative at x.

3. For the function $f(x) = x - \lfloor x \rfloor$ determine whether or not the one-sided derivatives exist at $x = 1$. In case of existence, determine the one-sided derivative.

6.2 Basic results on differentiation

It seems unlikely that a function with a discontinuity at $x = a$ can have a well-defined gradient (tangent to its graph) at the point $x = a$. This expectation is bourne out in the following Theorem.

Theorem 6.1. If f is differentiable at a, then f is continuous at a.

Proof. For $h \neq 0$ we have

$$f(a+h) - f(a) = h \times \frac{f(a+h) - f(a)}{h} \to 0 \times f'(a) = 0 \text{ as } h \to 0.$$

Hence $f(a+h) \to f(a)$ as $h \to 0$, i.e. f is continuous at a. $\square$

Note. It might be thought that the converse is also true. That this is not the case is shown by the example of $f(x) = |x|$ at $x = 0$ (see Example 6.2), where we have continuity without differentiability. However, it was thought for a long time that a function which is everywhere continuous must (in some sense) be differentiable almost everywhere. This conjecture was eventually proved false by Karl Weierstrass in 1872 who constructed a function that was continuous everywhere but differentiable nowhere - a sort of spiky-everywhere function. Weierstrass's function and others like it show that informal arguments based on sketching Cartesian graphs do not form a sound basis for a rigorous treatment of Calculus. What is required are precise definitions and proofs based on logical deduction principles.

We have seen in examples and exercises above how to calculate derivatives of simple functions such as x^2 and $\frac{1}{x}$. To make faster progress we need rules for calculating derivatives of functions. These are given in the following few theorems. We start with the combination rules for multiples, sums, products and quotients.

Theorem 6.2 (Combination rules for differentiation)**.** Suppose that f and g are functions differentiable at the point a. Then

(i) (multiple rule) if k is any constant, then kf is differentiable at a with derivative $kf'(a)$,

(ii) (sum rule) $f + g$ is differentiable at a with derivative $f'(a) + g'(a)$,

(iii) (product rule) fg is differentiable at a with derivative $f(a)g'(a)+g(a)f'(a)$,

(iv) (quotient rule) if $g(a) \neq 0$ then f/g is differentiable at a with derivative $\dfrac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}$. [Easiest to remember in words: "bottom times the derivative of the top minus top times the derivative of the bottom, all over the bottom squared".]

Proof. (i)

$$\frac{kf(a+h) - kf(a)}{h} = k\frac{f(a+h) - f(a)}{h} \to kf'(a) \text{ as } h \to 0.$$

(ii)

$$\begin{aligned}\frac{(f(a+h) + g(a+h)) - (f(a) + g(a))}{h} &= \frac{f(a+h) - f(a)}{h} \\ &\quad + \frac{g(a+h) - g(a)}{h} \\ &\to f'(a) + g'(a) \text{ as } h \to 0.\end{aligned}$$

(iii)

$$
\begin{aligned}
&\frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\
&= \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\
&= f(a+h)\frac{g(a+h) - g(a)}{h} + g(a)\frac{f(a+h) - f(a)}{h} \\
&\to f(a)g'(a) + g(a)f'(a) \text{ as } h \to 0 \text{ (noting that } f \text{ is continuous at } a).
\end{aligned}
$$

(iv) We first establish the result for the function $\frac{1}{g(x)}$ and then use the product rule. We have

$$
\begin{aligned}
\frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} &= \frac{g(a) - g(a+h)}{hg(a+h)g(a)} \\
&= \frac{-1}{g(a+h)g(a)} \times \frac{g(a+h) - g(a)}{h} \\
&\to \frac{-1}{(g(a))^2}g'(a) \text{ as } h \to 0 \text{ (noting that } g \text{ is continuous at } a).
\end{aligned}
$$

Now consider $\dfrac{f(x)}{g(x)} = f(x) \times \dfrac{1}{g(x)}$. Using the product rule we find that the derivative of $f(x)/g(x)$ equals

$$
\begin{aligned}
f(a) \times \frac{-1}{(g(a))^2}g'(a) + \frac{1}{g(a)} \times f'(a) &= \frac{g(a)f'(a)}{(g(a))^2} - \frac{f(a)g'(a)}{(g(a))^2} \\
&= \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}.
\end{aligned}
$$

□

If we write $u = f(x)$ and $v = g(x)$ then we can express the composites rules in other notations as follows:

(i) for $y = ku$, $\frac{dy}{dx} = k\frac{du}{dx}$, or $y' = ku'$,

(ii) for $y = u + v$, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$, or $y' = u' + v'$,

(iii) for $y = uv$, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, or $y' = uv' + vu'$,

(iv) for $y = u/v$, $\frac{dy}{dx} = (v\frac{du}{dx} - u\frac{dv}{dx})/v^2$, or $y' = (vu' - uv')/v^2$.

Theorem 6.3 (Composite, function-of-a-function, or chain rule)**.**
Suppose that g is differentiable at b, that $g(b) = a$, and that f is differentiable at a. Then the composite function $f \circ g$ is differentiable at b with derivative $(f \circ g)'(b) = f'(a)g'(b)$.
[If we write $y = f(z)$ and $z = g(x)$ then $y = (f \circ g)(x) = f(g(x))$. The result states that $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$, with the derivatives on the right hand side evaluated at the corresponding points a and b respectively. This is easy to remember because it *looks like* the dz terms cancel out, although this is not what is happening since both $\frac{dy}{dz}$ and $\frac{dz}{dx}$ are single composite symbols.]

Proof. Rather unusually I start with an incorrect proof. I will explain what is wrong with it. Then I will correct it and give a proper proof. So here's what you might try in the first instance, with $h \neq 0$ assumed.

$$\begin{aligned}\frac{(f \circ g)(b+h) - (f \circ g)(b)}{h} &= \frac{f(g(b+h)) - f(g(b))}{h} \\ &= \frac{f(g(b+h)) - f(g(b))}{g(b+h) - g(b)} \times \frac{g(b+h) - g(b)}{h} \qquad (6.1)\end{aligned}$$

We have $\dfrac{g(b+h) - g(b)}{h} \to g'(b)$ as $h \to 0$. Since g is continuous at b, we also have $g(b+h) - g(b) \to 0$ as $h \to 0$. So if we define $k(h) = g(b+h) - g(b)$ then $k(h) \to 0$ as $h \to 0$. But

$$f(g(b+h)) - f(g(b)) = f(g(b) + k(h)) - f(g(b)) = f(a + k(h)) - f(a),$$

and so

$$\frac{f(g(b+h)) - f(g(b))}{g(b+h) - g(b)} = \frac{f(a + k(h)) - f(a)}{k(h)} \to f'(a) \text{ as } h \to 0.$$

Consequently

$$\frac{(f \circ g)(b+h) - (f \circ g)(b)}{h} \to f'(a)g'(b) \text{ as } h \to 0.$$

The error lies in equation 6.1 in red above. It is possible for $g(b+h)$ to equal $g(b)$ for values of h other than $h = 0$. In such cases we must avoid dividing by $k(h) = g(b+h) - g(b)$. The difficulty lies in the fact that $\dfrac{f(a+k) - f(a)}{k}$ is undefined for $k = 0$. However it is true that $\dfrac{f(a+k) - f(a)}{k} \to f'(a)$ as $k \to 0$. So we start a correct proof by defining a new function ϕ that coincides with $\dfrac{f(a+k) - f(a)}{k}$ if $k \neq 0$, but is also defined with a "natural" value at $k = 0$.

$$\text{Define } \phi(k) = \begin{cases} \dfrac{f(a+k) - f(a)}{k} & \text{if } k \neq 0 \\ \\ f'(a) & \text{if } k = 0 \end{cases}$$

Then ϕ is continuous at 0 because $[f(a+k) - f(a)]/k \to f'(a)$ as $k \to 0$.

With $k(h) = g(b+h) - g(b)$ we have

$$\begin{aligned} \frac{(f \circ g)(b+h) - (f \circ g)(b)}{h} &= \frac{f(g(b+h)) - f(g(b))}{h} \\ &= \frac{f(g(b) + k(h))) - f(g(b))}{h} \\ &= \frac{f(a + k(h)) - f(a)}{h} \\ &= \phi(k(h)) \times \frac{k(h)}{h} \\ &= \phi(k(h)) \times \frac{g(b+h) - g(b)}{h} \qquad (6.2) \end{aligned}$$

Equation 6.2 is correct in all cases. If $k(h) = g(b+h) - g(b) \neq 0$ then it is the same as equation 6.1. But if $k(h) = g(b+h) - g(b) = 0$ then it is correct because both sides are zero. [Think about this!]

Since g is continuous at b we have $k(h) = g(b+h) - g(b) \to 0$ as $h \to 0$. Noting that ϕ is continuous at 0, equation 6.2 gives

$$\begin{aligned} \frac{(f \circ g)(b+h) - (f \circ g)(b)}{h} &= \phi(k(h)) \times \frac{g(b+h) - g(b)}{h} \\ &\to \phi(0) g'(b) \text{ as } h \to 0, \end{aligned}$$

and $\phi(0)g'(b) = f'(a)g'(b)$. □

The theorem can be applied to a function of a function of a function (etc). For example if $y = f(t)$, $t = g(u)$, and $u = h(x)$ then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{du}\frac{du}{dx}$. It is this that gives rise to the name "chain rule".

Equipped with the previous theorems, we can show that a wide variety of functions are differentiable. We start with integer powers.

Theorem 6.4.
If n is a non-negative integer then $f_n(x) = x^n$ is differentiable everywhere. If n is a negative integer then $f_n(x) = x^n$ is differentiable everywhere except at $x = 0$. In each case the derivative is nx^{n-1} (0 if $n = 0$).

Proof. Suppose first that $n = 0$. Choose $a \in \mathbb{R}$ and for $h \neq 0$ consider $[f_0(a+h) - f_0(a)]/h = [1-1]/h = 0 \to 0$ as $h \to 0$.

Next consider the case $n = 1$. Choose $a \in \mathbb{R}$ and for $h \neq 0$ consider $[f_1(a+h) - f_1(a)]/h = [a+h-a]/h = 1 \to 1$ as $h \to 0$.

Now suppose inductively that for some $k \geq 1$, $f_k(x) = x^k$ is differentiable at a with derivative ka^{k-1}. Then by the product rule $f_{k+1}(x) = f_k(x)f_1(x)$ is differentiable at a with derivative $f_k(a)f_1'(a) + f_1(a)f_k'(a) = a^k + aka^{k-1} = (k+1)a^k$. It follows by induction that $f_n(x)$ is differentiable at each point $a \in \mathbb{R}$ with derivative na^{n-1}.

To deal with $n < 0$ we use the quotient rule. So suppose $n < 0$ and put $m = -n$. Then by the quotient rule $f_n(x) = \dfrac{1}{x^m}$ is differentiable everywhere except at $x = 0$ with derivative

$$\frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1}. \qquad \square$$

Corollary 6.4.1. If $f(x)$ is a polynomial, then f is differentiable on $\mathbb{R}$. If $g(x)$ is a polynomial then the rational function $f(x)/g(x)$ is differentiable everywhere on $\mathbb{R}$ except for those points x (if any) where $g(x) = 0$.

Proof. The result follows from Theorem 6.4 by repeated application of the multiple, sum and quotient rules. $\square$

Theorem 6.5 (The Inverse Function Theorem for differentiable functions)**.**
Suppose that f is differentiable and strictly increasing on the closed interval $[a, b]$. If $\xi \in [a, b]$ and $f'(\xi) \neq 0$ then the inverse function g is differentiable at $\eta = f(\xi)$ with derivative $g'(\eta) = \dfrac{1}{f'(\xi)}$.
[The inverse function is usually denoted as f^{-1}, but g is used here in the proof to simplify the notation. If $\xi = a$ or b, then the derivatives are one-sided. You are strongly advised to look back to Theorem 5.12 in the previous chapter, the inverse function theorem for continuous functions.]

Proof. As in Theorem 5.12, we define $\alpha = f(a)$ and $\beta = f(b)$. By that previous theorem, g is continuous and strictly increasing on $[\alpha, \beta]$. Initially, take $\xi \in (a, b)$ and let $\eta = f(\xi)$, so that $\eta \in (\alpha, \beta)$ and $g(\eta) = \xi$. We suppose that $f'(\xi) \neq 0$. Figure 6.3 provides an illustration.

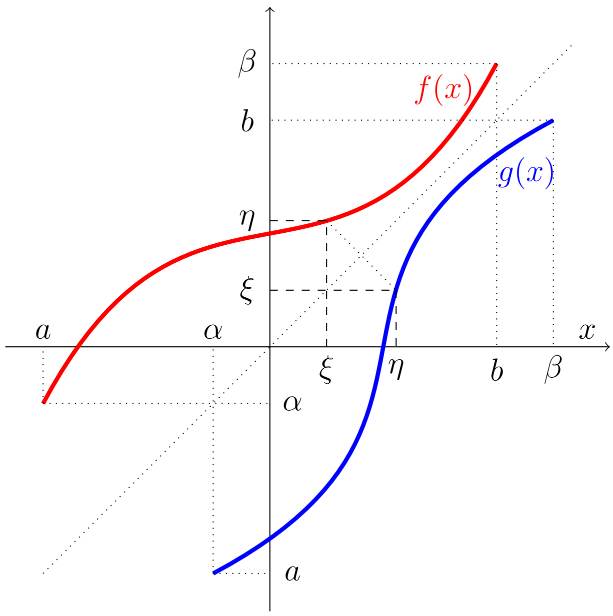


Figure 6.3: Inverse functions and differentiability

For $k \neq 0$ and such that $\eta + k \in (\alpha, \beta)$ define h by the equation

$$
\begin{aligned}
\eta + k &= f(\xi + h), \quad \text{in other words} \\
g(\eta + k) &= \xi + h \quad \text{i.e.} \\
h &= g(\eta + k) - \xi \\
&= g(\eta + k) - g(\eta)
\end{aligned}
$$

Figure 6.4 shows h and k and their role in determining the derivative of f at ξ, and the derivative of g at η. You are advised to look at Figure 6.4 before reading the next paragraph.

Note that h is uniquely determined by k (we could write $h = h(k)$ to emphasise that h is a function of k). Since $k \neq 0$ and g is strictly increasing, it follows that $h \neq 0$. Moreover, since g is continuous at η, we have $h = g(\eta+k) - g(\eta) \to 0$ as $k \to 0$ and

$$\frac{g(\eta + k) - g(\eta)}{k} = \frac{h}{k} = \frac{h}{f(\xi + h) - f(\xi)}.$$

As k tends to zero, h also tends to zero and so the right hand side of the expression tends to $\dfrac{1}{f'(\xi)}$. Hence

$$\frac{g(\eta + k) - g(\eta)}{k} \to \frac{1}{f'(\xi)} \text{ as } k \to 0.$$

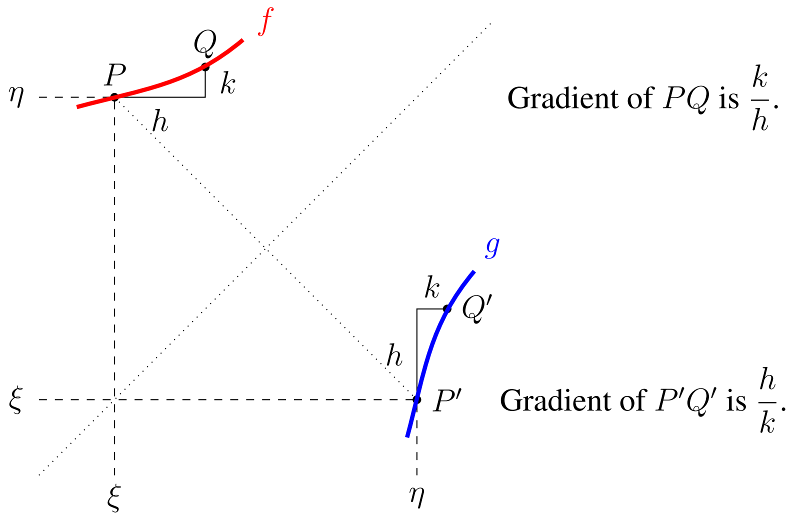


Figure 6.4: A close-up view of h and k

Consequently g is differentiable at η and $g'(\eta) = \dfrac{1}{f'(\xi)}$.

If ξ is one of the end points a or b then η is a corresponding end point, α or β. The same argument as above applies to the one sided derivatives, taking $k > 0$ or $k < 0$ as appropriate. $\square$

Although we have stated the result for a strictly *increasing* function f, it remains true if we replace "increasing" by "decreasing" throughout.

Corollary 6.5.1. Suppose that f is differentiable and strictly decreasing on the closed interval $[a, b]$. If $\xi \in [a, b]$ and $f'(\xi) \neq 0$ then the inverse function g is differentiable at $\eta = f(\xi)$ with derivative $g'(\eta) = \dfrac{1}{f'(\xi)}$.

Proof. Note that $\beta = f(b) < f(a) = \alpha$ in this case. Apply Theorem 6.5 to the function $h = -f$ with image set $[-\alpha, -\beta]$. If k is the inverse of h as guaranteed by that theorem, then g (the inverse of f) is given by $g(y) = k(-y)$. We get

$$g'(\eta) = -k'(-\eta) = -\frac{1}{h'(\xi)} = \frac{1}{f'(\xi)}. \qquad \square$$

Note. If we write $y = f(x)$, and $x = g(y)$ for the inverse, then the results may be expressed as $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, or $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$. So here again $\frac{dy}{dx}$ behaves a bit like a fraction with the dy and the dx *appearing* to cancel, although that is not what is happening. If it were really a case of just cancelling, there would be no need for the proofs presented above.

Corollary 6.5.2. Suppose that n is a positive integer. Then the function $g(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is differentiable on $(0, \infty)$ with derivative $\left(\frac{1}{n}\right)x^{\frac{1}{n}-1}$.

Proof. The function $f(x) = x^n$ is continuous and strictly increasing on $\mathbb{R}$ and so certainly continuous and strictly increasing on any closed interval $[0, b]$. The function g is the inverse of f and so differentiable on $[f(0), f(b)] = [0, b^n]$ except at the point 0. Since $b > 0$ is arbitrary, we deduce that g is differentiable on $(0, \infty)$. If $\eta = f(\xi) = \xi^n$ then $\xi = \eta^{\frac{1}{n}}$ and

$$g'(\eta) = \frac{1}{f'(\xi)} = \frac{1}{n\xi^{n-1}} = \frac{1}{n\eta^{1-\frac{1}{n}}} = \left(\frac{1}{n}\right)\eta^{\frac{1}{n}-1}.$$

[If we write $r = \frac{1}{n}$ then the derivative of x^r is just rx^{r-1}.] □

Corollary 6.5.3. If r is rational then $f(x) = x^r$ is differentiable on $(0, \infty)$ with derivative rx^{r-1}.

Proof. We may assume that $r = m/n$ in its lowest terms with m, n integers and $n > 0$. Then we take $g(x) = x^m$ and $h(x) = x^{\frac{1}{n}}$, so that $f(x) = g(h(x)) = (g \circ h)(x)$. By the composite rule, f is differentiable on $(0, \infty)$, with derivative

$$f'(x) = g'(h(x))h'(x) = m(x^{\frac{1}{n}})^{m-1} \times \left(\frac{1}{n}\right)x^{\frac{1}{n}-1} = \left(\frac{m}{n}\right)x^{\frac{m}{n}-1} = rx^{r-1}. \quad \square$$

Notes.

1. If $r = m/n$ is in its lowest terms with m, n integers, $n > 0$ and n *odd*, then $f(x) = x^r$ is differentiable on $(-\infty, 0)$ with derivative rx^{r-1}. Of course n has to be odd because negative numbers do not have an n^{th} root if n is even. You are asked to prove this result for $m = 1$ in one of the questions in Exercises 6.2. The result for general m then follows using the composite rule as in Corollary 6.5.3. Summarising: for a rational number r, the derivative of x^r is rx^{r-1} whenever the function and derivative make sense.

2. If α is irrational, we haven't yet got a definition of x^α so, for irrational powers, so questions of continuity and differentiability must wait. We will address this issue in the next chapter.

Exercises for Section 6.2

1. For each of the following functions determine the derivative. State any values for which the function is not differentiable. You can assume that $\sin(x)$ is differentiable with derivative $\cos(x)$, that $\cos(x)$ is differentiable with derivative $-\sin(x)$, and that $\exp(x)$ is differentiable with derivative $\exp(x)$.

a) $x^2 + x + 3$,	b) $(x^2 - 1)/(x^2 + 1)$,	c) $\sin(x)/\cos(x)$,
d) $\sin(x^2)$,	e) $\cos(\sqrt{x})$,	f) $\exp(\cos(x^2))$,
g) $\exp(1/x^2)$,	h) $\sin(x \exp(x^2))$,	i) $\cos(\exp(x) + x^2)$.

2. Suppose that $f(x) = x^{\frac{1}{n}}$, where n is an odd positive integer. Prove that f is differentiable on $(-\infty, 0)$ with derivative $\frac{1}{n}x^{\frac{1}{n}-1}$. [Hint: for $x < 0$, $f(x) = -(-x)^{\frac{1}{n}}$. Use the composite rule and Corollary 6.5.2 with $g(z) = -z^{\frac{1}{n}}$ and $z(x) = -x$.]

3. The function $f(x)$ is given by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$. Determine where $f(x)$ is differentiable, and state the value of of its derivative. You may assume that $\sin(x)$ is differentiable on $\mathbb{R}$ and has its usual properties.

6.3 Rolle's Theorem and the Mean Value Theorem

The two main results in this section show that the average rate of change of a differentiable function on a closed interval must be the actual rate of change at some point in the interval. Rolle's theorem deals with the case when the average rate of change is zero. From Rolle's theorem we obtain the Mean Value Theorem, which deals with the general case. But first there are some preliminary items.

In the previous chapter we defined what we meant by saying that a function f is strictly increasing or strictly decreasing *on an interval*. Here we define what is meant by saying that a function f is strictly increasing or decreasing *at a point*. We also take the opportunity to define local maxima and minima.

Definition 6.4. Suppose that $f(x)$ is continuous at the point $x = a$.
We say that f is *strictly increasing* at the point a if $\exists\delta > 0$ s.t. $\forall x \in (a - \delta, a)$ we have $f(x) < f(a)$, while $\forall x \in (a, a + \delta)$ we have $f(a) < f(x)$.
We say that f is *strictly decreasing* at the point a if $\exists\delta > 0$ s.t. $\forall x \in (a - \delta, a)$ we have $f(x) > f(a)$, while $\forall x \in (a, a + \delta)$ we have $f(a) > f(x)$.

We say that f has a *local maximum* at the point a if $\exists\delta > 0$ s.t. $\forall x \in (a - \delta, a)$ and $\forall x \in (a, a + \delta)$ we have $f(x) < f(a)$.
We say that f has a *local minimum* at the point a if $\exists\delta > 0$ s.t. $\forall x \in (a - \delta, a)$ and $\forall x \in (a, a + \delta)$ we have $f(a) < f(x)$.

Theorem 6.6. Suppose that f is differentiable at a. If $f'(a) > 0$ then f is strictly increasing at a. If $f'(a) < 0$ then f is strictly decreasing at a.

Proof. We give the proof for $f'(a) > 0$, the other case is similar.

Since f is differentiable at a, given $\epsilon > 0, \exists \delta > 0$ s.t. if $0 < |h| < \delta$, then $\Big|\dfrac{f(a+h)-f(a)}{h} - f'(a)\Big| < \epsilon$, i.e. $-\epsilon < \dfrac{f(a+h)-f(a)}{h} - f'(a) < \epsilon$. So if we take $\epsilon = f'(a)$, then for a corresponding δ we have

$$-f'(a) < \frac{f(a+h)-f(a)}{h} - f'(a),$$

and this gives $0 < \dfrac{f(a+h)-f(a)}{h}$, provided that $0 < |h| < \delta$. It follows that if $-\delta < h < 0$ then $f(a+h) < f(a)$, while if $0 < h < \delta$ then $f(a+h) > f(a)$. Hence f is strictly increasing at a. □

Corollary 6.6.1. If f is differentiable at a and it has a local maximum or a local minimum at a, then $f'(a) = 0$.

Proof. If $f'(a) > 0$ or if $f'(a) < 0$ then f is strictly increasing or strictly decreasing at a and so cannot have a local maximum or a local minimum at a. □

Cautions. The corollary is *not* saying that if $f'(a) = 0$ then f has a local maximum or a local minimum at a. If we consider the function $f(x) = x^3$, it is easy to see that it is strictly increasing on $\mathbb{R}$, and therefore has no local maxima or minima - see Figure 6.5. However, $f'(x) = 3x^2$, so $f'(0) = 0$. Note also that the corollary only applies to functions that are differentiable at a; there may be local maxima and minima of f at points where f is not differentiable. A further caution is that a function may have several local maxima and minima; indeed it is possible to have a local minimum greater than a local maximum as in Figure 6.6. And finally, these are *local* maxima and minima, so there may be a maximum or minimum value of f on a closed interval at one of the end points, or the function may have no overall maximum or minimum.

Despite the cautionary remarks, points where the derivative is zero clearly merit some attention, so we give them a name in the following definition.

Definition 6.5. If f is differentiable at a and $f'(a) = 0$ then we say that a is a *stationary point* of f.

One possible reason for this terminology is that for a distance-time graph, places where $f'(a) = 0$ represent points where the moving body is (at least temporarily) at rest, i.e. stationary.

Clearly we need to be able to distinguish local maxima and minima from other stationary points of f, and to distinguish between local maxima and local minima. There are two common tests. Before we get to them we will state and prove Rolle's Theorem, and the Mean Value Theorem. These two theorems have important

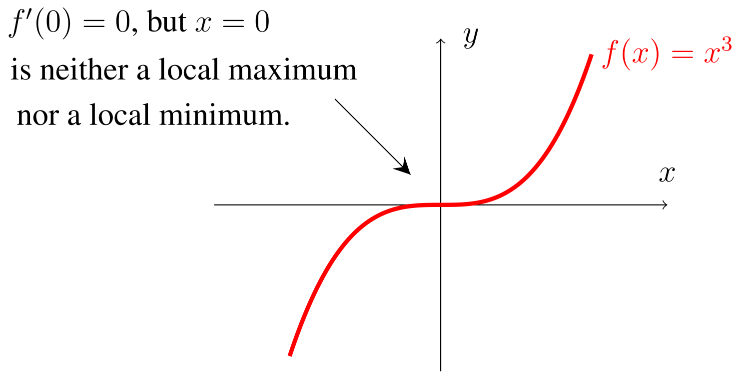


Figure 6.5: The graph of $f(x) = x^3$.

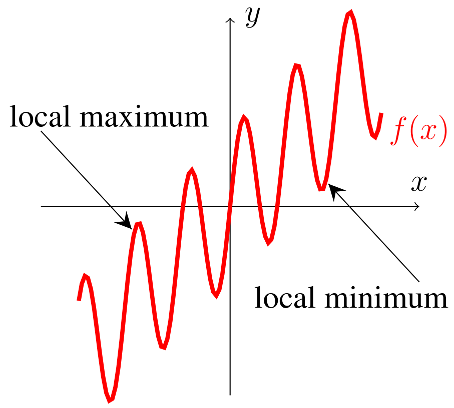


Figure 6.6: A function with several local maxima and minima.

consequences including, but not limited to, tests for local maxima and minima. Rolle's theorem is illustrated in Figure 6.7.

Theorem 6.7 (Rolle's Theorem)**.** Suppose that f is a function that is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there exists a point $\xi \in (a, b)$ such that $f'(\xi) = 0$, i.e. ξ is a stationary point.
[Observe that the average gradient across $[a, b]$ is $[f(b) - f(a)]/(b - a) = 0$.]

Proof. Since f is continuous on $[a, b]$, it is bounded on $[a, b]$ (see Theorem 5.10 in the previous chapter). Put

$$M = \sup_{x\in[a,b]} f(x), \quad \text{and} \quad m = \inf_{x\in[a,b]} f(x).$$

If $M = m$ then f is constant on $[a, b]$, and so $f'(x) = 0$ at every point $x \in (a, b)$. In such cases we can put $\xi = (a + b)/2$.

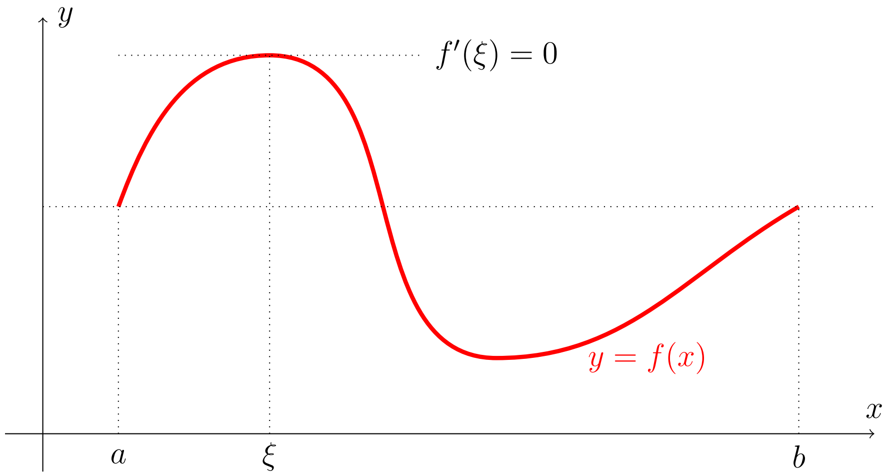


Figure 6.7: Illustrating Rolle's Theorem

If $M \neq m$ then either $M > f(a) = f(b)$ or $m < f(a) = f(b)$ (or perhaps both).

Suppose first that $M > f(a) = f(b)$. By Theorem 5.11 from the previous chapter, there exists a point $\xi \in [a, b]$ for which $f(\xi) = M$. Clearly $\xi \neq a, b$, so $\xi \in (a, b)$ and f is differentiable at ξ. If $f'(\xi) > 0$ then f is strictly increasing at ξ, and so there is a subinterval of (a, b) immediately to the right of ξ in which $f(x) > f(\xi) = M$. This contradicts the fact that M is the supremum. If $f'(\xi) < 0$ then f is strictly decreasing at ξ, and so there is a subinterval of (a, b) immediately to the left of ξ in which $f(x) > f(\xi) = M$. This also contradicts the fact that M is the supremum. Hence we must have $f'(\xi) = 0$.

The other alternative is that $m < f(a) = f(b)$. In this case, arguing in a similar fashion, there is a point $\xi \in (a, b)$ for which $f(\xi) = m$, and then $f'(\xi) = 0$. □

Theorem 6.8 (The Mean Value Theorem)**.** Suppose that f is a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a point $\xi \in (a, b)$ such that $f'(\xi) = [f(b) - f(a)]/(b - a)$. Figure 6.8 illustrates the result.
[Observe that the average (mean) gradient across $[a, b]$ is $[f(b) - f(a)]/(b - a)$.]

Proof. Put $g(x) = f(x) - \lambda x$ where λ is a constant chosen to make $g(a) = g(b)$. This entails

$$f(a) - \lambda a = f(b) - \lambda b,$$

which gives $\lambda = [f(b) - f(a)]/(b - a)$.

Then g satisfies the conditions of Rolle's Theorem, and so there exists a point $\xi \in (a, b)$ such that $g'(\xi) = 0$. But $g'(\xi) = f'(\xi) - \lambda$. Hence $f'(\xi) = \lambda = [f(b) - f(a)]/(b - a)$. □

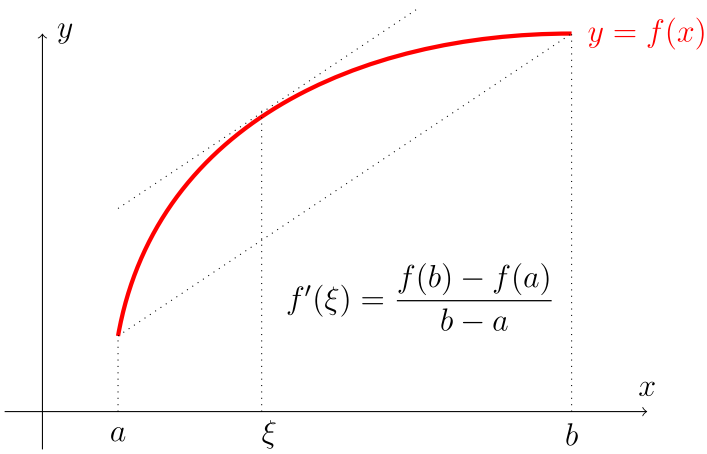


Figure 6.8: Illustrating the Mean Value Theorem

The result can also be expressed as $f(b) - f(a) = (b - a)f'(\xi)$ or as $f(b) = f(a) + (b - a)f'(\xi)$. Sometimes the result is referred to as the *First* Mean Value Theorem. Of course this suggests there must be a *Second* lurking in the wings, and maybe more. That will have to wait a few pages. Before that we have some important consequences of the Mean Value Theorem.

Corollary 6.8.1. Suppose that f is a function that is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f'(x) = 0$ for every $x \in (a, b)$. Then f is constant on $[a, b]$.

Proof. Suppose that α and β are any points in $[a, b]$ with $\alpha < \beta$. Then f satisfies the conditions of the Mean Value Theorem on the interval $[\alpha, \beta]$ and so $f(\beta) - f(\alpha) = (\beta - \alpha)f'(\xi)$ for some $\xi \in (\alpha, \beta)$. But $f'(\xi) = 0$ and so we have $f(\beta) = f(\alpha)$. It follows that f is constant on $[a, b]$. $\square$

Corollary 6.8.2. Suppose that g and h are functions that are continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $g'(x) = h'(x)$ for every $x \in (a, b)$. Then there exists a constant c such that $g(x) = h(x) + c$ for every $x \in [a, b]$.

Proof. Just apply the previous corollary to the difference $f(x) = g(x) - h(x)$. $\square$

This corollary really says that if we know the derivative of a function, then we know the function itself up to an additive constant. This result looks ahead to integration and we make the following definition

Definition 6.6. Suppose that F and f are defined on an interval I, and that F is differentiable on the interval I with derivative f, i.e. $F'(x) = f(x)$ for all $x \in I$. Then we say that F is a *primitive* of f on the interval I. In view of the previous corollary, if F_1 and F_2 are any primitives of the function f on an interval I then $F_1(x) - F_2(x)$ must take a constant value on I.

Corollary 6.8.3. Suppose that f is a function that is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f'(x) > 0$ for every $x \in (a, b)$. Then f is strictly increasing on $[a, b]$.
[If the condition $f'(x) > 0$ is replaced by $f'(x) < 0$, then the conclusion is that f is strictly decreasing on $[a, b]$.]

Proof. Suppose that α and β are any points in $[a, b]$ with $\alpha < \beta$. Then f satisfies the conditions of the Mean Value Theorem on the interval $[\alpha, \beta]$ and so $f(\beta) - f(\alpha) = (\beta - \alpha)f'(\xi)$ for some $\xi \in (\alpha, \beta)$. But $f'(\xi) > 0$ and so we have $f(\beta) - f(\alpha) > 0$, i.e. $f(\beta) > f(\alpha)$. It follows that f is strictly increasing on $[a, b]$.
[The decreasing case is similar.] □

We can now give the first derivative test for identifying local maxima and minima amongst the stationary points of a differentiable function.

Corollary 6.8.4 (First derivative test)**.** Suppose that the function f is differentiable on an open interval containing the point c and that c is a stationary point of f, i.e. $f'(c) = 0$. Then f has a local maximum at c if

(i) there is some interval I_l immediately to the left of c (say $I_l = (c - \delta, c)$) such that if $x \in I_l$ then $f'(x) > 0$, and

(ii) there is some interval I_r immediately to the right of c (say $I_r = (c, c + \gamma)$) such that if $x \in I_r$ then $f'(x) < 0$.

[If the inequalities (i) $f'(x) > 0$, and (ii) $f'(x) < 0$ are reversed then the stationary point is a local minimum.]

Proof. The previous corollary shows that f is strictly increasing on $(c - \delta, c]$ and strictly decreasing on $[c, c + \gamma)$. Hence c is a local maximum of f.
[The local minimum case follows using a similar argument.] □

Example 6.4. Show that $f(x) = x^4$ has a local minimum at $x = 0$.

Solution. We have $f'(x) = 4x^3$, so $f'(0) = 0$. Hence $x = 0$ is a stationary point of f. If $x < 0$ then $f'(x) < 0$, while if $x > 0$ then $f'(x) > 0$. So by the first derivative test, f has a local minimum at $x = 0$.
Of course no-one would ever bother with this argument in such a simple case because $f(x) = x^4 > 0$ unless $x = 0$, so f obviously has a local minimum at $x = 0$. However it does provide a simple illustration of the test.

For completeness we state the following definition.

Definition 6.7. Suppose that the function f is differentiable on an open interval containing the point c and that c is a stationary point of f, i.e. $f'(c) = 0$. If f is strictly increasing, or strictly decreasing at c (which ensures that c is neither a local maximum nor a local minimum of f), then we say that c is an *inflexion point* of f.

A trivial example is that $f(x) = x^3$ has an inflexion point at $x = 0$ (see Figure 6.5).

Remark. An inflexion point can be defined more generally as a point $x = a$ on the graph of a differentiable function where the graph crosses the tangent to the graph, so not requiring the point a to be a stationary point. As an easy example, think about rotating the graph of $f(x) = x^3$ 45 degrees clockwise about the origin. Under this more general definition $x = 0$ remains an inflexion point even though it is no longer a stationary point. But we will not spend any time on inflexion points in this book.

You will notice we called Corollary 6.8.4 the *first* derivative test. So what's the *second*? - There surely must be one. To explain this, we need to define higher derivatives.

Definition 6.8 (Higher derivatives)**.** Given a function f defined and differentiable on some interval, the derived function f' may itself be differentiable at some or all of the points in the same interval. Its derivative is then denoted by f'' or $f^{(2)}$. This is called the second derivative of f. If we write $y = f(x)$ then f' is denoted by $\frac{dy}{dx}$ and f'' by $\frac{d^2y}{dx^2}$. Note that it is $\frac{d^2y}{dx^2}$ and not $\frac{dy^2}{dx^2}$. The reason being that we regard the symbol $\frac{d}{dx}$ as an *operator*, i.e. an *instruction* to differentiate, so the symbol for differentiating twice is $\frac{d}{dx}\frac{d}{dx} = \frac{d}{dx^2}^2$. In a similar way we define higher order derivatives for $n \geq 3$: $f^{(n)}$, also denoted by $\frac{d^ny}{dx^n}$.

Sometimes it can be useful to concentrate on this operational aspect of differentiation and you may see the operator symbol $\frac{d}{dx}$ replaced by D or, if we wish to emphasise the role of x, by D_x. We won't be pursuing that here, but just to suggest how it might help, consider an expression such as $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y$. Using the D notation this can be written as $(D^2 + 3D + 2)y$, or even factorised as $(D + 2)(D + 1)y$.

To understand what these higher derivatives represent consider the graph shown in Figure 6.9 that gives the distance someone has travelled in a vehicle during a time interval. The x-coordinate represents the time from the start, and the y coordinate represents the distance travelled. The vehicle starts from rest, picks up speed and eventually slows before coming to rest again. The average speed is the total distance travelled divided by the total time. But the speed varies. Of course the speed is the rate of change of position. In this case the speed of travel is slowing between point P (at time a) and point Q (at time $a+h$) because the graph is flattening as you move from left to right from P to Q. The average speed between P and Q is $[f(a+h)-f(a)]/h$, but the speed at P itself (given by the slope of the tangent at P) is $f'(a)$. Similarly, the speed at Q is $f'(a+h)$, which for this graph is clearly less than the speed at P. So there is some deceleration (negative acceleration) as we move from left to right from P to Q. Acceleration is just the rate of change of velocity and so the acceleration at P is $f''(a)$, which in this case is negative. In the same way, the third derivative $f^{(3)}(a)$ would measure the rate of change of acceleration at P.

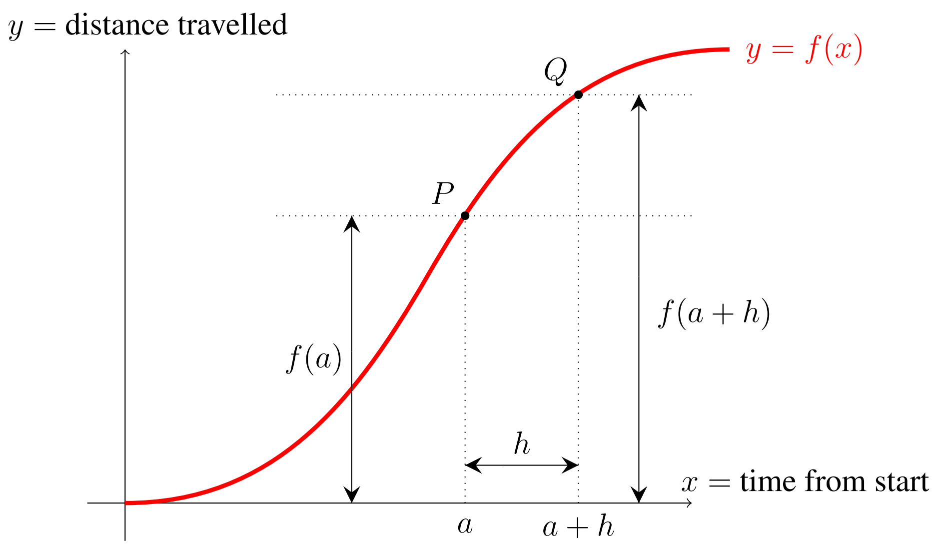


Figure 6.9: Explaining higher derivatives

Economists and politicians often talk about inflation, which is the rate of change of prices (a first derivative). When inflation is high there is often a focus on whether or not it is rising or falling. So a politician faced with high inflation is likely to make a big fuss about inflation falling (a second derivative), hoping to fool people into thinking things are getting better, rather than the more accurate view that at least they aren't getting worse quite as fast as they were. If inflation is falling or rising rapidly, there may even be some sly reference to the third derivative if it looks favourable.

The moral is that higher derivatives do have meaning. But read what politicians and economists say with great care. In this book were are concerned with how to calculate higher derivatives and what uses they might have. One of the uses is connected with locating maximum and minimum values of a function.

As previously mentioned there are two commonly used tests for identifying local maxima and minima among the stationary points of a differentiable function

f. We can now give the second test, which involves the sign (positive or negative) of the second derivative $f''(a)$. This second test is often easier to apply but it fails if $f''(a) = 0$. The result is yet another corollary of the Mean Value Theorem.

Corollary 6.8.5 (Second derivative test)**.** Suppose that the function f is differentiable on an open interval containing the point c, that c is a stationary point of f, i.e. $f'(c) = 0$, and that f has a second derivative at c. Then f has a local maximum at c if $f''(c) < 0$, and a local minimum at c if $f''(c) > 0$. If $f''(c) = 0$ then the test is inconclusive - the point c may be a local maximum, a local minimum, or neither.

Proof. Suppose that $f''(c) < 0$, then by Theorem 6.6, f' is strictly decreasing at c. Sice $f'(c) = 0$, there is some interval $I_l = (c - \delta, c)$ immediately to the left of c in which $f'(x) > 0$, and some interval $I_r = (c, c + \gamma)$ immediately to the right of c in which $f'(x) < 0$. It then follows from the first derivative test (Corollary 6.8.4) that f has a local maximum at c.

The proof for a local minimum is similar. That the test fails when $f''(c) = 0$ can be seen by considering in turn the functions (i) x^3, (ii) x^4, and (iii) $-x^4$. Each has a stationary point at $x = 0$ but respectively this stationary point is (i) neither a local maximum nor a local minimum, (ii) a local minimum, and (iii) a local maximum. □

This test is easy to use, easier than the first derivative test in many cases. However, it is subject to failure if $f''(c) = 0$, so the first derivative test is more discriminating. There are also higher derivative tests that can resolve cases where $f''(c) = 0$, but we won't consider them here. The first derivative test works in almost all practical cases. But it is possible to find rather pathological functions where even that test fails. An example of such a function is given in the Exercises for this section.

Rolle's Theorem can also be used to prove the following generalisation of the Mean Value Theorem known as Cauchy's Mean Value Theorem. This result is useful in many cases for determining the limiting value (if any) of an expression such as $\dfrac{f(x)}{g(x)}$ as x tends to some finite value c when f and g are differentiable functions having the somewhat disagreeable property that $f(c) = g(c) = 0$. Such expressions are sometimes called *indeterminate forms*, possibly not a good name for them.

Theorem 6.9 (Cauchy's Mean Value Theorem)**.** Suppose that f and g are functions that are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists a point $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Note first that the condition that $g'(x) \neq 0$ for all $x \in (a, b)$ ensures that $g(b) \neq g(a)$, because if $g(b) = g(a)$ then we could apply Rolle's theorem to deduce that $g'(x)$ is zero at some point $x \in (a, b)$. Then put $h(x) = f(x) - \lambda g(x)$ where λ is a constant chosen to make $h(a) = h(b)$. This entails

$$f(a) - \lambda g(a) = f(b) - \lambda g(b),$$

which gives $\lambda = [f(b) - f(a)]/[g(b) - g(a)]$.

Then h satisfies the conditions of Rolle's Theorem, and so there exists a point $\xi \in (a, b)$ such that $h'(\xi) = 0$. But $h'(\xi) = f'(\xi) - \lambda g'(\xi)$. Hence

$$\frac{f'(\xi)}{g'(\xi)} = \lambda = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \square$$

Remark. You might think that we could have proved this result by applying the Mean Value Theorem twice. After all, there is a point ξ such that $f'(\xi) = [f(b)-f(a)]/(b-a)$ and there is a point ξ such that $g'(\xi) = [g(b) - g(a)]/(b - a)$, so why can't we just divide these two expressions and get $f'(\xi)/g'(\xi) = [f(b) - f(a)]/[g(b) - g(a)]$? The problem is that the ξ that works for f may not be the same ξ that works for g. If we call these ξ_f and ξ_g respectively, to make it clear that they can be different, then all we'd get would be $f'(\xi_f)/g'(\xi_g) = [f(b) - f(a)]/[g(b) - g(a)]$, which is not what we are trying to prove.

An important corollary to Cauchy's Mean Value Theorem is known as L'Hôpital's rule for indeterminate forms.

Corollary 6.9.1 (L'Hôpital's Rule)**.** Suppose that f and g are functions that are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$, except possibly at c itself. Suppose also that $c \in (a, b)$ and that $f(c) = g(c) = 0$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided that the latter limit exists.

Proof. The functions f and g will satisfy the conditions of Cauchy's Mean Value Theorem on any interval $[c, x]$ where $c < x \leq b$. So for any such x there is a corresponding $\xi \in (c, x)$ for which

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f(x)}{g(x)}.$$

[Note $\xi \neq c$, so $g'(\xi) \neq 0$.] But $\xi \to c$ as $x \to c+$ because $c < \xi < x$. Hence

$$\lim_{x \to c+} \frac{f(x)}{g(x)} = \lim_{x \to c+} \frac{f'(x)}{g'(x)},$$

provided that the latter limit exists. This deals with limits on the right at c, and a similar argument deals with limits on the left at c. So we may conclude that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided that the latter limit exists. □

Note the condition that $f(c) = g(c) = 0$. Do not try to apply this result if this is not the case. Indeed, there is no need to try to apply the result if $g(c) \neq 0$. If f and g are continuous at c then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$ if $g(c) \neq 0$. One might reflect that L'Hôpital's rule is an example of something that mostly works precisely when you need it (but doesn't work at all when you don't need it!).

If you find that $\frac{f'(x)}{g'(x)}$ is itself an indeterminate form at $x = c$, you can reapply the result (but check the conditions) and consider $\frac{f''(x)}{g''(x)}$.

Example 6.5. Find the limiting value as x tends to zero of (i) $\dfrac{\sin(x)}{x}$, and (ii) $\dfrac{1 - \cos(x)}{x^2}$. You can assume that $\sin(x)$ and $\cos(x)$ are differentiable with derivatives $\cos(x)$ and $-\sin(x)$ respectively, and that $\sin(0) = 0$ and $\cos(0) = 1$.

Solution.
(i) Both $\sin(x)$ and x take the value 0 at $x = 0$ and the derivative of x is non-zero. So L'Hôpital's rule gives

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1},$$

provided that the latter limit exists, which it does since $\cos(x)$ is continuous at 0, and the value of the limit is $\cos(0) = 1$. So $\dfrac{\sin(x)}{x} \to 1$ as $x \to 0$.
(ii) Both $1 - \cos(x)$ and x^2 take the value 0 at $x = 0$ and the derivative of x^2 is $2x$, which is non-zero except at $x = 0$. So L'Hôpital's rule gives

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x}.$$

provided that the latter limit exists, which it does by part (i) and has the value $\frac{1}{2}$. So $\dfrac{1 - \cos(x)}{x^2} \to \dfrac{1}{2}$ as $x \to 0$.

The final result in this section is a generalisation of the product rule to higher derivatives. How might you find the tenth derivative of a product such as $x^3 \sin(x)$? Well, you could go laboriously through finding the first derivative, then the second, then the third, and so on. But it is easy to get a general formula.

If we use the D notation to denote the differential operator $\frac{d}{dx}$ and look at the first few derivatives of a general product, the pattern should become clear and is then easily proved. So suppose that $f(x)$ and $g(x)$ are functions that are differentiable multiple times in some interval. Then by the product rule,

$$D(fg) = D(f)g + fD(g).$$

Differentiating again gives

$$D^2(fg) = \left[D^2(f)g + D(f)D(g)\right] + \left[D(f)D(g) + fD^2(g)\right].$$

Collecting like terms together gives

$$D^2(fg) = D^2(f)g + 2D(f)D(g) + fD^2(g).$$

If you differentiate again (go on - try it) you will get

$$D^3(fg) = D^3(f)g + 3D^2(f)D(g) + 3D(f)D^2(g) + fD^3(g).$$

Hopefully this is beginning to remind you of expressions like

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

suggesting that the numbers which appear as coefficients (in this instance 1, 3, 3, 1) are simply the binomial coefficients. This is indeed the case and the general result is known as Leibniz' Theorem.

Theorem 6.10 (Leibniz' Theorem)**.** Suppose that both the functions $f(x)$ and $g(x)$ are n times differentiable at $x = a$. Then the n^{th} derivative of the product $f(x)g(x)$ evaluated at a is given by

$$D^n(fg) = D^n(f)g + \binom{n}{1}D^{n-1}(f)D(g) + \binom{n}{2}D^{n-2}(f)D^2(g) + \ldots + fD^n(g),$$

where $D^r(f)$ denotes the r^{th} derivative of f evaluated at a, and likewise $D^r(g)$ denotes the r^{th} derivative of g evaluated at a. In alternative notation the result can be expressed as

$$(fg)^{(n)}(a) = f^{(n)}(a)g(a) + \binom{n}{1}f^{(n-1)}(a)g'(a) + \binom{n}{2}f^{(n-2)}(a)g''(a) + \\ \ldots + f(a)g^{(n)}(a).$$

Proof. The proof is by induction. The initial case for $n = 1$ is just the product rule. We will assume that the result holds for some integer $n = k \geq 1$ and, on this assumption, prove that it holds for $n = k + 1$. In order to do this we need a result about combining the binomial coefficients $\binom{k}{r-1}$ and $\binom{k}{r}$. We have

$$\begin{aligned}\binom{k}{r-1} + \binom{k}{r} &= \frac{k!}{(r-1)!(k-r+1)!} + \frac{k!}{(r)!(k-r)!} \\ &= \frac{k!}{(r-1)!(k-r)!}\Big[\frac{1}{k-r+1} + \frac{1}{r}\Big] \\ &= \frac{k!}{r!(k-r+1)!}\Big[r + k - r + 1\Big] \\ &= \frac{(k+1)!}{r!(k-r+1)!} = \binom{k+1}{r}.\end{aligned}$$

Having this, we proceed with the induction by assuming the result for $n = k$ and differentiating once more to get a result for $n = k + 1$. This gives

$$\begin{aligned}D^{k+1}(fg) &= \Big[D^{k+1}(f)g + D^k(f)D(g)\Big] \\ &\quad + \binom{k}{1}\Big[D^k(f)D(g) + D^{k-1}(f)D^2(g)\Big] \\ &\quad + \binom{k}{2}\Big[D^{k-1}(f)D^2(g) + D^{k-2}(f)D^3(g)\Big] \\ &\quad + \ldots + \Big[D(f)D^k(g) + fD^{k+1}(g)\Big].\end{aligned}$$

If we now group together like terms we get

$$\begin{aligned}D^{k+1}(fg) &= D^{k+1}(f)g + D^k(f)D(g)\Big[1 + \binom{k}{1}\Big] \\ &\quad + D^{k-1}(f)D^2(g)\Big[\binom{k}{1} + \binom{k}{2}\Big] \\ &\quad + D^{k-2}(f)D^3(g)\Big[\binom{k}{2} + \binom{k}{3}\Big] \\ &\quad + \ldots + D(f)D^k(g)\Big[\binom{k}{k-1} + 1\Big] + fD^{k+1}(g).\end{aligned}$$

Using the result about combining binomial coefficients gives

$$\begin{aligned}D^{k+1}(fg) = D^{k+1}(f)g &+ \binom{k+1}{1}D^k(f)D(g) + \binom{k+1}{2}D^{k-1}(f)D^2(g) \\ &+ \ldots + fD^{k+1}(g),\end{aligned}$$

which completes the induction. $\square$

Note. There is an easy, purely combinatorial, argument for establishing the identity $\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$. Since $\binom{k}{r}$ is the number of ways of selecting r distinct objects from k distinct objects when the order of choice doesn't matter, we can select r objects from $k+1$ objects in two parts. In the first part we fix our first choice as one of the objects (say object A) and choose the remaining $r-1$ objects from k in $\binom{k}{r-1}$ ways. This covers all choices that contain the selected object (A). In the second part we set aside the selected object and choose r objects from the remaining k objects in $\binom{k}{r}$ ways. It follows that $\binom{k+1}{r} = \binom{k}{r-1} + \binom{k}{r}$.

Example 6.6. Find the tenth derivative of $x^3 \sin(x)$, assuming that the derivative of $\sin(x)$ is $\cos(x)$, and the derivative of $\cos(x)$ is $-\sin(x)$.

Solution. We apply Leibniz' Theorem with $f(x) = \sin(x)$ and $g(x) = x^3$, so that $g^{(n)}(x) = 0$ if $n \geq 4$, while $f^{(4)}(x) = f(x) = \sin(x)$. Hence

$$\begin{aligned}(fg)^{(10)}(x) &= f^{(10)}(x)g(x) + 10f^{(9)}(x)g'(x) + 45f^{(8)}(x)g''(x) + \\ &\quad 120f^{(7)}(x)g^{(3)}(x) \\ &= -x^3\sin(x) + 30x^2\cos(x) + 270x\sin(x) - 720\cos(x).\end{aligned}$$

EXERCISES 6.3

1. Prove that the function $f(x) = x^3 - 3x^2 + x + 1$ has precisely one local maximum and one local minimum, and precisely three zeros.
2. Suppose that $y = f(x)$ is a differentiable function that satisfies the differential equation $\frac{dy}{dx} + 2xy = x$. Multiply this equation by $\exp(x^2)$ and show that it can be written in the form $\frac{d}{dx}\big(y\exp(x^2)\big) = \frac{d}{dx}\big(\frac{1}{2}\exp(x^2)\big)$. Deduce that $y = \frac{1}{2} + A\exp(-x^2)$, where A is some constant. If it is known that $y = 1$ when $x = 0$ (i.e. $f(0) = 1$) determine the value of the constant A. [You may assume all the usual properties of the exponential function, such as $\exp'(x) = \exp(x)$, $\exp(x+y) = \exp(x)\exp(y)$, and $\exp(0) = 1$.]
3. The function f is defined as follows.

$$f(x) = \begin{cases} x^2\Big(2 - \sin(\frac{1}{x})\Big) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

   Prove that f is differentiable on $\mathbb{R}$ and that $f'(0) = 0$, so that 0 is a stationary point of f. Prove also that $f'(x)$ changes sign infinitely often in any interval $(a, 0)$ immediately to the left of 0, and in any interval $(0, b)$ immediately to the right of 0, so that the first derivative test cannot be applied to determine the nature of the stationary point at $x = 0$. What is the nature of this stationary point? [You may assume that the sine and cosine functions have all their usual properties including derivatives and bounds.]

4. Evaluate the following limits. You may assume the usual properties of the functions $\sin(x), \cos(x)$ and the exponential function $\exp(x)$.

a) $\displaystyle\lim_{x\to 1} \frac{x^3 - 3x^2 + x + 1}{\sin(\pi x)}$, b) $\displaystyle\lim_{x\to 0} \frac{\exp(x) - 1}{\exp(2x) - 1}$,

c) $\displaystyle\lim_{x\to 0} \frac{(\exp(x) - 1)^3}{\sin(x) - x}$, d) $\displaystyle\lim_{x\to 0} \frac{x + \cos(x) - 1}{\sin(x) + 1}$.

5. Assuming the usual properties of the functions $\sin(x), \cos(x)$ and the exponential function $\exp(x)$, determine the tenth derivative of $\exp(x)\sin(x)$.

6.4 Taylor's Theorem

In the previous section you saw the Mean Value Theorem, sometimes called the *First* Mean Value Theorem. The obvious question is "what's the *Second*?". The First says that, subject to certain conditions, there exists $\xi \in (a, b)$ such that

$$f(b) = f(a) + (b - a)f'(\xi).$$

The Second says that, subject to certain conditions, there exists $\xi \in (a, b)$ such that

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2} f''(\xi).$$

Having seen this you can probably anticipate that there is an n^{th} Mean Value Theorem. This says that, subject to certain conditions, there exists $\xi \in (a, b)$ such that

$$\begin{aligned} f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2!} f''(a) + \dots \\ + \frac{(b - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + \frac{(b - a)^n}{n!} f^{(n)}(\xi). \end{aligned}$$

With the appropriate conditions attached, this is known as Taylor's Theorem. If we write $a + h$ in place of b and allow that h might be positive or negative or zero (so that $b = a + h$ might be greater than a or less than a or even equal to a) then the result has the form

$$f(a + h) = T_{f,a,n,}(h) + R_{f,a,n}(h).$$

Here $T_{f,a,n}(h)$ is called the *Taylor Polynomial* in the variable h for the function f at the point a and having n terms (so degree $n - 1$) and is given by

$$T_{f,a,n}(h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n - 1)!} f^{(n-1)}(a),$$

and $R_{f,a,n}(h) = f(a+h) - T_{f,a,n}(h)$ is called the n^{th}-*Remainder Term* in the variable h for the function f at the point a. It can be given by

$$R_{f,a,n}(h) = \frac{h^n}{n!} f^{(n)}(\xi) \text{ for some } \xi \text{ between } a \text{ and } a+h.$$

If we have in mind a particular function f and a particular point a, we abbreviate $T_{f,a,n,}(h)$ to $T_n(h)$ and $R_{f,a,n}(h)$ to $R_n(h)$. The point a is sometimes called the *centre of the expansion*.

Before we state Taylor's Theorem and the conditions under which it holds, we just mention that the principal difficulty using it is the lack of knowledge about ξ; all we know is that ξ is some point between a and $a+h$. There are several alternative forms for the Remainder Term that attempt to address this issue. We give a form due to Schlömilch which is capable of producing the form we have described, but which can give a more useful form in certain cases.

Theorem 6.11. Taylor's Theorem] Suppose that f and its derivatives up to order $n-1$ are continuous on $[a, a+h]$ and f is differentiable n times on $(a, a+h)$. Then for any positive integer p, $f(a+h) = T_n(h) + R_n(h)$, where

$$T_n(h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a),$$

and $R_n(h)$ can be expressed in the form

$$R_n(h) = \frac{h^n(1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta h), \text{ for some } \theta \in (0,1).$$

Notes. Before presenting a proof, we make a few comments.
1. θ will depend on f, a, h, p and n.
2. The version stated assumes $h > 0$. The result remains true if $h < 0$ provided we replace $[a, a+h]$ by $[a+h, a]$ and $(a, a+h)$ by $(a+h, a)$. The result is trivially true if $h = 0$.
3. The form given for $R_n(h)$ is due to Schlömilch. If we take $p = n$ it gives

$$R_n(h) = \frac{h^n}{n!} f^{(n)}(a+\theta h) = \frac{h^n}{n!} f^{(n)}(\xi) \quad \text{(Lagrange's form)},$$

where $\xi = a + \theta h$ lies between a and $a+h$.
3. If we take $p = 1$ in Schlömilch's Remainder we get

$$R_n(h) = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) \quad \text{(Cauchy's form)},$$

where $\theta \in (0,1)$.
5. Lagranges's form is probably the easiest to remember. Cauchy's form is useful

for obtaining the binomial theorem for rational and negative powers, which we cover below in this section. It is also useful for obtaining the logarithmic series, which we cover in the next chapter. You might reasonably ask how anyone could come up with something as complicated looking as Schlömilch's Remainder. I confess that I do not know the details of this case, but complicated formulae are often the result of quite lengthy attempts to unify existing results (thinking here of the Lagrange and Cauchy remainders). That is why they look less than obvious.

Proof. We deal with the case $h > 0$, the proof for $h < 0$ is similar and requires only minor modifications. We start by defining a function $r(x)$ for $x \in [0, h]$. This function will be continuous on $[0, h]$ and differentiable on $(0, h)$. We put

$$r(x) = f(a+h) - \Big[f(a+x) + (h-x)f'(a+x) + \frac{(h-x)^2}{2!} f''(a+x) + \dots \\ + \frac{(h-x)^{n-1}}{(n-1)!} f^{(n-1)}(a+x) \Big]$$

Observe that $r(0)$ is precisely the remainder term that we are trying to estimate, namely $f(a+h) - T_n(h) = R_n(h)$. Also note that $r(h) = 0$.

The conditions on f ensure that r is differentiable on $(0, h)$. Although not instantly obvious it is easy to check that $r'(x)$ is given by the very simple formula

$$r'(x) = -\frac{(h-x)^{n-1}}{(n-1)!} f^{(n)}(a+x).$$

Next define the function $s(x)$ by putting

$$s(x) = r(x) - \Big(\frac{h-x}{h}\Big)^p r(0).$$

Then s is continuous on $[0, h]$ and differentiable on $(0, h)$. Moreover, $s(0) = s(h) = 0$. Hence s satisfies the conditions of Rolle's Theorem (Theorem 6.7) on the interval $[0, h]$. So there exists a point $\xi \in (0, h)$ such that $s'(\xi) = 0$. However,

$$s'(\xi) = r'(\xi) + \frac{p}{h}\Big(\frac{h-\xi}{h}\Big)^{p-1} r(0).$$

Consequently

$$r(0) = -\frac{h}{p}\Big(\frac{h}{h-\xi}\Big)^{p-1} r'(\xi) = \frac{h}{p}\Big(\frac{h}{h-\xi}\Big)^{p-1} \frac{(h-\xi)^{n-1}}{(n-1)!} f^{(n)}(a+\xi).$$

This reduces to

$$r(0) = \frac{h^p(h-\xi)^{n-p}}{p(n-1)!} f^{(n)}(a+\xi).$$

Since $\xi \in (0, h)$, we may write $\xi = \theta h$ where $\theta \in (0, 1)$. Recalling that $r(0)$ is just the remainder term $R_n(h)$ we get

$$R_n(h) = \frac{h^p(h - \theta h)^{n-p}}{p(n-1)!} f^{(n)}(a + \theta h) = \frac{h^n(1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a + \theta h),$$

as required. □

Comments.
1. If we take $f^{(0)}(h)$ to be $f(h)$ then we can write $T_n(h)$ in the form

$$T_n(h) = \sum_{r=0}^{n-1} \frac{h^r}{r!} f^{(r)}(a).$$

2. Suppose that f is a function with derivatives of all orders on $[a, a+h]$ for which we can show that the remainder term $R_n(h) \to 0$ as $n \to \infty$. Then we have

$$f(a+h) = \sum_{r=0}^{\infty} \frac{h^r}{r!} f^{(r)}(a).$$

This follows immediately from Taylor's Theorem because $R_n(h)$ is the difference between $f(a + h)$ and $T_n(h)$, and $T_n(h)$ is just the n^{th} partial sum of the series. The infinite series is then called the Taylor Series expansion for $f(a + h)$ about the point a.
3. If f is a function with derivatives of all orders on $[a, a+h]$, then it certainly has derivatives of all orders on any subinterval $[a, a + x]$ where $0 < x \leq h$. Hence if we can show that $R_n(x) \to 0$ as $n \to \infty$ for each $x \in (0, h]$ it will follow that for each $x \in (0, h]$

$$f(a+x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(a).$$

4. Both the preceding comments tacitly assume that $h > 0$, but they remain true if $h < 0$ provided that $[a, a+h]$ and $[a, a+x]$ are replaced by $[a+h, a]$ and $[a+x, a]$ respectively , where $x \in [h, 0]$. (The results are trivially true if $h = 0$.)
5. Allowing the previous comments, if f is a function with derivatives of all orders on $[a-h, a+h]$ and we can show that $R_n(x) \to 0$ as $n \to \infty$ for each $x \in [-h, h]$, then it will follow that for each $x \in [-h, h]$

$$f(a+x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(a).$$

In such circumstances the series is called the Taylor Series for $f(a + x)$ about the point a, and its radius of convergence is at least h. We re-state this comment as a corollary to Theorem 6.11.

Corollary 6.11.1 (Taylor Series)**.** Suppose that f is a function with derivatives of all orders on $[a-h, a+h]$. For $x \in [-h, h]$ define $R_n(x)$ by

$$R_n(x) = f(a+x) - \sum_{r=0}^{n-1} \frac{x^r}{r!} f^{(r)}(a).$$

Then, by Taylor's Theorem, for any positive integer p, $R_n(x)$ may be expressed in the form

$$R_n(x) = \frac{x^n(1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta x),$$

where θ may depend on f, a, x, p and n, but lies in the interval $(0, 1)$.
If $R_n(x) \to 0$ as $n \to \infty$ for each $x \in [-h, h]$, then for $x \in [-h, h]$

$$f(a+x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(a),$$

and the power series has radius of convergence at least h. Such a power series is known as the Taylor series for $f(a+x)$ about the point a.

If we put $a = 0$ in Taylor's Theorem (or in Corollary 6.11.1) we obtain what is known as Maclaurin's Theorem (or a Maclaurin Series). At first sight this looks like a special case of expansion about zero, but it is easily shown to be equivalent. For example, if we have a function f that we wish to expand about a, we can simply define $g(x) = f(a+x)$ and then use the Maclaurin version for g, noting that $g^{(r)}(0) = f^{(r)}(a)$, etc. Despite this equivalence we will state the Maclaurin version as a separate corollary.

Corollary 6.11.2 (Maclaurin Series)**.** Suppose that f is a function with derivatives of all orders on $[-h, h]$. For $x \in [-h, h]$ define $R_n(x)$ by

$$R_n(x) = f(x) - \sum_{r=0}^{n-1} \frac{x^r}{r!} f^{(r)}(0).$$

Then, by Taylor's Theorem, for any positive integer p, $R_n(x)$ may be expressed in the form

$$R_n(x) = \frac{x^n(1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(\theta x),$$

where θ may depend on f, x, p and n, but lies in the interval $(0, 1)$.
If $R_n(x) \to 0$ as $n \to \infty$ for each $x \in [-h, h]$, then for $x \in [-h, h]$

$$f(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0),$$

and the power series has radius of convergence at least h. Such a power series is known as the Maclaurin Series for $f(x)$.

Beware. In both the preceding corollaries it is vital to observe the condition that the remainder term $R_n(x) \to 0$ as $n \to \infty$ for each $x \in [-h, h]$. If $\big(R_n(x)\big)$ is a non-convergent sequence then the Taylor (or Maclaurin) series will not converge. An easy example of this is the Maclaurin series for $f(x) = \frac{1}{1+x}$, which we will see is non-convergent if $|x| \geq 1$. It is even possible that $\big(R_n(x)\big)$ does converge but to a non-zero limit. In such cases the corresponding series converges but the sum of the series will not be equal to the function value $f(a+x)$ (or $f(x)$). Such cases are a bit harder to find but we give one in the Exercises for this section.

Note. We will allow ourselves to speak of $\sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0)$ as *the Maclaurin series of* $f(x)$ even if we don't know whether or not it converges to $f(x)$. This just gives a name to the series. Investigating its convergence is a separate issue. A similar comment applies to a Taylor series centred at a point a.

On a notational aspect, the use of T_n and R_n is not universal practice. Some authors use different letters.

Even if your lecturer uses T_n and R_n, you need to look at how they are defined. We have defined

$$T_n(h) = \sum_{r=0}^{n-1} \frac{h^r}{r!} f^{(r)}(a).$$

If h is replaced by $x - a$ this gives

$$T_n(x-a) = \sum_{r=0}^{n-1} \frac{(x-a)^r}{r!} f^{(r)}(a),$$

but some authors prefer to call this sum $T_n(x)$ rather than $T_n(x-a)$. Similarly we defined

$$R_n(h) = f(a+h) - T_n(h),$$

but with $x - a$ replacing h this gives

$$R_n(x-a) = f(x) - T_n(x-a).$$

The same authors would call this $R_n(x)$ rather than $R_n(x-a)$. Amidst this possible confusion there is a bright spot: if $a = 0$ (i.e. for Maclaurin Series) the two usages are identical.

You might be experiencing some impatience at this point. We have spent the last six pages discussing Taylor's theorem, its consequences, and a lot of warnings. Now we will get around to using it. Most of our examples will use the Maclaurin version. We start with the exponential function.

Theorem 6.12. Suppose that there is a differentiable function $f(x)$ such that for every $x \in \mathbb{R}$, $f'(x) = f(x)$, and $f(0) = 1$. Then for every $x \in \mathbb{R}$,

$$f(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

Proof. The Maclaurin Series of $f(x)$ is

$$\sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0),$$

and this converges with sum $f(x)$ provided that the remainder term $R_n(x) \to 0$ as $n \to \infty$. But $f'(x) = f(x)$ implies that for each positive integer r, $f^{(r)}(x) = f(x)$, and since $f(0) = 1$ we get $f^{(r)}(0) = 1$. So the Maclaurin Series reduces to

$$\sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

The Lagrange form of the remainder is

$$R_n(x) = \frac{x^n}{n!} f^{(r)}(\xi) = \frac{x^n}{n!} f(\xi),$$

where ξ lies between 0 and x. Since f is differentiable, it is continuous and therefore bounded on the finite closed interval I with end points at 0 and x. If M is such a bound (i.e $|f(z)| < M$ for $z \in I$) then

$$|R_n(x)| \leq= \frac{|x|^n}{n!} M \to 0 \text{ as } n \to \infty.$$

[Recall that if x is any Real Number, then $x^n/n! \to 0$ as $n \to \infty$.]
Hence for every $x \in \mathbb{R}$,

$$f(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

□

Of course the function described in the theorem above is really $\exp(x)$, but so far we haven't actually defined $\exp(x)$. In the next chapter this omission will be rectified by *defining* $\exp(x)$ as the Maclaurin Series that we have just obtained. Indeed, the same process will be used to define $\sin(x)$ and $\cos(x)$. We will show how the familiar properties of these functions can be obtained directly from these defining series. The inverse function theorem (Theorem 6.5) can then be used to define inverses of these functions, and Maclaurin Series obtained for these inverses.

In Corollary 6.5.3 we proved that for any Rational Number r the function x^r is differentiable on $(0, \infty)$ with derivative rx^{r-1}. Here it becomes convenient to use something other than r to denote the exponent. We will use α. The Corollary can be used to obtain a Maclaurin Series for $f(x) = (1 + x)^\alpha$ when α is any Rational Number and $x \in (-1, 1)$. This result is generally known as the binomial series. It involves generalising the binomial coefficients $\binom{n}{r}$ to the case when the integer n is replaced by a non-integer α. For a positive integer n we have

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!},$$

so if the integer n is replaced by a non-integer α (when $\alpha!$ is undefined), it is natural to define

$$\binom{\alpha}{r} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-r+1)}{r!},$$

where r remains a non-negative integer. Using this notation and taking $\binom{\alpha}{0} = 1$, we can state the following result.

Theorem 6.13 (The Binomial Theorem for Rational Powers)**.**
Suppose that $f(x) = (1 + x)^\alpha$, where α is a Rational Number that is not zero and not a positive integer, i.e. α is either a negative integer or a non-integer. Then the Maclaurin Series of $f(x)$ is $\sum_{r=0}^{\infty} \binom{\alpha}{r} x^r$ and this converges absolutely to $f(x) = (1 + x)^\alpha$ if $|x| < 1$ and diverges if $|x| > 1$.
[If α is zero or a positive integer then $\binom{\alpha}{r} = 0$ for $r > \alpha$ and the series is actually finite and the result is none other than the familiar(?) binomial theorem (see Section 3.1) for non-negative integer powers, so that is why this case is excluded.]

Proof. Assume first that $|x| < 1$. Differentiating $f(x) = (1 + x)^\alpha$ r times gives

$$f^{(r)}(x) = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-r+1)(1+x)^{\alpha-r}.$$

In particular

$$f^{(r)}(0) = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-r+1).$$

Hence the Maclaurin Series is

$$\sum_{r=0}^{\infty} \frac{x^r}{r!}\alpha(\alpha-1)(\alpha-2)\dots(\alpha-r+1) = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r,$$

and this converges to $f(x) = (1+x)^\alpha$ if and only if the remainder term $R_n(x)$ converges to zero as n tends to infinity. Cauchy's formula for the remainder tells us that $R_n(x)$ can be expressed in the following form for some $\theta \in (0,1)$.

$$\begin{aligned} R_n(x) &= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x) \\ &= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} \alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)(1+\theta x)^{\alpha-n} \\ &= \frac{x^n}{(n-1)!}\Big(\frac{1-\theta}{1+\theta x}\Big)^{n-1}(1+\theta x)^{\alpha-1}\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1). \end{aligned}$$

But for $|x|<1$ we have $0<(1-\theta)/(1+\theta x)<1$, and $1-|x|<1+\theta x<1+|x|$ so that $(1+\theta x)^{\alpha-1}$ is bounded above by the maximum of $(1-|x|)^{\alpha-1}$ and $(1+|x|)^{\alpha-1}$ (which one depends on whether $\alpha<1$ or $\alpha>1$). So if M denotes this maximum we obtain

$$|R_n(x)| \le \Big|M\frac{x^n}{(n-1)!}\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)\Big| = a_n, \text{ say.}$$

To prove that the remainder converges to zero, we consider the ratio a_{n+1}/a_n. This looks a good strategy since most of the complicated expression cancels out. We get

$$\frac{a_{n+1}}{a_n} = \frac{|x||\alpha-n|}{n} \to |x| \text{ as } n\to\infty.$$

So the limit of the ratio is $|x|<1$, and if we pick a number y between $|x|$ and 1, then for all sufficiently large n we will have $a_{n+1}/a_n < y$. To be a bit more specific, if we put $y = (1+|x|)/2$ then $|x|<y<1$, and so there exists N such that for $n\ge N$,

$$\frac{a_{n+1}}{a_n} < y < 1.$$

Hence for $n\ge N$, $a_n \le a_N y^{n-N} \to 0$ as $n\to\infty$. It follows that the Maclaurin Series converges to $(1+x)^\alpha$ for $|x|<1$. By Theorem 4.12 it follows that the convergence is absolute.

It remains to prove that the series diverges for $|x|>1$. In this case the general term of the series is $\binom{\alpha}{r}x^r = b_r$, say. We will show that (b_r) is not a null sequence and so the series cannot converge. We use the same sort of strategy that we used for $|x|<1$.

$$\Big|\frac{b_{r+1}}{b_r}\Big| = \frac{|x||\alpha-r|}{r+1} \to |x| \text{ as } r\to\infty.$$

If we put $y=(1+|x|)/2$ then $1<y<|x|$, and so there exists R such that for $r\ge R$,

$$\Big|\frac{b_{r+1}}{b_r}\Big| > y > 1.$$

Hence for $r \geq R$, $|b_r| \geq |b_R|y^{r-R}$ and since b_R is non-zero (indeed none of the terms of the series are zero), we have $|b_r| \to \infty$ as $r \to \infty$. Thus (b_r) is a non-null sequence, and so the series cannot converge.

We just remark that the two parts of this result, convergence for $|x| < 1$ and divergence for $|x| > 1$, tell us that the radius of convergence of the series is 1. $\square$

We established convergence for $|x| < 1$ and divergence for $|x| > 1$. What happens for $|x| = 1$ is rather complicated. In fact, for $x = 1$ the series converges if $\alpha > -1$ and diverges if $\alpha < -1$, while for $x = -1$ the series converges if $\alpha > 0$ and diverges if $\alpha < 0$. But we won't prove those assertions here. Of course if $\alpha < 0$ then $(1+x)^\alpha$ is undefined for $x = -1$, so it can't have a Maclaurin Series valid at this point.

Example 6.7. Find the square root of 1.5 correct to 3 decimal places.

Solution. We apply the Maclaurin Series (i.e. the binomial series) for $(1 + x)^\alpha$ with $x = \frac{1}{2}$ and $\alpha = \frac{1}{2}$. Taking terms up to x^5 gives the approximate value

$$1 + \frac{1}{4} - \frac{1}{32} + \frac{1}{128} - \frac{5}{2048} + \frac{7}{8192} = 1.22498 \text{ to 5 decimal places.}$$

The remainder term $R_6(\frac{1}{2})$ can be estimated using the Lagrange form of the remainder (it's the easiest version). This gives

$$R_6(\tfrac{1}{2}) = -\frac{21}{1024}\xi^6, \text{ where } 0 < \xi < \tfrac{1}{2}.$$

So $R_6(\frac{1}{2})$ is negative and lies between 0 and $-21/65536 = -0.00032$ to 5 decimal places. It follows that the square root of 1.5 lies between (approximately) 1.22466 and 1.22498. ["approximately" because rounding errors may affect the fifth decimal place.] The value correct to 3 decimal places is 1.225.

You can use the result to determine y^α for values of y outside the interval $(0, 2)$. For example, to compute the square root of 20 we can write $20 = 16 \times 1.25$, so $20^{\frac{1}{2}} = 4 \times (1.25)^{\frac{1}{2}}$, and then use the Maclaurin Series for $(1 + x)^{\frac{1}{2}}$ with $x = \frac{1}{4}$. However, using Maclaurin Series like this, or with similar tricks, is probably not the best way of computing roots.

EXERCISES 6.4

1. Sketch $f(x) = \sin(x)$ and Taylor polynomials $T_n(x) = \sum_{r=0}^{n-1} \frac{x^r}{r!} f^{(r)}(0)$ for the expansion of $\sin(x)$ about the point $a = 0$ for $n = 4, 6$ and 8 (i.e. of orders $3, 5$ and 7) on the same set of axes. The purpose of this question is to give you some idea of how closely Taylor polynomials can approximate a function. Assume that the functions $\sin(x)$ and $\cos(x)$ have all their usual properties. [You can use a graph plotting package.]

2. With the same notation as the previous question, calculate $T_8(0.1)$ to 16 decimal places. Show that the remainder term $R_8(0.1)$ lies between 0 and -2.481×10^{-13} (use Lagrange's form for the remainder). Check that $\sin(x)$ is indeed given by $T_8(0.1)$, correct to 12 decimal places.

3. With $f(x) = \sin(x)$, find an approximation (correct to 6 decimal places) to the value of $\sin(50^{\,o})$ by expanding the function in a Taylor series about the value $a = \pi/4$. Again, assume that the functions $\sin(x)$ and $\cos(x)$ have all their usual properties.

4. Show that if $|x| > 1$ and if α is a Rational Number then

$$(1+x)^\alpha = x^\alpha \sum_{r=0}^{\infty} \binom{\alpha}{r} \left(\frac{1}{x}\right)^r .$$

Take the first 5 terms of the series to get an approximate value for $\sqrt{5}$. Compare this with what you get on a calculator.

5. This lengthy question concerns the Maclaurin series of the function

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

You will show that the Maclaurin series converges, but not to $f(x)$. You may assume that $\exp(x) > 0$ for every $x \in \mathbb{R}$, $\exp(0) = 1$, $\exp(x) > 1$ for $x > 0$, and that $\exp$ is differentiable on $\mathbb{R}$ with derivative $\exp'(x) = \exp(x)$. The first four parts of this question enable you to determine the derivatives $f^{(n)}(0)$. Easier arguments become possible once we have defined the exponential function and investigated some of its properties, which we will do formally in the next chapter.

(a) Use induction on n to prove that if $x > 0$ and if n is a non-negative integer then $g_n(x) = \exp(x) - \frac{x^n}{n!}$ is strictly positive.

(b) Use the result of part (a) to prove that if $u > 0$ and if n is a non-negative integer then $\frac{\exp(u)}{u^n} > \frac{u^n}{(2n)!}$, and hence that $\frac{\exp(u)}{u^n} \to \infty$ as $u \to \infty$.

(c) Use the result of part (b) to prove that, for any non-negative integer n, $\frac{\exp(-1/x^2)}{x^{2n}} \to 0$ as $x \to 0$.

(d) Use the result of part (c) to prove that, for any non-negative integer r, $\frac{\exp(-1/x^2)}{x^r} \to 0$ as $x \to 0$.

(e) Use the result of part (d) to prove that f is differentiable on $\mathbb{R}$ with derivative

$$f'(x) = \begin{cases} 2x^{-3}\exp(-1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases} .$$

Then show that f is twice differentiable on $\mathbb{R}$ with derivative

$$f''(x) = \begin{cases} (4x^{-6} - 6x^{-4}) \exp(-1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

More generally, prove that for $n \geq 1$ the function f is n times differentiable on $\mathbb{R}$ with a derivative of the form

$$f^{(n)}(x) = \begin{cases} \phi_n(x) \exp(-1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases},$$

where $\phi_n(x)$ is a finite sum of multiples of negative powers of x.

(f) Deduce that the Maclaurin series of $f(x)$ converges, but not to $f(x)$ (except for the trivial case $x = 0$).

(g) Sketch the graph of $f(x)$ for $x \in [-5, 5]$. [You can use a graph plotting package.]

6. Suppose that $s(x)$ and $c(x)$ are differentiable functions satisfying the conditions

$$s'(x) = c(x),\ c'(x) = -s(x),\ s(0) = 0,\ c(0) = 1.$$

Obtain the Maclaurin series for $s(x)$ and $c(x)$, and prove that they are valid for all $x \in \mathbb{R}$. [Of course these functions are really $\sin(x)$ and $\cos(x)$.]

6.5 Power series and differentiation

Wouldn't it be nice if we could differentiate an infinite series of differentiable functions term by term? Given $F(x) = \displaystyle\sum_{n=0}^{\infty} f_n(x)$, where each function f_n is differentiable, it would be nice to be able to assert that F is itself differentiable and that its derivative is $F'(x) = \displaystyle\sum_{n=0}^{\infty} f_n'(x)$. After all, this is what happens for a finite sum. Well, the bad news is that is not always true for an infinite sum, but the good news is that it is true for power series. The precise result for power series is given in Theorem 6.15 below. I just remind you that every power series has a radius of convergence R (which can be 0 or ∞) and that a power series with radius of convergence $R > 0$ converges *absolutely* within its radius of convergence. The initial step in proving the main result is the following theorem.

Theorem 6.14. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. Then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also has radius of convergence R.

Proof. We establish the result by showing (a) that $\sum_{n=1}^{\infty} na_n x^{n-1}$ diverges if $|x| > R$ and (b) that $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges if $|x| < R$.

(a) In the case $R = \infty$, there is nothing to prove. So assume that $R \neq \infty$. Take any x such that $|x| > R$ and suppose that $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges. Then $na_n x^{n-1} \to 0$ as $n \to \infty$, and so there exists a constant $A > 0$ such that $|na_n x^{n-1}| < A$ for every $n \in \mathbb{N}$, and hence $|a_n x^n| < A|x|$. Now take any y between $|x|$ and R, for example $y = (|x| + R)/2$, so that $|x| > y > R$. Then

$$|a_n y^n| = |a_n x^n| \left|\frac{y}{x}\right|^n < A|x| \left|\frac{y}{x}\right|^n.$$

Hence, using the comparison test with the geometric series $\sum_{n=0}^{\infty} \left|\frac{y}{x}\right|^n$, which has common ratio $\left|\frac{y}{x}\right| < 1$, we see that $\sum_{n=1}^{\infty} |a_n y^n|$ converges. But since $y > R$, this contradicts the assumption that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. It follows that $\sum_{n=1}^{\infty} na_n x^{n-1}$ diverges if $|x| > R$.

(b) In the case $R = 0$ there is nothing to prove. So assume that $R > 0$. Take any x such that $0 < |x| < R$. Again choose y between $|x|$ and R, so that $|x| < y < R$. Then $\sum_{n=0}^{\infty} a_n y^n$ converges so that $a_n y^n \to 0$ as $n \to \infty$, and consequently there exists a constant $A > 0$ such that $|a_n y^n| < A$ for every $n \in \mathbb{N}$. It follows that

$$|na_n x^{n-1}| = n\frac{|a_n y^n|}{|x|} \left|\frac{x}{y}\right|^n < n\frac{A}{|x|} \left|\frac{x}{y}\right|^n.$$

Taking n^{th} roots, this gives

$$|na_n x^{n-1}|^{\frac{1}{n}} < n^{\frac{1}{n}} \left(\frac{A}{|x|}\right)^{\frac{1}{n}} \left|\frac{x}{y}\right| \to \left|\frac{x}{y}\right| \text{ as } n \to \infty.$$

Since $\left|\frac{x}{y}\right| < 1$, it follows from Cauchy's n^{th} root test that $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges for any x with $|x| < R$. □

Of course the result can be reapplied repeatedly. A single repetition gives the following corollary.

Corollary 6.14.1. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. Then $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ also has radius of convergence R.

We obtained the result of Theorem 6.14 with differentiation in mind. So far we haven't mentioned integration, the subject of a future chapter. But with integration in mind we state the following corollary.

Corollary 6.14.2. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. Then $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also has radius of convergence R.

Proof. Define $b_0 = 0$, and $b_{n+1} = \frac{a_n}{n+1}$ for $n = 0, 1, 2, \ldots$. Then the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ can be expressed as $\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} b_n x^n$. By Theorem 6.14, if this series has radius of convergence R', then the series $\sum_{n=0}^{\infty} n b_n x^{n-1}$ also has radius of convergence R'. However,

$$\sum_{n=0}^{\infty} n b_n x^{n-1} = \sum_{n=1}^{\infty} n \frac{a_{n-1}}{n} x^{n-1} = \sum_{n=0}^{\infty} a_n x^n.$$

So $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R'. But, by assumption, the radius of convergence of this series is R, so we deduce that $R' = R$. $\square$

Comment. Once we have proved the next theorem it will follow that $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is a primitive for $f(x) = \sum_{n=0}^{\infty} a_n x^n$, i.e. $F'(x) = f(x)$ for $|x| < R$.

Now we come to the main result of this section.

Theorem 6.15 (Term-by-term differentiation of power series)**.**
Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ with derivative $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
[This clearly requires that $\sum_{n=1}^{\infty} n a_n x^{n-1}$ should also have radius of convergence R, which explains the need for Theorem 6.14.]

Proof. We will prove the result by showing that for each $x \in (-R, R)$,

$$\left| \frac{f(x+h) - f(x)}{h} - \sum_{n=1}^{\infty} n a_n x^{n-1} \right| \to 0 \text{ as } h \to 0.$$

In order to do this, we need an estimate for the expression $\dfrac{(x+h)^n - x^n}{h} - n x^{n-1}$. This expression is zero for $n = 1$. For $n \geq 2$ an estimate can be obtained from the *Second* Mean Value Theorem, i.e. Taylor's Theorem with $n = 2$ (that's the n of Taylor's theorem), and Lagrange's form of the remainder, which asserts that under suitable conditions on a function g we have

$$g(x+h) = g(x) + h g'(x) + \frac{h^2}{2} g''(\xi), \text{ for some } \xi \text{ between } x \text{ and } x+h.$$

We apply this result to the function $g(x) = x^n$, which certainly satisfies the conditions of Taylor's Theorem for every positive integer $n \geq 2$. This gives

$$(x+h)^n = x^n + n h x^{n-1} + \frac{n(n-1)}{2} h^2 \xi^{n-2}, \text{ for some } \xi \text{ between } x \text{ and } x+h.$$

From this it follows that

$$\frac{(x+h)^n - x^n}{h} - nx^{n-1} = \frac{h}{2}n(n-1)\xi^{n-2}, \quad \text{for some } \xi \text{ between } x \text{ and } x+h. \tag{6.3}$$

We can now complete the proof. Take any $x \in (-R, R)$ and then take y between $|x|$ and R, so that $|x| < y < R$, giving $x \in (-y, y)$. Now take any $h \neq 0$ such that $x + h \in (-y, y)$. Note that for any number ξ between x and $x + h$ we have $|\xi| < y$. Then

$$\begin{aligned}
\Big|\frac{f(x+h)-f(x)}{h} - \sum_{n=1}^{\infty} na_n x^{n-1}\Big| &= \Big|\sum_{n=2}^{\infty} a_n\Big[\frac{(x+h)^n - x^n}{h} - nx^{n-1}\Big]\Big| \\
&\le \sum_{n=2}^{\infty} |a_n|\Big|\frac{(x+h)^n - x^n}{h} - nx^{n-1}\Big| \\
&\le \sum_{n=2}^{\infty} |a_n|\Big|\frac{h}{2}\Big|n(n-1)y^{n-2} \quad \text{by equation 6.3} \\
&= \Big|\frac{h}{2}\Big|\sum_{n=2}^{\infty} n(n-1)|a_n y^{n-2}|.
\end{aligned}$$

Since $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, it follows (see Corollary 6.14.1) that $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ also has radius of convergence R. So we deduce that $\sum_{n=2}^{\infty} n(n-1)|a_n y^{n-2}|$ converges to a finite sum, S, say. Hence

$$\Big|\frac{f(x+h)-f(x)}{h} - \sum_{n=1}^{\infty} na_n x^{n-1}\Big| \le \Big|\frac{h}{2}\Big|S \to 0 \text{ as } h \to 0.$$

Therefore

$$\frac{f(x+h)-f(x)}{h} \to \sum_{n=1}^{\infty} na_n x^{n-1} \text{ as } h \to 0,$$

which establishes that f is differentiable at each $x \in (-R, R)$ with derivative $\sum_{n=1}^{\infty} na_n x^{n-1}$. □

Corollary 6.15.1. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then f has derivatives of all orders on $(-R, R)$, and these derivatives can be obtained by term-by-term differentiation of the series.

Just before moving on to the Exercises for this section we mention that the same series can be represented in slightly different ways. Consider for example the series $S = \sum_{n=1}^{\infty} a_n$. The n that appears here is a dummy variable. It can be replaced by any other (sensible) dummy variable. If we choose to replace n

by $r + 1$, then r must vary from 0 to ∞ in order that n varies from 1 to ∞. So we can write $S = \sum_{r=0}^{\infty} a_{r+1}$, and having done that we can change the dummy variable from r back to n again, giving the somewhat strange looking result that $S = \sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$. Of course nothing here alters the fact that $S = a_1 + a_2 + a_3 + \ldots$. As another example consider $T = \sum_{n=0}^{\infty} a_{2n+1}$. Replacing n by $r - 1$ here gives $T = \sum_{r=1}^{\infty} a_{2r-1}$, and then changing r back to n leaves us with $T = \sum_{n=0}^{\infty} a_{2n+1} = \sum_{n=1}^{\infty} a_{2n-1}$. A further point is that terms that are zero can be omitted, so if $a_0 = 0$ then $\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} a_n$.

EXERCISES 6.5

1. If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, prove that $f'(x) = f(x)$.

2. If $s(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $c(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$, prove that $s'(x) = c(x)$ and $c'(x) = -s'(x)$.

3. If $l(x) = \sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^n}{n}$, prove that $l'(x) = \frac{1}{1+x}$ for $|x| < 1$.

Chapter 7

Familiar Functions

In this chapter we give rigorous definitions of the familiar functions of Real Analysis: $\exp, \sin, \cos$ and their inverses. You will have seen references to these definitions in earlier chapters and indeed we have already covered some of the proofs. But here we bring it all together. We start with the exponential function.

7.1 The exponential function

Definition 7.1. For each $x \in \mathbb{R}$ we define

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

If we put $a_n = x^n/n!$ then for $x \neq 0$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty.$$

It follows by D'Alembert's ratio test that the radius of convergence of the series defining $\exp(x)$ is infinity, so that the series converges (absolutely) for all values of x. Hence the definition is a good for every $x \in \mathbb{R}$.
Note. The ratio test is valid for complex a_n. So the series can be used to define $\exp(z)$ for $z \in \mathbb{C}$.

We proceed to verify all the usual properties of $\exp$.

Theorem 7.1 (Product rule)**.** For every $x, y \in \mathbb{R}, \exp(x) \cdot \exp(y) = \exp(x+y)$. [The result was one of the exercises in Section 4.4 of Chapter 4.]

Proof. The result follows from Theorem 4.19, which deals with the Cauchy product of two absolutely convergent series. To start rather informally, we write

$$\begin{aligned}\exp(x)\cdot\exp(y) &= \Big(1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots\Big)\times\\ &\qquad\Big(1+y+\frac{y^2}{2!}+\ldots+\frac{y^n}{n!}+\ldots\Big)\\ &= 1+(1\cdot y+x\cdot 1)+\Big(1\cdot\frac{y^2}{2!}+xy+\frac{x^2}{2!}\cdot 1\Big)+\ldots\\ &= 1+(y+x)+\frac{(y+x)^2}{2!}+\ldots\end{aligned}$$

In rather more detail, the general term in the Cauchy product is

$$1\cdot\frac{y^n}{n!}+\frac{x}{1!}\cdot\frac{y^{n-1}}{(n-1)!}+\ldots\frac{x^i}{i!}\cdot\frac{y^{n-i}}{(n-i)!}+\ldots+\frac{x^n}{n!}\cdot 1.$$

This can be written as

$$\frac{1}{n!}\Big(y^n+xy^{n-1}\frac{n!}{1!(n-1)!}+\ldots+x^iy^{n-i}\frac{n!}{i!(n-i)!}\;\ldots+x^n\Big),$$

or more suggestively as

$$\frac{1}{n!}\Big(x^0y^n\binom{n}{0}+x^1y^{n-1}\binom{n}{1}+\ldots+x^iy^{n-i}\binom{n}{i}+\ldots+x^ny^0\binom{n}{n}\Big).$$

But the sum

$$x^0y^n\binom{n}{0}+x^1y^{n-1}\binom{n}{1}+\ldots+x^iy^{n-i}\binom{n}{i}+\ldots+x^ny^0\binom{n}{n}$$

is just the binomial expansion of $(x+y)^n$. It follows that

$$\exp(x)\cdot\exp(y)=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!}=\exp(x+y). \qquad \square$$

Theorem 7.2 (Differentiation)**.** The function $\exp(x)$ is differentiable on $\mathbb{R}$ and $\exp'(x)=\exp(x)$. [The result was one of the exercises in Section 6.5 of Chapter 6.]

Proof. The result follows from Theorem 6.15 which asserts that a power series may be differentiated term-by-term within its radius of convergence. Applying term-by-term differentiation gives

$$\exp'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}=\exp(x). \qquad \square$$

Theorem 7.3 (Basic properties of $\exp$)**.**

(a) $\exp(0) = 1$ and $\exp(x) > 0$ for all $x \in \mathbb{R}$.

(b) $\exp(x)$ is strictly increasing on $\mathbb{R}$.

(c) $\exp(x) \to +\infty$ as $x \to +\infty$.

(d) $\exp(x) \to 0$ as $x \to -\infty$.

(e) For any Rational Number k, $\frac{\exp(x)}{x^k} \to +\infty$ as $x \to +\infty$. [This really says that $\exp(x)$ grows faster than any power of x as x increases.]

Proof.

(a) Put $x = 0$ in the defining power series to see that $\exp(0) = 1$. Clearly if $x > 0$ then each term $\frac{x^n}{n!}$ is positive, and so $\exp(x) > 0$ if $x > 0$. But $\exp(x) \cdot \exp(-x) = \exp(x - x) = \exp(0) = 1$, so if $x < 0$ we have $\exp(x) = \frac{1}{\exp(-x)} > 0$.

(b) For every $x \in \mathbb{R}$ we have $\exp'(x) = \exp(x) > 0$. It follows that $\exp(x)$ is strictly increasing on $\mathbb{R}$.

(c) For $x > 0$, from the power series definition, $\exp(x) > x \to +\infty$ as $x \to +\infty$.

(d) For $x < 0$, $\exp(x) = \frac{1}{\exp(-x)} \to 0$ as $x \to -\infty$.

(e) Given a Rational number k, let n denote the first positive integer greater than k. Then for $x > 0$ we have

$$\frac{\exp(x)}{x^k} > \frac{x^n/n!}{x^k} = \frac{x^{n-k}}{n!} \to +\infty \text{ as } x \to +\infty. \qquad \square$$

The graph of the function $\exp(x)$ is illustrated in Figure 7.1.

Definition 7.2. We define Euler's number e to be $\exp(1)$, so that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots.$$

An approximate value for e can be obtained from the defining series. This gives $e \approx 2.718\,281\,828$ (correct to 9 decimal places). In fact e is an irrational number, as we prove in the next theorem.

Theorem 7.4. The number e is irrational.

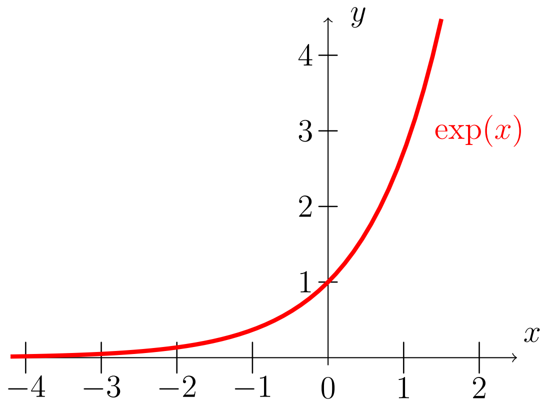


Figure 7.1: The exponential function.

Proof. Suppose that e is a Rational Number. Then we may write $e = \frac{p}{q}$ where p and q are positive integers without a common factor. Since e lies between 2 and 3, it is not an integer, so we may assume that $q \neq 1$. Consider the equation

$$e = \frac{p}{q} = 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{q!} + \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \ldots.$$

Multiply this by $q!$ to get

$$q!\frac{p}{q} = q!\Big(1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{q!}\Big) + \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \ldots.$$

This can be written as

$$q!\Big(\frac{p}{q} - 1 - 1 - \frac{1}{2!} - \ldots - \frac{1}{q!}\Big) = \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \ldots.$$

The left-hand side of this equation is an integer. The right-hand side is greater than zero and less than the sum of the geometric series

$$\frac{1}{(q+1)} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \ldots = \Big(\frac{1}{q+1}\Big)\frac{1}{1 - \frac{1}{q+1}} = \frac{1}{q} < 1.$$

So we have a contradiction and conclude that e is not a Rational Number. □

Comment. It can be shown that e^r is an irrational number for each non-zero Rational Number r. [Aigner, Martin; Ziegler, Günter M. (1998). Proofs from The Book (4th ed.). Berlin, New York: Springer-Verlag. pp. 27–36. doi:10.1007/978-3-642-00856-6. ISBN 978-3-642-00855-9.]

Theorem 7.5. If $x \in \mathbb{R}$ and if r is a Rational Number then $(\exp(x))^r = \exp(rx)$. In particular (taking $x = 1$),

$$e^r = \exp(r).$$

Proof. The result is trivial if $r = 0$ or if $r = 1$. So suppose initially that r is a positive integer greater than 1. Then using Theorem 7.1 $(r-1)$ times we get

$$\begin{aligned}(\exp(x))^r &= (\exp(x))(\exp(x))\ldots(\exp(x)) \;\; (r \text{ factors})\\ &= \exp(rx).\end{aligned}$$

Next suppose that r is a negative integer. Then

$$\begin{aligned}(\exp(x))^r &= \frac{1}{(\exp(x))^{-r}}\\ &= \frac{1}{\exp(-rx)} \quad (\text{since } -r \text{ is a positive integer})\\ &= \exp(rx).\end{aligned}$$

Next suppose that $r = \frac{1}{q}$, where q is a positive integer. Then

$$\begin{aligned}\big(\exp(rx)\big)^q &= \Big(\exp\Big(\frac{x}{q}\Big)\Big)^q\\ &= \exp\Big(\frac{qx}{q}\Big) \quad (\text{since } q \text{ is a positive integer})\\ &= \exp(x), \quad \text{and taking } q^{\text{th}} \text{ roots of this gives}\\ \exp(rx) &= \big(\exp(x)\big)^{\frac{1}{q}}\\ &= (\exp(x))^r.\end{aligned}$$

Finally suppose that $r = \frac{p}{q}$, where p is an integer and q is a positive integer. Then

$$\begin{aligned}\exp(rx) &= \exp\Big(\frac{px}{q}\Big)\\ &= \Big(\exp\Big(\frac{x}{q}\Big)\Big)^p \quad (\text{since } p \text{ is an integer})\\ &= \big(\exp(x)\big)^{\frac{p}{q}} \quad (\text{since } q \text{ is a positive integer})\\ &= (\exp(x))^r.\end{aligned}$$

□

Comment. We have just proved that $e^r = \exp(r)$ for any Rational Number r. But we can't assert that $e^\alpha = \exp(\alpha)$ for an irrational number α, because we have no definition of irrational powers such as e^α. But $\exp(\alpha)$ is well-defined, so we do the natural thing and make the following definition.

Definition 7.3. If α is an irrational number, then we *define* e^α to be $\exp(\alpha)$.

In view of this definition and the preceding theorem we have $e^x = \exp(x)$ for any Real Number x, be it rational or irrational. So Theorem 7.1 can be written as $e^x \cdot e^y = e^{(x+y)}$ and Theorem 7.2 can be written as $y = e^x \implies \frac{dy}{dx} = e^x$.

Comment. We have just defined our first irrational power, namely an irrational power of the number e. But we still have no definition of a^x if $a \neq e$ and x is irrational. We will deal with this issue after discussing the logarithm function, which is the inverse of the exponential function. In Chapter 3 we showed that the sequence $\Big((1 + \frac{1}{n})^n)\Big)$ converges to a limit between 2 and 3. We will prove that the limit is e, again after discussing the logarithm function.

EXERCISES 7.1

1. Use the defining series to obtain the value of e correct to 4 decimal places and prove that your value is that accurate.

7.2 The logarithm function

The logarithm function $\log_e(x)$ is defined as the inverse of the exponential function $\exp(x)$, which is continuous and strictly increasing. Recall results about inverse functions from earlier chapters, especially Chapter 5, Theorem 5.12 and Chapter 6, Theorem 6.5. The exponential function has domain $\mathbb{R}$ and its image set is the interval $(0, \infty)$ because $\exp(x) \to 0$ as $x \to -\infty$ and $\exp(x) \to +\infty$ as $x \to +\infty$. Therefore, $\log_e(x)$ has domain $(0, \infty)$ and image set $\mathbb{R}$. Note that this means that $\log_e(x)$ is undefined for $x \leq 0$. Because $\exp(x)$ is strictly increasing, continuous and differentiable, so is $\log_e(x)$.

If $\eta = \exp(\xi)$ then $\xi = \log_e(\eta)$ and

$$\log_e'(\eta) = \frac{1}{\exp'(\xi)} = \frac{1}{\exp(\xi)} = \frac{1}{\eta}.$$

Put another way, if $y = \log_e(x)$ then $\frac{dy}{dx} = \frac{1}{x}$.

Some authors denote the function $\log_e$ simply as $\log$, others use $\ln$. We will call it "log(arithm) to the base e", others call it the "natural log(arithm)", which explains the alternative notation $\ln$. Key properties of the logarithm function can be deduced fairly easily from properties of the exponential function. Figure 7.2 illustrates the graph of $\log_e(x)$.

Theorem 7.6 (Basic properties of $\log_e$)**.**

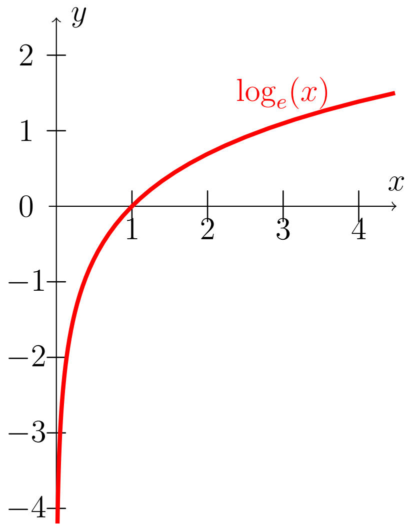


Figure 7.2: The logarithm function.

(a) $\log_e(1) = 0$, $\log_e(e) = 1$, and if $x, y > 0$ then $\log_e(xy) = \log_e(x) + \log_e(y)$, and $\log_e\left(\frac{1}{x}\right) = -\log_e(x)$

(b) For any Rational Number r and $x > 0$, $\log_e\left(x^r\right) = r\log_e(x)$

(c) $\log_e(x) \to +\infty$ as $x \to +\infty$.

(d) $\log_e(x) \to -\infty$ as $x \to 0+$.

(e) For any positive Rational Number k, $\frac{\log_e(x)}{x^k} \to 0$ as $x \to +\infty$. [This really says that $\log_e(x)$ grows slower than any positive power of x as x increases.]

Proof.

(a) Since $\exp(0) = 1$, $\log_e(1) = 0$.
Since $\exp(1) = e$, $\log_e(e) = 1$.
For $x, y > 0$ put $X = \log_e(x)$ and $Y = \log_e(y)$, so that $x = \exp(X)$ and $y = \exp(Y)$. Then $xy = \exp(X) \cdot \exp(Y) = \exp(X + Y)$, so $\log_e(xy) = X + Y = \log_e(x) + \log_e(y)$.
With the same notation, $\dfrac{1}{x} = \dfrac{1}{\exp(X)} = \exp(-X)$, so $\log_e\left(\frac{1}{x}\right) = -X = -\log_e(x)$.

(b) Suppose that $r = \frac{p}{q}$ where p is an integer and q is a positive integer. Then with $X = \log_e(x)$,

$$(x^r)^q = x^p = \big(\exp(X)\big)^p = \exp(pX) = \left(\exp\left(\frac{pX}{q}\right)\right)^q.$$

Taking q^{th} roots gives $x^r = \exp(rX)$, and then taking logarithms gives $\log_e\left(x^r\right) = rX = r\log_e(x)$.

(c) Choose $A \in \mathbb{R}$. We will prove that if $x > \exp(A)$ then $\log_e(x) > A$. To do this, observe that since $\exp$ is strictly increasing, if $\exp(X) > \exp(A)$ then $X > A$. With $X = \log_e(x)$ this implies that if $\exp(\log_e(x)) > \exp(A)$ then $\log_e(x) > A$, i.e. if $x > \exp(A)$, then $\log_e(x) > A$.

(d) Choose $A \in \mathbb{R}$. We will prove that if $0 < x < \exp(A)$ then $\log_e(x) < A$. To do this, observe that since $\exp$ is strictly increasing, if $\exp(X) < \exp(A)$ then $X < A$. With $X = \log_e(x)$ this implies that if $\exp(\log_e(x)) < \exp(A)$ then $\log_e(x) < A$, i.e. if $x < \exp(A)$, then $\log_e(x) < A$.

(d) From part (e) of Theorem 7.3 we have

$$\frac{\exp(kX)}{X} \to +\infty \text{ as } X \to +\infty.$$

With $X = \log_e(x)$, and noting that $X \to +\infty$ as $x \to +\infty$, we obtain

$$\frac{\exp(k\log_e(x))}{\log_e(x)} \to +\infty \text{ as } x \to +\infty,$$

$$\text{i.e. } \frac{x^k}{\log_e(x)} \to +\infty \text{ as } x \to +\infty,$$

which gives $\dfrac{\log_e(x)}{x^k} \to 0$ as $x \to +\infty$. □

Comment. In days gone by (pre the 1960s), logarithms were used extensively to simplify multiplications of ugly numbers. Without any calculator how would you multiply (for example) 376.8 by 27.91? You could do it by long multiplication, but much easier to consult a book of tables, look up the logarithms of the two numbers, add them, and then look up the "anti-logarithm" (effectively the exponential) of the sum. Because addition is much simpler to do by hand than multiplication, this was by far the best method. The logarithms used were to base 10, rather than to base e, and the associated function is $\log_{10}(x)$, the inverse of the function 10^x. Base 10 is preferable for calculations since we count in the decimal system using base 10. We will now explain how to define functions such as 10^x and $\log_{10}(x)$, more generally a^x and $\log_a(x)$ for any $a > 0$. Note that we already know what a^r means for an Rational Number r, and we already know what e^x means even when x is irrational (it means $\exp(x)$).

Definition 7.4. If $a > 0$ and x is an irrational number, we define

$$a^x = \exp(x\log_e(a)).$$

We must check a few things for consistency. First, if $a = e$, this definition gives $e^x = \exp(x \log_e(e))$, but $\log_e(e) = 1$, so it gives $e^x = \exp(x)$, a definition we had earlier. Second, if x is rational then $a^x = \Big(\exp(\log_e(a))\Big)^x = \exp(x \log_e(a))$ by Theorem 7.5. Hence

$$a^x = \exp(x \log_e(a)) \text{ for any } x \in \mathbb{R}.$$

If we take logarithms of this we get another useful result

$$\log_e(a^x) = \log_e \Big(\exp(x \log_e(a))\Big) = x \log_e(a).$$

Using the results in this and in the previous section it is easy to verify all the usual properties of a^x for $a > 0$. In particular

$$\begin{aligned} a^x \cdot a^y &= a^{(x+y)}, \\ (a^x)^y &= a^{xy}, \\ \frac{d}{dx}(a^x) &= \log_e(a) \cdot a^x. \end{aligned}$$

Moreover, if $x > 0$ and α is an irrational number then

$$\begin{aligned} \frac{d}{dx}(x^\alpha) &= \frac{d}{dx}(\exp(\alpha \log_e(x)) \\ &= \big(\frac{\alpha}{x}\big)\big(\exp(\alpha \log_e(x)\big) \text{ by the composite rule} \\ &= \big(\frac{\alpha}{x}\big)x^\alpha = \alpha x^{\alpha-1}. \end{aligned}$$

By combining this with the result of Corollary 6.5.3 from Chapter 6, we find that the derivative of x^α is $\alpha x^{\alpha-1}$ whenever the function and derivative make sense, whether or not α is a Rational or irrational number.

In Figure 7.3 we show the graphs of $2^x, e^x$ and 4^x on the same set of axes. You will see they are somewhat similar, but the gradients at $x = 0$ are respectively less than 1, equal to 1, and greater than 1. Another way of characterising e is that it is the unique positive number a for which a^x has gradient 1 at $x = 0$.

It is also easy to show that for $a > 1$ the function a^x has an inverse. We denote this inverse by $\log_a(x)$, it is defined and differentiable on $(0, \infty)$. Key values are $\log_a(1) = 0$ and $\log_a(a) = 1$. For $x, y > 0$, we have $\log_a(xy) = \log_a(x) + \log_a(y)$ and for any $\alpha \in \mathbb{R}$, $\log_a(x^\alpha) = \alpha \log_a(x)$.

We can get a formula for $\log_a(x)$ in terms of $\log_e(x)$ as follows. Put $y = \log_a(x)$. Then $x = a^y$, so $\log_e(x) = \log_e(a^y) = y \log_e(a) = \log_a(x) \log_e(a)$. Hence

$$\log_a(x) = \frac{\log_e(x)}{\log_e(a)}.$$

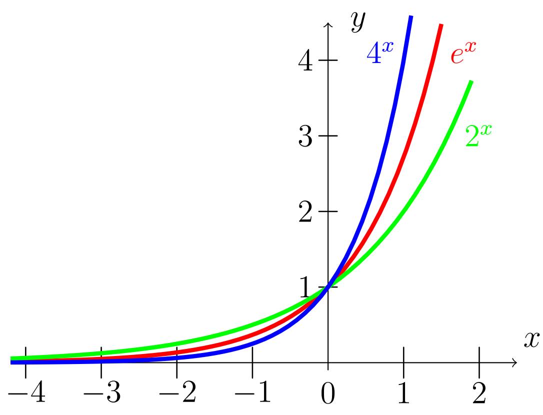


Figure 7.3: The functions $2^x, e^x, 4^x$.

Use of logarithm tables.
In days of yore tables were produced showing $\log_{10}(x)$ for $x \in (1, 10)$. About 20 A4 pages would be required to display a tabulation from $x = 1.001$ to $x = 9.999$ in steps of 0.001, with the values of $\log_{10}(x)$ given correct to 6 decimal places. To calculate 376.8×27.91, we read $\log_{10}(3.768) = 0.576111$ and $\log_{10}(2.791) = 0.445760$. Adding these values gives 1.021871. Look-up the number with logarithm 0.021871, which is 1.05165 (some interpolation needed here), multiply by 10 to take account that the logarithm was 1.021871 rather than 0.021871, to get 10.5165. Finally take account of the fact that we dropped a factor 10^3 to get the answer $10\,516.5$. This compares with the precise value $10\,516.488$.

Because we normally count in the decimal system, some computations are much easier using logarithms to base 10 than to base e. For example to calculate the cube root of 83.67, we look up the logarithm (to base 10) of 8.367, which is 0.922570, add 1 (using $\log_{10}(83.67) = \log_{10}(10) + \log_{10}(8.367)$) to get 1.922570, and then divide by 3 (for the cube root) to get 0.640857. Finally look up the number with this logarithm to get the answer 4.3738. The process of extracting an integer power of 10 from 83.67 is very easy, but if we used logarithms to base e we'd either have to extract an integer power of e, which is considerably more difficult, or know the value of $\log_e(10)$ and undertake an awkward addition.

That limit again.
We said in Chapter 3, section 3.6 that we would prove that

$$\left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty.$$

We actually prove a bit more than this in the following result.

Theorem 7.7. For any $a \in \mathbb{R}$,

$$\left(1 + \frac{a}{x}\right)^x \to e^a \text{ as } x \to +\infty.$$

This entails

$$\left(1 + \frac{a}{n}\right)^n \to e^a \text{ as } n \to \infty,$$

where n takes positive integer values. And taking $a = 1$ in this gives

$$\left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty.$$

Proof. We start by considering the limiting value (if any) of $\dfrac{\log_e(1 + ay)}{y}$ as $y \to 0$. This is an indeterminate form since both $\log_e(1 + ay)$ and y are differentiable functions of y with common value 0 at $y = 0$. These functions satisfy the conditions for applying L'Hôpital's rule, their derivatives are respectively $\frac{a}{1+ay}$ and 1 and so we find that

$$\lim_{y \to 0} \frac{\log_e(1 + ay)}{y} = \lim_{y \to 0} \frac{\frac{a}{1+ay}}{1},$$

provided that the latter limit exists, which it does because $1 + ay \to 1$ as $y \to 0$. So we have

$$\frac{\log_e(1 + ay)}{y} \to a \text{ as } y \to 0.$$

If we write $y = \frac{1}{x}$ then $y \to 0$ as $x \to +\infty$, so that

$$x \log_e \left(1 + \frac{a}{x}\right) \to a \text{ as } x \to +\infty.$$

The exponential function $\exp$ is continuous at a, so

$$\exp\left(x \log_e \left(1 + \frac{a}{x}\right)\right) \to e^a \text{ as } x \to +\infty.$$

But

$$\exp\left(x \log_e \left(1 + \frac{a}{x}\right)\right) = \exp\left(\log_e \left(\left(1 + \frac{a}{x}\right)^x\right)\right) = \left(1 + \frac{a}{x}\right)^x.$$

Hence

$$\left(1 + \frac{a}{x}\right)^x \to e^a \text{ as } x \to +\infty. \qquad \square$$

Comment. The result also holds good for $x \to -\infty$. To see this, note that, in the proof, $y \to 0$ as $x \to -\infty$. Alternatively, replace x by $-x$ and a by $-a$ in the result.

Although it would be nice to give a Maclaurin series for $\log_e(x)$, we cannot do this because $\log_(x)$ is undefined for $x \leq 0$, so it is hopeless trying to get a power series for $\log_e(x)$ valid in some open interval containing 0. So we do the next best thing and try for a Maclaurin series for $\log_e(1+x)$. Hopefully that might converge for $|x| < 1$, although we can't allow $|x| > 1$ because then $1 + x$ can be less than 0.

Theorem 7.8. For $-1 < x \leq 1$,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

Proof. Note that if $f(x) = \log_e(1+x)$ then $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$, $f''(x) = (-1)(1+x)^{-2}$, $f^{(3)}(x) = (-2)(-1)(1+x)^{-3}$, and more generally for $n \geq 1$, $f^{(n)}(x) = (-1)^{n-1}\big((n-1)!\big)(1+x)^{-n}$. It follows that for $n \geq 1$, we have $f^{(n)}(0) = (-1)^{n-1}(n-1)!$, while $f(0) = \log_e(1) = 0$. Consequently the Maclaurin series for f is

$$\sum_{n=1}^{\infty} (-1)^{n-1}(n-1)! \frac{x^n}{n!} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

However, we still have to prove that this converges to $f(x)$. To do this we examine the remainder term $R_n(x)$ in the Taylor expansion of the function about the point 0. We split the argument into two cases. The first deals with $|x| < 1$, and the second deals with the special case $x = 1$.

In the case $|x| < 1$ we use Cauchy's formula for the n^{th} remainder, which tells us that $R_n(x)$ can be expressed in the following form for some $\theta \in (0, 1)$.

$$\begin{aligned} R_n(x) &= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x) \\ &= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} (-1)^{n-1}\big((n-1)!\big)(1+\theta x)^{-n} \\ &= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{-1}. \end{aligned}$$

But for $|x| < 1$ we have $0 < (1-\theta)/(1+\theta x) < 1$, and $1 - |x| < 1 + \theta x$ so that $(1+\theta x)^{-1}$ is bounded above by the $(1-|x|)^{-1}$. It follows that

$$|R_n(x)| \leq \frac{|x|^n}{1-|x|} \to 0 \text{ as } n \to \infty.$$

Hence the Maclaurin series for $\log_e(1+x)$ converges to $\log_e(1+x)$ for $|x| < 1$.

In the special case $x = 1$ we use Lagrange's formula for the n^{th} remainder, which tells us that $R_n(x)$ can be expressed in the following form for some $\xi \in (0,1)$.

$$\begin{aligned} R_n(1) &= \frac{1^n}{n!} f^{(n)}(\xi) \\ &= \frac{1}{n!}(-1)^{n-1}\big((n-1)!\big)(1+\xi)^{-n} \\ &= \frac{(-1)^{n-1}}{n}(1+\xi)^{-n} \end{aligned}$$

It follows that $|R_n(1)| \leq \frac{1}{n} \to 0$ as $n \to \infty$. Hence the Maclaurin series for $\log_e(1+x)$ converges to $\log_e(1+x)$ for $x = 1$. In other words, the sum of the alternating harmonic series is $\log_e(2)$:

$$\log_e(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}. \qquad \square$$

EXERCISES 7.2

1. Prove that for $a, b > 1$, $\log_a(b) = \frac{1}{\log_b(a)}$.
2. Prove that for $|x| < 1$,

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\Big(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\Big) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.$$

Hence show that if $y > 0$ then

$$\log_e(y) = 2\sum_{n=0}^{\infty} \frac{1}{2n+1} \cdot \left(\frac{y-1}{y+1}\right)^{2n+1}.$$

Use this series to obtain an approximate value for $\log_e(2)$ with an error less than 10^{-5}. How many terms of the alternating harmonic series would be needed to achieve similar accuracy?

3. Prove that the rearranged alternating harmonic series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots,$$

where each positive term is followed by two negative terms, converges with sum $\frac{1}{2}\log_e(2)$.

7.3 Circular or trigonometric functions

The principal functions treated in this section are the sine and cosine functions. These are known as circular or trigonometric functions. We define them using power series, just as we did for the exponential function. After deducing their main properties we examine the connection with the familiar geometric definitions of these functions in terms of circles and triangles. From the Real Analysis point of view these geometric definitions are not sufficiently precise because they rely on appealing to pictures and intuition.

Definition 7.5. For each $x \in \mathbb{R}$ we define

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots, \quad \text{and}$$
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

If we put $a_n = (-1)^n x^{2n+1}/(2n+1)!$ then for $x \neq 0$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x^2|}{(2n+3)(2n+2)} \to 0 \text{ as } n \to \infty.$$

It follows by D'Alembert's ratio test that the radius of convergence of the series defining $\sin(x)$ is infinity, so that the series converges (absolutely) for all values of x.

Similarly, if we put $b_n = (-1)^n x^{2n}/(2n)!$ then for $x \neq 0$

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{|x^2|}{(2n+2)(2n+1)} \to 0 \text{ as } n \to \infty.$$

It follows by D'Alembert's ratio test that the radius of convergence of the series defining $\cos(x)$ is infinity, so that the series converges (absolutely) for all values of x.

Hence these definitions of $\sin(x)$ and $\cos(x)$ are good for every $x \in \mathbb{R}$. Some properties are easily seen from the defining series. We have $\sin(0) = 0$, $\cos(0) = 1$, $\sin$ is an odd function, i.e. $\sin(-x) = -\sin(x)$, while $\cos$ is an even function, i.e. $\cos(-x) = \cos(x)$.

Note. The ratio test is valid for complex a_n and b_n. So the series can be used to define $\sin(z)$ and $\cos(z)$ for $z \in \mathbb{C}$.

We proceed to verify all the usual properties of $\sin$ and $\cos$.

Theorem 7.9 (Addition formulae)**.** For every $x, y \in \mathbb{R}$,

$$\begin{aligned} &\text{(a)} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y), \\ &\text{(b)} \quad \cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y). \end{aligned}$$

[If you had a very strong stomach for algebra you may have proved the first of these in one of the exercises in Section 4.4 of Chapter 4.]

Proof. The result follows from Theorem 4.19, which deals with the Cauchy product of two absolutely convergent series. You are strongly advised to look back at the general term in the product $\exp(x)\exp(y)$ from the proof of Theorem 7.1. The algebra here is similar, but nastier.

(a) The $(n+1)^{\text{th}}$ term in the Cauchy product of $\sin(x)\cos(y)$ is

$$\begin{aligned} u_n &= x \cdot \frac{(-1)^n y^{2n}}{(2n)!} - \frac{x^3}{3!} \cdot \frac{(-1)^{n-1} y^{2n-2}}{(2n-2)!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} \cdot 1 \\ &= (-1)^n \Big[\frac{xy^{2n}}{(2n)!} + \frac{x^3 y^{2n-2}}{3!(2n-2)!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} \Big] \\ &= \frac{(-1)^n}{(2n+1)!} \Big[\binom{2n+1}{1} xy^{2n} + \binom{2n+1}{3} x^3 y^{2n-2} + \\ &\qquad \ldots + \binom{2n+1}{2n+1} x^{2n+1} \Big] \end{aligned}$$

The $(n+1)^{\text{th}}$ term in the Cauchy product of $\cos(x)\sin(y)$ is

$$\begin{aligned} v_n &= 1 \cdot \frac{(-1)^n y^{2n+1}}{(2n+1)!} - \frac{x^2}{2!} \cdot \frac{(-1)^{n-1} y^{2n-1}}{(2n-1)!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} \cdot y \\ &= (-1)^n \Big[\frac{y^{2n+1}}{(2n+1)!} + \frac{x^2 y^{2n-1}}{2!(2n-1)!} + \ldots + \frac{x^{2n} y}{(2n)!} \Big] \\ &= \frac{(-1)^n}{(2n+1)!} \Big[\binom{2n+1}{0} y^{2n+1} + \binom{2n+1}{2} x^2 y^{2n-1} + \\ &\qquad \ldots + \binom{2n+1}{2n} x^{2n} y \Big] \end{aligned}$$

Adding these expressions for u_n and v_n gives

$$\begin{aligned} u_n + v_n = w_n &= \frac{(-1)^n}{(2n+1)!} \Big[\sum_{r=0}^{2n+1} \binom{2n+1}{r} x^r y^{2n+1-r} \Big] \\ &= \frac{(-1)^n}{(2n+1)!} (x+y)^{2n+1} \end{aligned}$$

Hence the resulting series for $\sin(x)\cos(y) + \cos(x)\sin(y)$ is

$$\sum_{n=0}^{\infty} w_n = \sum_{n=0}^{\infty} (-1)^n \frac{(x+y)^{2n+1}}{(2n+1)!} = \sin(x+y).$$

(b) The result for $\cos(x+y)$ can be proved in the same way as for part (a). You will learn more by doing it than by reading another page of algebra here. So try it for yourself. There is an easy way of getting it from part (a) by differentiation that we will cover after the next theorem (Theorem 7.10). $\square$

Theorem 7.10 (Differentiation)**.** The functions $\sin(x)$ and $\cos(x)$ are differentiable on $\mathbb{R}$, $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$. [These results formed one of the exercises in Section 6.5 of Chapter 6.]

Proof. The result follows from Theorem 6.15 which asserts that a power series may be differentiated term-by-term within its radius of convergence. Applying term-by-term differentiation gives

$$\begin{aligned}
\sin'(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos(x) \\
\cos'(x) &= \sum_{n=1}^{\infty} (-1)^n \frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} \\
&= -\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = -\sin(x)
\end{aligned}$$

$\square$

Corollary 7.10.1. For every $x, y \in \mathbb{R}$,

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

As a consequence of this, for each $x \in \mathbb{R}$, $\cos^2(x) + \sin^2(x) = 1$, $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$.

Proof. By Theorem 7.9 part(a), we have

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Differentiating with respect to x while holding y constant gives

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

If we replace y by $-x$ and bear in mind that $\cos$ is an even function, while $\sin$ is an odd function, we get

$$\cos(x - x) = \cos(x)\cos(x) + \sin(x)\sin(x).$$

But $\cos(x - x) = \cos(0) = 1$, so we find that $\cos^2(x) + \sin^2(x) = 1$. It follows that $\cos^2(x) \leq 1$ and that $\sin^2(x) \leq 1$, which gives $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$. □

Remark. The addition formulae give rise to half-angle formulae:

$$\begin{aligned}\sin(2x) &= 2\sin(x)\cos(x),\\ \cos(2x) &= \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1.\end{aligned}$$

Finding π.

How is π defined? At school it would have been defined using circles, as the ratio of the circumference to the diameter. But this really won't do for a rigorous course of Real Analysis. So here we will define π by using properties of the functions $\sin$ and $\cos$. To be precise we will define $\pi/2$ as the least positive number for which $\cos(x)$ is zero. In order to do this we must show that $\cos(x)$ is zero for some positive value(s) of x. The first step is to show that $\cos(x)$ is strictly decreasing on the interval $[0, 2]$.

Theorem 7.11. The function $\sin(x)$ is strictly positive for $x \in (0, 2)$ and the function $\cos(x)$ is strictly decreasing on $[0, 2]$.

Proof. Since $\cos'(x) = -\sin(x)$, it suffices to show that $\sin(x)$ is strictly positive for $x \in (0, 2)$. We have

$$\begin{aligned}\sin(x) &= \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \ldots\\ &= x\left(1 - \frac{x^2}{3 \times 2}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{7 \times 6}\right) + \ldots.\end{aligned}$$

For $x \in (0, 2)$ each of the bracketed terms is strictly positive, and so $\sin(x) > 0$. □

Theorem 7.12. There is a positive number, which we will denote as $\pi/2$, lying between $\sqrt{2}$ and $\sqrt{3}$ at which the cosine function takes the value zero. Moreover $\cos(x) > 0$ for $x \in [0, \pi/2)$.

Proof.

$$\cos(\sqrt{2}) = \left(1 - \frac{2}{2}\right) + \frac{2^2}{4!}\left(1 - \frac{2}{6 \times 5}\right) + \frac{2^4}{8!}\left(1 - \frac{2}{10 \times 9}\right) + \ldots > 0.$$

$$\cos(\sqrt{3}) = \left(1 - \frac{3}{2} + \frac{3^2}{4!}\right) - \frac{3^3}{6!}\left(1 - \frac{3}{8 \times 7}\right) - \frac{3^5}{10!}\left(1 - \frac{3}{12 \times 11}\right) + \dots.$$

But

$$\left(1 - \frac{3}{2} + \frac{3^2}{4!}\right) = -\frac{1}{2} + \frac{3}{8} < 0,$$

which gives $\cos(\sqrt{3}) < 0$.

Because the cosine function is continuous, by the Intermediate Value Function it has a zero between $\sqrt{2}$ and $\sqrt{3}$. Since $\cos(0) = 1$ and $\cos$ is strictly decreasing on $[0, 2]$, it follows that this is the only zero in $[0, 2]$. Denoting this zero as $\pi/2$ we have, in particular, that $\cos(x) > 0$ for $x \in [0, \pi/2)$, and (of course) $\cos(\pi/2) = 0$. $\square$

To 4 decimal places this locates π between $2\sqrt{2} = 2.8284$ and $2\sqrt{3} = 3.4642$, so not very precise. We will get a more precise estimate when we discuss the inverse functions.

Theorem 7.13 (Properties of π)**.**

(a) $\sin(\pi/2) = 1, \quad \sin(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$.

(b) $\sin(x + \pi/2) = \cos(x)$ and $\cos(x + \pi/2) = -\sin(x) \ \ (\forall x \in \mathbb{R})$,

(c) $\sin(x + \pi) = -\sin(x)$ and $\cos(x + \pi) = -\cos(x) \ \ (\forall x \in \mathbb{R})$,

(d) Both $\sin$ and $\cos$ are periodic with period 2π, i.e.

$$\sin(x + 2\pi) = \sin(x) \ \text{ and } \ \cos(x + 2\pi) = \cos(x) \ \ (\forall x \in \mathbb{R}).$$

Moreover there is no smaller period, i.e if $0 < a \leq 2\pi$ and if $\sin(x + a) = \sin(x) \ \ (\forall x \in \mathbb{R})$, then $a = 2\pi$, and likewise for $\cos$.

Proof. (a) Since $\cos(\pi/2) = 0$ and for all $x \in \mathbb{R}$, $\sin^2(x) + \cos^2(x) = 1$, it follows that $\sin^2(\pi/2) = 1$. Because $\sin(\pi/2) > 0$, we deduce that $\sin(\pi/2) = 1$.
We also have $\cos(\pi/2) = \cos^2(\pi/4) - \sin^2(\pi/4) = 1 - 2\sin^2(\pi/4)$. Consequently $2\sin^2(\pi/4) = 1$ and because $\sin(\pi/4) > 0$ it follows that $\sin(\pi/4) = \frac{1}{\sqrt{2}}$.
But $\cos^2(\pi/4) = 1 - \sin^2(\pi/4) = \frac{1}{2}$, and noting that $\cos(\pi/4) > 0$, we get $\cos(\pi/4) = \frac{1}{\sqrt{2}}$.

(b) These follow from the addition formulae:

$$\sin(x + \pi/2) = \sin(x)\cos(\pi/2) + \cos(x)\sin(\pi/2) = \cos(x),$$
$$\cos(x + \pi/2) = \cos(x)\cos(\pi/2) - \sin(x)\sin(\pi/2) = -\sin(x).$$

Note in particular that $\sin(\pi) = \cos(\pi/2) = 0$ and $\cos(\pi) = -\sin(\pi/2) = -1$.

(c) Again from the addition formulae:

$$\sin(x+\pi) = \sin(x)\cos(\pi) + \cos(x)\sin(\pi) = -\sin(x),$$
$$\cos(x+\pi) = \cos(x)\cos(\pi) - \sin(x)\sin(\pi) = -\cos(x).$$

Note in particular that $\sin(2\pi) = -\sin(\pi) = 0$ and $\cos(2\pi) = -\cos(\pi) = 1$.

(d) Again from the addition formulae:

$$\sin(x+2\pi) = \sin(x)\cos(2\pi) + \cos(x)\sin(2\pi) = \sin(x),$$
$$\cos(x+2\pi) = \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi) = \cos(x).$$

To prove that there is no smaller period, suppose that $0 < a < 2\pi$. Then $0 < \frac{a}{4} < \frac{\pi}{2} < 2$ and hence $\cos(a/4) > 0$ and $\sin(a/4) > 0$. We deduce that

$$\sin\left(\frac{a}{2}\right) = 2\sin\left(\frac{a}{4}\right)\cos\left(\frac{a}{4}\right) > 0.$$

Therefore

$$\cos(a) = \cos^2\left(\frac{a}{2}\right) - \sin^2\left(\frac{a}{2}\right) < 1 - 0 = 1 = \cos(0).$$

Hence a is not a period for the cosine function. If it were a period for the sine function then $\sin(x+a) = \sin(x)$ for all $x \in \mathbb{R}$, and differentiation would give $\cos(x+a) = \cos(x)$, i.e. a would have to be a period for the cosine function, which it isn't. Hence a is not a period for the sine function. Thus 2π is the smallest period for both sin and cos.

□

At this point we have enough information to sketch the graphs of $\sin(x)$ and $\cos(x)$. These are shown in Figure 7.4.

We can define the remaining circular or trigonometric functions (tangent, cotangent, secant, cosecant) as follows.

Definition 7.6.

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \mathrm{cosec}(x) = \frac{1}{\sin(x)}.$$

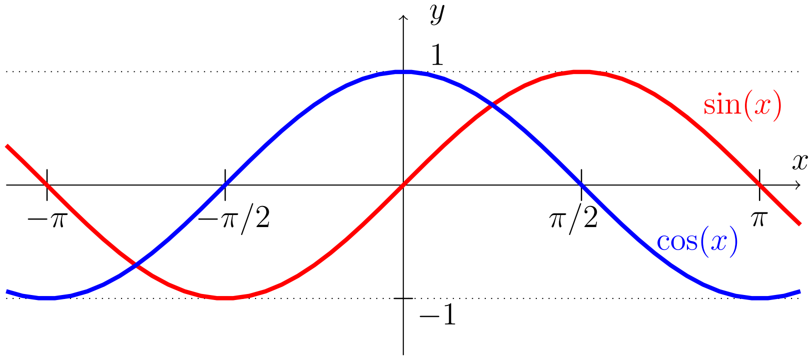


Figure 7.4: $\sin(x)$ and $\cos(x)$.

Of course these definitions are only valid when they make sense. For example, $\tan(\pi/2)$ is undefined because $\cos(\pi/2) = 0$. Note that the three functions with the co prefix (cosine, cotangent, cosecant) decrease on $(0, \pi/2)$, while the other three (sine, tangent, secant) increase on this interval. The only one of these with which we will really concern ourselves is $\tan(x)$. From the earlier results on $\sin$ and $\cos$, it is easy to see that $\tan$ is an odd function, which is periodic with (shortest) period π, and that it is undefined at all odd multiples of $\pi/2$. Since $\tan(x) = \sin(x)/\cos(x)$, by the quotient rule we have

$$\tan'(x) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}.$$

This can be expressed in several alternative forms. We can write it as $\cos^2(x)/\cos^2(x) + \sin^2(x)/\cos^2(x) = 1 + \tan^2(x)$, or we can recognise that $\cos^2(x)+\sin^2(x) = 1$ and write it as $1/\cos^2(x)$ and this can be written as $\sec^2(x)$. So to summarise:

$$\tan'(x) = 1 + \tan^2(x) = \frac{1}{\cos^2(x)} = \sec^2(x).$$

The graph of $\tan(x)$ is illustrated in Figure 7.5.

Reconciling analytical and geometric definitions.
The next set of familiar functions to be discussed are the inverse circular functions. However, before we move on to these we will spend a few pages showing that the functions we have defined as $\sin$ and $\cos$ by means of power series really are the familiar functions that you met at school. To do this we really need to set rigorous analysis aside and examine the geometric definitions of these functions. I will refer to the familiar geometric versions as $s(x)$ and $c(x)$ with the aim of showing that these are indeed given by the power series.

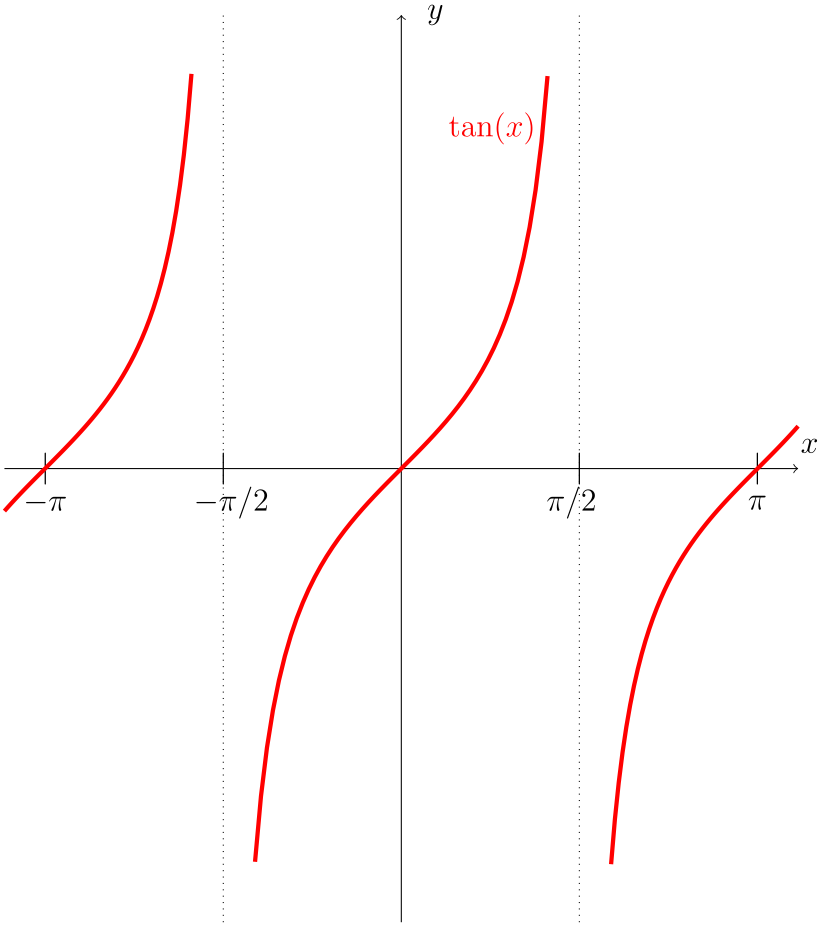


Figure 7.5: $\tan(x)$.

The functions $s(\theta)$ and $c(\theta)$ are defined in Figure 7.6 as the y and x coordinates (respectively) of a point P on the circumference of a circle of unit radius centred on the origin O and such that the radial line OP makes an angle θ (radians) with the x-axis.

Using these geometric definitions we will argue that $s(\theta)$ and $c(\theta)$ are differentiable and that $s'(\theta) = c(\theta)$ and $c'(\theta) = -s(\theta)$. From these it is an easy step to show that s and c have the same power series representations as our definitions for $\sin(\theta)$ and $\cos(\theta)$. We will assume that s and c obey all the addition formulae you grew to know and love in your earlier existence. These can be proved using geometric diagrams involving triangles, and proofs may be found in most elementary textbooks dealing with these functions. In particular we assume that

$$s(a+b) = s(a)c(b) + c(a)s(b) \text{ and } c(a+b) = c(a)c(b) - s(a)s(b).$$

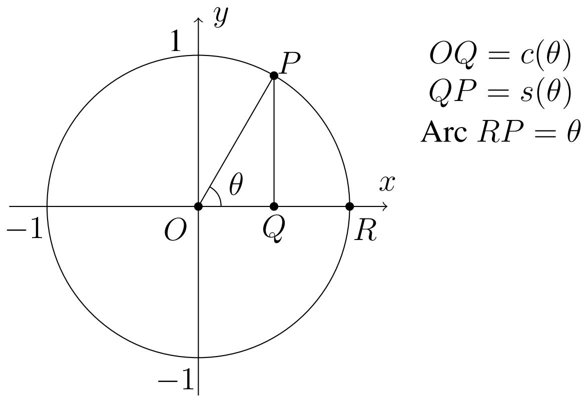


Figure 7.6: $s(\theta)$ and $c(\theta)$.

From these it can be deduced that

$$s(a)-s(b)=2c\Big(\frac{a+b}{2}\Big)s\Big(\frac{a-b}{2}\Big) \quad \text{and}$$
$$c(a)-c(b)=-2s\Big(\frac{a+b}{2}\Big)s\Big(\frac{a-b}{2}\Big).$$

Looking at triangle OQP in Figure 7.6, Pythagoras' Theorem gives $s^2(\theta)+c^2(\theta)=1$, so that $|s(\theta)|\le 1$ and $|c(\theta)|\le 1$ for any $\theta\in\mathbb{R}$. Also note from the geometry of Figure 7.6 that

$$s(\theta)\to 0 \text{ as } \theta\to 0, \qquad c(\theta)\to 1 \text{ as } \theta\to 0.$$

Consequently $s(\theta+h)=s(\theta)c(h)+c(\theta)s(h)\to s(\theta)$ as $h\to 0$, and $c(\theta+h)=c(\theta)c(h)-s(\theta)s(h)\to c(\theta)$ as $h\to 0$, so that both s and c are continuous functions.

The main step in the argument is to show that $s(\theta)/\theta\to 1$ as $\theta\to 0$. To do this consider Figure 7.7 in which we assume that $|\theta|<\pi/2$ and that $\theta\neq 0$. The Figure is drawn with $\theta>0$, but the argument applies, with minor changes, for $\theta<0$.

Arguing from Figure 7.7, we see that QP is shorter than arc RP, hence $s(\theta)<\theta$, giving

$$\frac{s(\theta)}{\theta}<1.$$

Also, arc RP is shorter than the combined length of RS plus SP. But SP is shorter than ST since ST is the hypotenuse of the triangle SPT. So arc RP is shorter than $RS+ST=RT$. Triangles OQP and ORT are similar, so $QP/OQ=RT/OR$, i.e. $s(\theta)/c(\theta)=RT/1$, giving $RT=s(\theta)/c(\theta)$. Because arc RP is

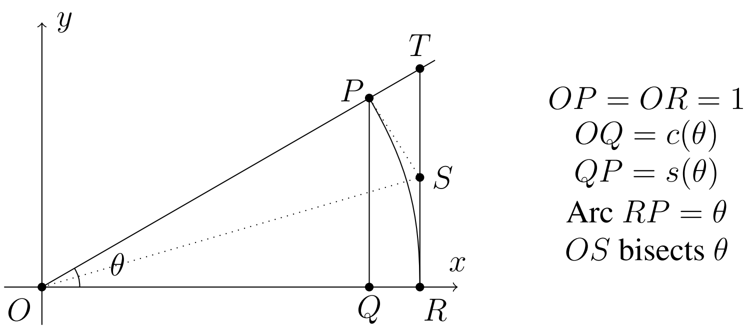


Figure 7.7: Estimating $s(\theta)/\theta$

shorter than RT we get $\theta < s(\theta)/c(\theta)$. From this we obtain

$$c(\theta) < \frac{s(\theta)}{\theta}.$$

So we now have

$$c(\theta) < \frac{s(\theta)}{\theta} < 1.$$

Since $c(\theta) \to 1$ as $\theta \to 0$, we deduce that $s(\theta)/\theta \to 1$ as $\theta \to 0$.

We are now well equipped to prove that $s(\theta)$ is differentiable with derivative $c(\theta)$. To do this take $h \neq 0$, and use the formula for $s(a) - s(b)$ to get

$$\frac{s(\theta + h) - s(\theta)}{h} = \frac{2c\big(\theta + \frac{h}{2}\big)s\big(\frac{h}{2}\big)}{h} = c\Big(\theta + \frac{h}{2}\Big)\frac{s\big(\frac{h}{2}\big)}{\frac{h}{2}} \to c(\theta) \cdot 1 \text{ as } h \to 0,$$

since c is continuous and $s(h/2)/(h/2) \to 1$ as $h \to 0$. Hence $s(\theta)$ is differentiable with derivative $c(\theta)$.

Similarly for $h \neq 0$

$$\frac{c(\theta + h) - c(\theta)}{h} = \frac{-2s\big(\theta + \frac{h}{2}\big)s\big(\frac{h}{2}\big)}{h} = -s\Big(\theta + \frac{h}{2}\Big)\frac{s\big(\frac{h}{2}\big)}{\frac{h}{2}} \to -s(\theta) \cdot 1 \text{ as } h \to 0,$$

since s is continuous and $s(h/2)/(h/2) \to 1$ as $h \to 0$. Hence $c(\theta)$ is differentiable with derivative $-s(\theta)$.

It follows from $s' = c$ and $c' = -s$ that both s and c have derivatives of all orders. Using Taylor's Theorem to expand about the point $a = 0$ with Lagrange's form of the remainder, we find that for the function $s(\theta)$ the n^{th} remainder R_n can be expressed as $R_n = \dfrac{\theta^n}{n!}s^{(n)}(\xi)$ for some ξ. But $s^{(n)}$ is one of $s, c, -s, -c$, so $|R_n| \leq \left|\dfrac{\theta^n}{n!}\right| \to 0$ as $n \to \infty$. Hence $s(\theta)$ is given by the

Maclaurin series $s(\theta) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} s^{(n)}(0)$. Bearing in mind that $s(0) = 0,\ c(0) = 1$ this gives $s(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} = \sin(\theta)$. In the same way we prove that $c(\theta) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} = \cos(\theta)$.

EXERCISES 7.3

1. Use the addition formulae to prove the following identities.

(a) $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$,

(b) $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$,

(c) $\tan(x + y) = \dfrac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$,

(d) $\tan(x - y) = \dfrac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$.

(For part (c) we assume that none of $x, y, x + y$ is an odd multiple of $\pi/2$. Similarly for part (d) with $x - y$ replacing $x + y$.)

2. Prove that for all $x \in \mathbb{R}$, $\sin(\pi/2 - x) = \cos(x)$ and that $\cos(\pi/2 - x) = \sin(x)$.

3. Prove that

$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) \quad \text{and}$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right).$$

4. Prove that $\tan(x)$ is strictly increasing on $(-\pi/2, \pi/2)$, that $\tan(\pi/4) = 1$, $\tan(x+\pi/2) = -\cot(x)$, and $\tan(x+\pi) = \tan(x)$ (so that $\tan$ is periodic with period π). Prove also that there is no smaller (positive) period of the $\tan$ function.

7.4 Inverse circular functions

First a comment on notation: we will denote the inverse circular functions by using the prefix "arc", so $\arcsin$ is the inverse of the $\sin$ function. There are alternative notations, the most common of which is to append a -1 superscript as in $\sin^{-1}$. Unfortunately it is easy to confuse $\sin^{-1}(x)$ with $\dfrac{1}{\sin(x)}$, especially as $\sin^2(x)$ is taken to mean $\big(\sin(x)\big)^2$.

Each of the functions $\sin, \cos$ and $\tan$ is a many-one function. To get injective (one-to-one) functions suitable for defining inverses, we need to restrict their domains. It would be nice to do this symmetrically about zero, and this works for both $\sin$ and $\tan$, which are odd functions. For these functions we will restrict the domains to $[-\pi/2, \pi/2]$ and $(-\pi/2, \pi/2)$ respectively (the end points $-\pi/2$ and $\pi/2$ are excluded for $\tan$ because $\tan(x)$ is undefined for $x = \pm\pi/2$). However, for $\cos$ this symmetric approach does not work because $\cos$ is an even function and so still many-one on $[-\pi/2, \pi/2]$, for example $\cos(-\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$. Instead, for $\cos(x)$ we restrict the domain to $[0, \pi]$, on which the function is injective (one-to-one).

Both $\sin$ and $\tan$ are strictly increasing on their restricted domains, and also continuous and differentiable there. The function $\cos$ is strictly decreasing on $[0, \pi]$ and again continuous and differentiable there. So all three restricted functions have inverses whose properties follow from general results about inverse functions covered in earlier chapters, especially Chapter 5, Theorem 5.12 and Chapter 6, Theorem 6.5.

The inverse sine function $\arcsin$ is strictly increasing on $[-1, 1]$ with image set $[-\pi/2, \pi/2]$, it is continuous on its domain and differentiable on $(-1, 1)$ with derivative given by $\arcsin'(x) = \dfrac{1}{\cos(y)}$, where $\sin(y) = x$, so that $\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$, and note that we take the positive square root since $\cos(y) > 0$ for $y \in (-\pi/2, \pi/2)$. Hence

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \quad \text{for} \quad x \in (-1, 1).$$

The inverse cosine function $\arcsin$ is strictly decreasing on $[-1, 1]$ with image set $[0, \pi]$, it is continuous on its domain and differentiable on $(-1, 1)$ with derivative given by $\arccos'(x) = \dfrac{1}{-\sin(y)}$, where $\cos(y) = x$, so that $\sin(y) = \sqrt{1 - \cos^2(y)} = \sqrt{1 - x^2}$, and note that we take the positive square root since $\sin(y) > 0$ for $y \in (0, \pi)$. Hence

$$\arccos'(x) = -\frac{1}{\sqrt{1 - x^2}} \quad \text{for} \quad x \in (-1, 1).$$

The inverse tangent function $\arctan$ is strictly increasing on $(-\infty, \infty) = \mathbb{R}$ with image set $(-\pi/2, \pi/2)$, it is continuous and differentiable on its domain with derivative given by $\arctan'(x) = \dfrac{1}{\sec^2(y)}$, where $\tan(y) = x$, so that $\sec^2(y) =$

$1 + \tan^2(y) = 1 + x^2$ (look back to the previous section where we discussed the derivative of $\tan$). Hence

$$\arctan'(x) = \frac{1}{1+x^2} \quad \text{for} \quad x \in \mathbb{R}.$$

Figure 7.8 shows the graphs of $\arcsin(x)$ and $\arccos(x)$, and Figure 7.9 shows the graph of $\arctan(x)$.

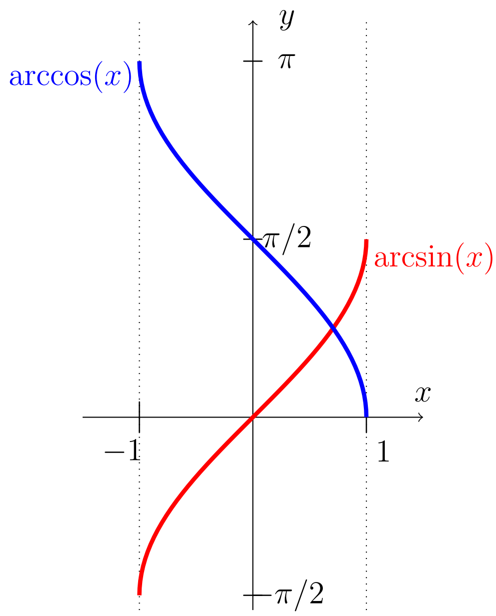


Figure 7.8: $\arcsin(x)$ and $\arccos(x)$.

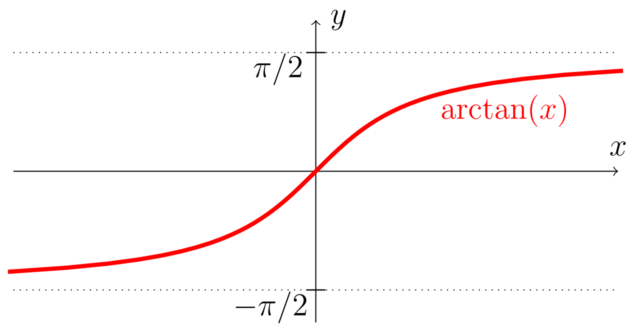


Figure 7.9: $\arctan(x)$.

Calculating the value of π.
The $\arctan$ function can be used to calculate the value of π to sufficient accuracy

for most practical purposes. The first step is to prove that $\tan(\pi/6) = 1/\sqrt{3}$. This can be done using the addition formulae.

$$\begin{aligned}
0 = \cos\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right)\\
&= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)\\
&= \left[\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)\right]\cos\left(\frac{\pi}{6}\right) - \left[2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)\right]\sin\left(\frac{\pi}{6}\right)\\
&= \cos\left(\frac{\pi}{6}\right)\left[\cos^2\left(\frac{\pi}{6}\right) - 3\sin^2\left(\frac{\pi}{6}\right)\right].
\end{aligned}$$

But $\cos\left(\frac{\pi}{6}\right) > 0$, so we have $3\sin^2\left(\frac{\pi}{6}\right) = \cos^2\left(\frac{\pi}{6}\right)$, giving $\tan^2\left(\frac{\pi}{6}\right) = \frac{1}{3}$, and since $\tan\left(\frac{\pi}{6}\right) > 0$, we get $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$. From this it follows that $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

Next we get a power series for $\arctan(x)$, valid for $|x| < 1$. We already have that for $|x| < 1$, by the binomial theorem

$$\arctan'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

Put

$$F(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

By Chapter 6, Corollary 6.14.2, this power series has radius of convergence $R = 1$. Hence F is differentiable on $(-1, 1)$ with derivative given by

$$F'(x) = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \arctan'(x).$$

It follows that both $F(x)$ and $\arctan(x)$ are primitives for $\arctan'(x)$ for $|x| < 1$. But then, from Chapter 6, Corollary 6.8.1, we find that $F(x) - \arctan(x)$ takes a constant value on $(-1, 1)$. Since $F(0) - \arctan(0) = 0$, the value of this constant is zero. Hence for $|x| < 1$,

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

We can now use the value $\arctan\big(\frac{1}{\sqrt{3}}\big) = \frac{\pi}{6}$ and the power series with $x = \frac{1}{\sqrt{3}}$ to obtain a numerical series for $\pi/6$. This gives

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\Big[1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \dots\Big] = \frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cdot 3^n}.$$

Hence, noting that $6/\sqrt{3} = (2\times 3)/\sqrt{3} = 2\sqrt{3}$, we obtain

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cdot 3^n}.$$

The terms of this series alternate in sign and their absolute values are strictly decreasing. Hence successive partial sums provide over-estimates and under-estimates for the value of π. If we take the partial sum as far as $n = 7$, the value is 3.14157 (to 5 decimal places), while going to $n = 8$ gives 3.14160 (to 5 decimal places), so we can conclude that $\pi = 3.1416$ correct to 4 decimal places.

Comment. We evaluated π using a series for $\arctan(1/\sqrt{3})$. We can get π to almost any accuracy needed for practical purposes from this series. However, getting millions of decimal digits of π has become a challenge for computer scientists and mathematicians in recent years, and this series is not adequate for this purpose. On the day I wrote this (International pi day, 14th march 2024), it seems that the first 105 trillion (1.05×10^{14}) digits were known. No doubt that, by the time you read this, the number will have grown again. In one of the Exercises for this section we show how one can get a series for π that converges more quickly than the one we have used. But considerable ingenuity is needed if one wants to get very large numbers of digits.

A summary of the history of the quest for π up to around 1995 is given in "The Quest for π" by David H. Bailey, Jonathan M. Borwein, Peter B. Borwein and Simon Plouffe in The Mathematical Intelligencer, vol. 19, no. 1 (Jan. 1997), pg. 50–57. This is a very readable paper and is available on the internet (currently at https://www.davidhbailey.com//dhbpapers/pi-quest.pdf).

EXERCISES 7.4

1. Obtain a power series in x for $\arcsin(x)$.

2. Prove that $\sin(\pi/6) = 1/2$. (Look at how we proved $\tan(\pi/6) = \frac{1}{\sqrt{3}}$.)

3. Use the results of the previous questions to prove that

$$\pi = 3\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2(2n+1)2^{4n}}.$$

By taking the first six terms of this series, get an approximate value for π and show that the error lies between 0 and 0.0001.

4. Prove that
$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$
This known as Machin's formula.
[Hint: Use the addition formulae for $\tan(x \pm y)$ from Exercises 7.3. Put $\theta = \arctan(\frac{1}{5})$ and show that $\tan(4\theta) = 120/119$, then put $\phi = \arctan(\frac{1}{239})$, and compute $\tan(4\theta - \phi)$.]

5. It will not have escaped your notice that series like the one for $\arctan(x)$ converge much more rapidly if x is "small" (like $\frac{1}{5}$ or $\frac{1}{239}$) rather than "large" (just less than 1). So the formula from the previous question (Machin's formula) gives a faster way to get a lot of decimal places for π. Get π correct to within 10^{-10} by this method.

7.5 Hyperbolic functions and their inverses

These functions may be less familiar to the reader than were the exponential, logarithm and circular functions. Here are the basic definitions.

Definition 7.7. For $x \in \mathbb{R}$ we define
$$\sinh(x) = \frac{\exp(x) - \exp(-x)}{2} \quad \text{and} \quad \cosh(x) = \frac{\exp(x) + \exp(-x)}{2}.$$

The other hyperbolic functions are defined in terms of these as follows.
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)},$$
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}, \quad \operatorname{cosech}(x) = \frac{1}{\sinh(x)}.$$

A few comments are in order. First on pronunciation, $\sinh$ is pronounced as "shine" or "sinch" (both are common). $\cosh$ is easy - just say "cosh". $\tanh$ is more difficult, either use "than" (the "th" is a soft version pronounced as in "thanks", rather than the harder "th" of "than") or use "tanch". Don't worry too much about the others as they are rarely used: "coth", "shec" and "coshec" will get you by.

Second, note that $\sinh$ is an odd function, in fact the odd part of $\exp$, while $\cosh$ is an even function, the even part of $\exp$, so that $\exp(x) = \cosh(x)+\sinh(x)$. Since $\exp(x) > 0$ for all $x \in \mathbb{R}$, it follows that $\cosh(x) > 0$ for all $x \in \mathbb{R}$. We will show shortly that $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$. On the other hand, $\sinh(x) = 0$ if and only if $\exp(x) = \exp(-x)$. But this can only happen if $\exp(2x) = 1$, and the only solution is therefore $x = 0$. Thus the definitions of $\coth(x)$ and $\operatorname{cosech}(x)$ need to be qualified by the statement that these functions are undefined for $x = 0$.

Third, why are these functions named so similarly to the circular functions $\sin, \cos$, etc? There *is* a **very** good reason that requires some knowledge of complex numbers. We will discuss this in the final section of this chapter. If you don't know anything about complex numbers you can skip it, but I hope your curiosity might be raised.

Fourth, why are they called hyperbolic functions? That is easier to explain. If we write $x = \cosh(t)$ and $y = \sinh(t)$ and then eliminate the parameter t (see below), we get $x^2 - y^2 = 1$. The resulting graph is an hyperbola. Compare this with the circular functions: taking $x = \cos(\theta)$ and $y = \sin(\theta)$, elimination of the parameter θ gives $x^2 + y^2 = 1$, corresponding to a circle.

Identities and derivatives.
Since these functions are defined in terms of the exponential function, they are continuous and differentiable on their domains. Some easy arithmetic gives $\cosh^2(x) - \sinh^2(x) = 1$ for all $x \in \mathbb{R}$. There are many other identities that are similar to those for the circular functions, and a few of these are covered in the exercises for this section.

It is also easily seen that $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$, while $\tanh'(x) = \dfrac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)}$. This can be written in various forms:

$$\tanh'(x) = 1 - \tanh^2(x) = \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x).$$

Power series.
From the power series for $\exp(x)$ we get

$$\begin{aligned}
\sinh(x) &= \frac{\exp(x) - \exp(-x)}{2} \\
&= \frac{1}{2}\Big[\Big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\Big) - \Big(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\Big)\Big] \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\cosh(x) &= \frac{\exp(x)+\exp(-x)}{2}\\ &= \frac{1}{2}\Big[\Big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\Big)+\Big(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots\Big)\Big]\\ &= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\\ &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}.\end{aligned}$$

These series are valid for all $x \in \mathbb{R}$. The series look rather like those for $\sin$ and $\cos$, but they lack the alternating signs on the individual terms. From the series for $\cosh(x)$ it is obvious that $\cosh(x) > 1$ for $x \neq 0$, while $\cosh(0) = 1$. The graphs of $\sinh(x)$ and $\cosh(x)$ are sketched in Figure 7.10, and that of $\tanh$ is sketched in Figure 7.11.

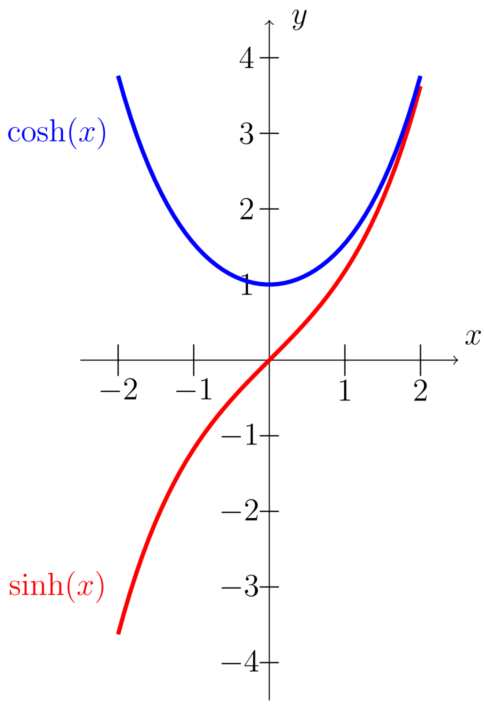


Figure 7.10: $\sinh(x)$ and $\cosh(x)$.

The graph of $\cosh(x)$ looks a bit like a parabola. However it is much steeper at the extremities. Indeed the derivative or $\cosh(x)$ is $\sinh(x)$ which behaves like $\exp(x)/2$ for large positive x, and this is much larger than the derivative of x^2, a

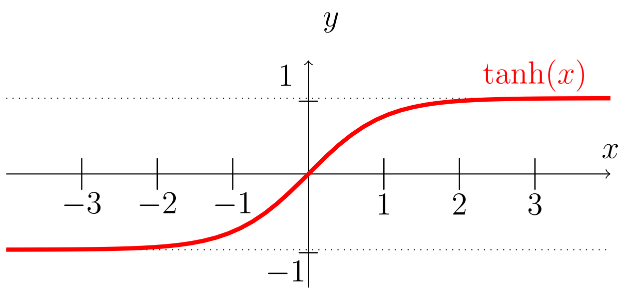


Figure 7.11: $\tanh(x)$.

measly $2x$. The shape of the $\cosh$ graph is called a *catenary*. It is the shape taken up by a heavy chain or cable, secured at both ends, and hanging freely under the influence of gravity.

Inverse hyperbolic functions.
First a comment on notation: we will denote the inverse hyperbolic functions by using the prefix "arg", so $\operatorname{argsinh}$ is the inverse of the $\sinh$ function. There are alternative notations, the most common of which is to append a -1 superscript as in $\sinh^{-1}$. Unfortunately it is easy to confuse $\sinh^{-1}(x)$ with $\dfrac{1}{\sinh(x)}$, especially as $\sinh^2(x)$ is taken to mean $\big(\sinh(x)\big)^2$.

The functions $\sinh$ and $\tanh$ are injective (one-to-one) and both are strictly increasing. So they have strictly increasing inverses that are continuous on their domains. The image set of $\sinh$ is $\mathbb{R}$, so this is the domain of $\operatorname{argsinh}$, while the image set of $\tanh$ is $(-1, 1)$ so this is the domain of $\operatorname{argtanh}$.

The function $\cosh$ is many-one (because it is an even function), so we must restrict its domain to get an injective function. We do this by taking the restricted domain as $[0, \infty)$. The restricted function is strictly increasing on this domain, so the inverse will also be strictly increasing, as well as continuous. The image set is $[1, \infty)$ and this is the domain of $\operatorname{argcosh}$.

Derivatives of the inverse hyperbolic functions.

Using the general result about differentiability of inverse functions (Chapter 6, Theorem 6.5) we find that, apart from a single exceptional point for $\operatorname{argcosh}(x)$, all three of these are differentiable on their domains. The exceptional point for $\operatorname{argcosh}(x)$ is the left-hand end point of its domain, namely $x = 1$, where the one-sided derivative does not exist (because $\cosh'(0) = \sinh(0) = 0$).

The function $\operatorname{argsinh}(x)$ is differentiable on $\mathbb{R}$ with derivative given by $\operatorname{argsinh}'(x) = \dfrac{1}{\cosh(y)}$, where $\sinh(y) = x$, so that $\cosh(y) = \sqrt{1 + \sinh^2(y)} =$

$\sqrt{1+x^2}$, and note that we take the positive square root since $\cosh(y) > 0$ for $y \in \mathbb{R}$. Hence

$$\operatorname{argsinh}'(x) = \frac{1}{\sqrt{1+x^2}} \quad \text{for } \ x \in \mathbb{R}.$$

The function $\operatorname{argcosh}(x)$ is differentiable on $(1,\infty)$ with derivative given by $\operatorname{argcosh}'(x) = \dfrac{1}{\sinh(y)}$, where $\cosh(y) = x$, so that $\sinh(y) = \sqrt{\cosh^2(y) - 1} = \sqrt{x^2-1}$, and note that we take the positive square root since $\sinh(y) > 0$ for $y > 0$. Hence

$$\operatorname{argcosh}'(x) = \frac{1}{\sqrt{x^2-1}} \quad \text{for } \ x \in (1,\infty).$$

The function $\operatorname{argtanh}(x)$ is differentiable on $(-1,1)$ with derivative given by $\operatorname{argtanh}'(x) = \dfrac{1}{1-\tanh^2(y)}$, where $\tanh(y) = x$. Hence

$$\operatorname{argtanh}'(x) = \frac{1}{1-x^2} \quad \text{for } \ x \in (-1,1).$$

The graphs of $\operatorname{argsinh}(x)$ and $\operatorname{argcosh}(x)$ are sketched in Figure 7.12, and that of argtanh is sketched in Figure 7.13.

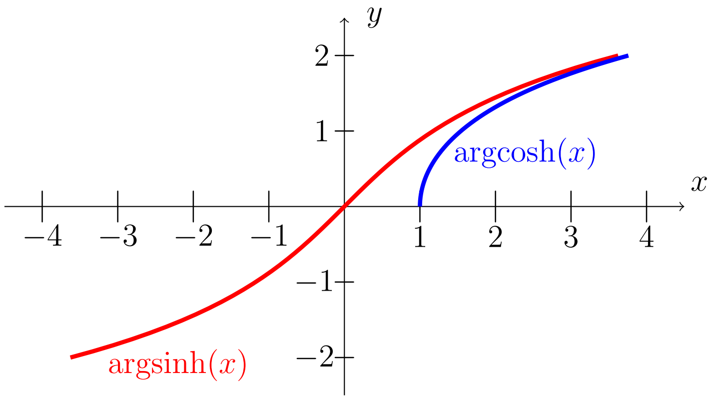


Figure 7.12: $\operatorname{argsinh}(x)$ and $\operatorname{argcosh}(x)$.

Because the hyperbolic functions are defined in terms of the exponential function, it is reasonable to ask if the inverse hyperbolic functions can be expressed in terms of the logarithm function (i.e. the inverse of the exponential function). This can be done quite easily.

Logarithmic formulae for the inverse hyperbolic functions.

Suppose that $y = \operatorname{argsinh}(x)$, so that $x = \sinh(y) = (\exp(y) - \exp(-y))/2$. This gives $(\exp(y))^2 - 2x\exp(y) - 1 = 0$. Solving this quadratic equation for

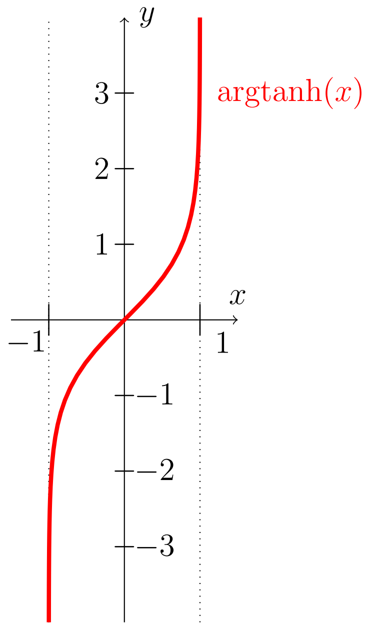


Figure 7.13: $\mathrm{argtanh}(x)$.

$\exp(y)$ gives $\exp(y) = x + \sqrt{x^2+1}$, where the positive square root must be taken since $\exp(y) > 0$ $(\forall y \in \mathbb{R})$. Inverting the exponential function, gives $y = \log_e\left(x + \sqrt{x^2+1}\right)$. Hence

$$\mathrm{argsinh}(x) = \log_e\left(x + \sqrt{x^2+1}\right) \text{ for } x \in \mathbb{R}.$$

Similarly if $y = \mathrm{argcosh}(x)$ (so that $x \geq 1$ and $y \geq 0$) then $x = \cosh(y) = (\exp(y) + \exp(-y))/2$. This gives $(\exp(y))^2 - 2x\exp(y) + 1 = 0$. Solving this quadratic equation for $\exp(y)$ gives $\exp(y) = x + \sqrt{x^2-1}$, where the positive square root must be taken since $\exp(y) \geq 1$ $(\forall y \in [0,\infty))$. [See one of the exercises below for a precise justification for rejecting the negative root.] Inverting the exponential function, gives $y = \log_e\left(x + \sqrt{x^2-1}\right)$. Hence

$$\mathrm{argcosh}(x) = \log_e\left(x + \sqrt{x^2-1}\right) \text{ for } x \in [1,\infty).$$

Finally if $y = \mathrm{argtanh}(x)$ (so that $x \in (-1,1)$) then $x = \tanh(y) = (\exp(y) - \exp(-y))/(\exp(y) + \exp(-y))$. This gives $x(\exp(2y) + 1) = \exp(2y) - 1$, which leads to $\exp(2y) = (1+x))/(1-x)$ and hence to $y = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$. Thus

$$\mathrm{argtanh}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \text{ for } x \in (-1,1).$$

EXERCISES 7.5

1. Prove the following identities.

a) $\cosh^2(x) - \sinh^2(x) = 1 \ \ (\forall x \in \mathbb{R})$.

b) $\sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b) \ \ (\forall a, b \in \mathbb{R})$.

c) $\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b) \ \ (\forall a, b \in \mathbb{R})$.

2. Prove that if $x > 1$ then $x - \sqrt{x^2 - 1} < 1$.

3. Obtain power series in x, valid for $|x| < 1$, for (a) $\operatorname{argsinh}(x)$ and (b) $\operatorname{argtanh}(x)$. Why is this not possible for $\operatorname{argcosh}(x)$?

7.6 Familiar functions and Complex Numbers

If you know nothing about complex numbers you should probably skip this section. However if you have some knowledge of how complex numbers work, then this is for you. It shines a light on why the exponential, circular and hyperbolic functions are so closely related.

Any complex number may be written in the form $x + iy$, where x, y are Real Numbers and i represents a square root of -1, so that $i^2 = -1$. The Real Number x is called the real part of z, and the Real Number y (*not* iy) is called the imaginary part of z. If $y = 0$ then we say that $z = x + i0 = x$ is real. If $x = 0$, then we say that $z = 0 + iy = iy$ is imaginary. [The term "imaginary" is used for historical reasons - there is nothing more imaginary about such numbers than there is about Real Numbers.]

The *modulus* of $z = x + iy$ (for $x, y \in \mathbb{R}$) is the non-negative Real Number $|z| = \sqrt{x^2 + y^2}$. Many results about power series apply when we allow the terms to be complex numbers. In particular, each power series has a radius of convergence R (which may be infinite). If $R > 0$ then for $|z| < R$ the series will converge absolutely and the resulting sum function $f(z)$ will be continuous and differentiable within the resulting circle of convergence.

The exponential function $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $R = \infty$. The addition formula continues to apply, so that $\exp(z) \cdot \exp(w) = \exp(z+w)$, and we can define $e^z = \exp(z)$ for complex z. If we replace z by iz in the series and note that $i^2 = -1,\ i^3 = -i,\ i^4 = 1$, etc., we get

$$\begin{aligned}
\exp(iz) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\
&= 1 + iz + i^2\frac{z^2}{2!} + i^3\frac{z^3}{3!} + i^4\frac{z^4}{4!} + i^5\frac{z^5}{5!} + i^6\frac{z^6}{6!} + i^7\frac{z^7}{7!} + \dots \\
&= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!} - i\frac{z^7}{7!} + \dots \\
&= \Big[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\Big] + i\Big[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\Big] \\
&= \cos(z) + i\sin(z).
\end{aligned}$$

Replacing z by $-z$ in this gives

$$\exp(-iz) = \cos(z) - i\sin(z).$$

Hence adding the expressions for $\exp(iz)$ and $\exp(-iz)$ gives

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} = \cosh(iz),$$

while subtracting gives

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i} = -i\sinh(iz).$$

So $\cos(z)$ is the even part of $\exp(iz)$ and $i\sin(z)$ is the odd part of $\exp(iz)$. If we replace z by iz in these identities we obtain $\cosh(z) = \cos(iz)$ and $\sinh(z) = -i\sin iz$. So you can see that in the field $\mathbb{C}$ of complex numbers, there is a very close connection between the hyperbolic and circular functions.

If z is a real number (i.e $z = x + i0$ where $x \in \mathbb{R}$) then we have

$$\exp(ix) = \cos(x) + i\sin(x).$$

But in these circumstances both $\cos(x)$ and $\sin(x)$ are Real Numbers, so that $\cos(x)$ is the real part of $\exp(ix)$, and $\sin(x)$ is the imaginary part of $\exp(ix)$.

The exponential function $\exp(z)$ is periodic with period $2\pi i$. This is because $\exp(2\pi i) = \cos(2\pi) + i\sin(2\pi) = 1$, and consequently

$$\exp(z + 2\pi i) = \exp(z)\cdot\exp(2\pi i) = \exp(z)\ \ \forall z \in \mathbb{C}.$$

A complication arising from this is that the domain of exp needs to be restricted in order to define an inverse function (a complex logarithm), and this is usually done by choosing the domain to be $\{z = x + iy : -\pi < y \leq \pi\}$. We will not investigate this more fully here. However, we have said sufficient to indicate that all the familiar functions in this Chapter are intimately related to one principal function, namely the exponential function.

To conclude this section we mention that $\exp(i\pi) = \cos(\pi) + i\sin(\pi) = -1$. This can be written in the following form, known as Euler's identity

$$e^{i\pi} + 1 = 0.$$

This beautiful formula connects the additive identity (0), the multiplicative identity (1), the two most famous mathematical constants (π and e), and a square root of -1 (i).

Chapter 8

Further chapter to be added

The Riemann Integral

Chapter 9

Answers to the Exercises

Exercises 2.2

1. Write the statement as

$$\exists p, q \in \mathbb{N} \text{ s.t. } p^2 = 2q^2.$$

The negation is

$$\forall p, q \in \mathbb{N}, p^2 \neq 2q^2.$$

2. Write P as

$$\forall n \in \mathbb{N}, \exists p > n \text{ s.t. } p \text{ and } p + 2 \text{ are both prime numbers.}$$

Then $\neg P$ can be expressed as

$$\exists n \in \mathbb{N} \text{ s.t. } \forall p > n, p \text{ and } p + 2 \text{ are not both prime numbers.}$$

Of course the phrase that "p and $p + 2$ are not both prime numbers" can be expressed in alternative forms such as "at most one of p and $p + 2$ is prime" or "at least one of p and $p + 2$ is composite".

3. Write Q as
$\exists D \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \exists p, q$ both prime numbers greater than n s.t. $|p - q| \leq D$.
Then $\neg Q$ can be expressed as

$$\forall D \in \mathbb{N}, \exists n \in \mathbb{N} \text{ s.t. } \forall p, q \text{ both prime numbers greater than } n, |p-q| > D.$$

Alternatively,
$\forall D \in \mathbb{N}, \exists n \in \mathbb{N}$ s.t. $\forall p, q$ both greater than n, if $|p - q| \leq D$ then p and q cannot both be prime numbers.

Exercises 2.3

1. $(-a)(-b) - ab$ can be written as $(-a)(-b) - ab + (-a)b - (-a)b$ because the associative law (A3) applies (so we don't have to specify how terms are bracketed) and because the last two terms cancel (A7), and 0 is an additive identity (A5). The expression can be re-ordered using the commutative law (A2) to write it as $(-a)(-b) + (-a)b - ab - (-a)b$.

We have $(-a)(-b) + (-a)b = (-a)((-b) + b)$ by the distributive law (A4), and since $(-b) + b = 0$ (by A7), this comes to $(-a)0$.

Now consider $-ab - (-a)b$. Since $[-ab - (-a)b] + [ab + (-a)b] = [-ab + ab] + [(-a)b - (-a)b] = 0 + 0 = 0$ (using A2, A3, A5 and A7), we have $-ab - (-a)b = -[ab + (-a)b] = -(a + (-a))b = -0b = -b0$ (using A2, A4 and A7).

So $(-a)(-b) - ab = (-a)0 - b0$. The terms on the right are each equal to 0. To see this consider $x0 = x(0 + 0) = x0 + x0$ (using A4 and A5), where x is any element of $\mathbb{R}$. By adding $-x0$ we get $0 = x0 - x0 = x0 + x0 - x0 = x0$ (using A3, A5 and A7), so $x0 = 0$ for any $x \in \mathbb{R}$. It follows that $(-a)(-b) - ab = 0 + 0 = 0$ (by A5). But then by A7, $(-a)(-b) = ab$. [It's a good job we don't teach it like this in schools!]

2. On division by 3, a positive integer n has a remainder of $0, 1$ or 2, so n has one of the three forms $n = 3r$, $n = 3r + 1$, or $n = 3r + 2$, where r is a positive Integer. But then n^2 has the corresponding form $9r^2$, $9r^2 + 6r + 1$, or $9r^2 + 12r + 4$, and the latter two possibilities are not divisible by 3 (they both leave remainder 1). So if n^2 is divisible by 3, n itself must be divisible by 3. Conversely, if n is divisible by 3 then n has the form $3r$ and so $n^2 = 9r^2 = 3(3r^2)$, so n^2 is divisible by 3.

3. Suppose that $\sqrt{3}$ is a Rational Number. Then $\sqrt{3} = p/q$ for some positive Integers p and q. By cancelling, we can assume that p and q have no common factors, in particular they are not both divisible by 3. By squaring we find that $3 = p^2/q^2$, so $p^2 = 3q^2$ and consequently p^2 is divisible by 3. So p itself is divisible by 3, meaning that $p = 3r$ for some Integer r. But then $p^2 = 9r^2$ and combining this with $p^2 = 3q^2$ we get $3q^2 = 9r^2$ and so $q^2 = 3r^2$. Thus q^2 is divisible by 3, and so q itself is divisible by 3. But this is a contradiction since p and q have no common factors. Hence $\sqrt{3}$ cannot be a Rational Number.

4. The quick answer is that $n = 2$ provides a counterexample since $n^2 = 4$ is divisible by 4, even though n itself is not divisible by 4. A more detailed explanation is that if n has a remainder 2 on division by 4 then n has the form $n = 4r + 2$ for some Integer r, so $n^2 = 16r^2 + 16r + 4$, which is divisible by 4.

5. If $\frac{1}{x}$ is a Rational Number then there are Integers $p \neq 0$ and $q \neq 0$ such that $\frac{1}{x} = \frac{p}{q}$. But then $x = \frac{q}{p}$ must be a Rational Number. So if x is irrational, $\frac{1}{x}$ must also be irrational.
If $z = a + bx$ with $b \neq 0$, then $x = (z - a)/b$. Consequently if a, b, z are all Rational Numbers, then so is x. Hence if x is irrational but a, b are Rational Numbers, then z must be irrational.

6. $3/7 = 0.\overline{428571}$.

7. If $x = 27.53\overline{27}$ then $100x = 2753.\overline{27}$. Write $100x$ above x, line up the decimal points and subtract:

$$\begin{aligned} 100x &= 2753.272727\ldots \\ x &= \quad 27.532727\ldots \\ 99x &= 2725.74 \end{aligned}$$

So $x = 272574/9900$ and we can take $p = 272574, q = 9900$, or cancel down to get $x = 15143/550$ and take $p = 15143, q = 550$.

8. The length of the recurring section cannot be any longer than 17 because there are only 17 possible remainders when dividing by 17. By calculator we find that $1/17 = 0.05882352941176470588235294117647 05\cdots$. By inspection, the recurring section is 0588235294117647 of length 16.

9. The set S contains all numbers of the form $\frac{n}{n+1}$ for $n = 1, 2, 3, \ldots$. All these numbers are less than 1, so S is bounded above by 1. It seems likely that this is the least upper bound since $\frac{n}{n+1}$ will get arbitrarily close to 1 as n increases. So we prove that 1 is the least upper bound by taking any $\epsilon > 0$ and showing that there is an element $\frac{n}{n+1}$ of S for which $\frac{n}{n+1} > 1 - \epsilon$ (so that $1 - \epsilon$ is not an upper bound). This inequality is equivalent to $1 - \frac{n}{n+1} < \epsilon$, that is $\frac{1}{n+1} < \epsilon$, and this will certainly be true if $n + 1 > \frac{1}{\epsilon}$. Indeed, if $n > \frac{1}{\epsilon}$ then $n + 1 > \frac{1}{\epsilon}$, and the corresponding element of S, namely $\frac{n}{n+1}$, will exceed $1 - \epsilon$.

10. The set T only contains negative numbers, so it is certainly bounded above by 0. It also contains a maximum element, namely $-\frac{1}{2}$, so this is the least upper bound of T.

11. By the Archimedean Axiom, there is some $q \in \mathbb{N}$ such that $q > \frac{1}{b-a}$. But then $qb - qa > 1$ and so the interval between qa and qb has length greater than 1 and must therefore contain some integer p. We have $qa < p < qb$, so that $r = \frac{p}{q}$ is a Rational Number satisfying $a < r < b$.

12. Take a Rational Number r satisfying $a < r < b$. By the previous result there is also a Rational Number s satisfying $a < s < r$ and another Rational Number t satisfying $r < t < b$. By the Archimedean axiom there is some $n \in \mathbb{N}$ such that $n > \frac{1}{t-s}$. But then $nt - ns > 1$, so the interval between ns and nt contains the (irrational) number $ns + \frac{1}{\sqrt{2}}$. Since $ns < ns + \frac{1}{\sqrt{2}} < nt$ we have $a < s < z < t < b$, where $z = s + \frac{1}{n\sqrt{2}}$ is an irrational number.

Exercises 2.4

1. For $n = 1$ the sum is $\sum_{i=1}^{1} i^2 = 1^2 = 1$, while $n(n+1)(2n+1)/6 = 1 \times 2 \times 3/6 = 1$. So the formula is correct for $n = 1$. Now assume it is correct when n is some positive integer k, i.e. $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$. We want to prove it is correct for the next value $n = k + 1$. We have

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\ &= k(k+1)(2k+1)/6 + (k+1)^2 \text{ (by the inductive assumption)} \\ &= (k+1)(k(2k+1) + 6(k+1))/6 \\ &= (k+1)(2k^2 + 7k + 6)/6 \\ &= (k+1)(k+2)(2k+3)/6 \\ &= n(n+1)(2n+1)/6 \text{ when } n = k+1.\end{aligned}$$

Hence, by induction, the formula holds for every positive integer n.

2. There are many ways to prove the result, one of which is to use induction. Using a calculator we find that $25! > 1.5 \times 10^{25}$ so $n! > 10^n$ for $n = 25$. Assuming that $k! > 10^k$ for some positive Integer $k \geq 25$, we find $(k+1)! = (k+1) \times k! \geq 26 \times 10^k > 10 \times 10^k = 10^{k+1}$. Hence, by induction, $n! > 10^n$ for $n \geq 25$.

3. The flaw in the "proof" is in the step from k to $k + 1$ when $k = 1$. If there are just two people in the room, X and Y, and we send out person X, then we are left with Y having birthday d. But if we bring X back into the room and send out Y, there is no-one left in the room that we are certain has birthday d. The moral is to be careful about the inductive step from k to $k + 1$, in particular that it doesn't make unwarranted assumptions about k.

Exercises 2.5

1. Consider $(\sqrt{x}-\sqrt{y})^2 \geq 0$. Expanding the bracket gives $x-2\sqrt{xy}+y \geq 0$. Rearranging this gives $(x+y)/2 \geq \sqrt{xy}$. In fact there is equality between the geometric and arithmetic means if and only if $x = y$ because if $x \neq y$ then $(\sqrt{x} - \sqrt{y})^2 > 0$

2. This can happen if pupil P's score is below the mean in Class X, but above the mean in Class Y. Moving a pupil will not affect the overall mean score across all Classes. So Mr. Fixit's bonus could be paid even though the overall mean score might have declined.

Exercises 3.2

1. The sequence is certainly bounded below by 0 since the terms are all positive. For large n we might expect $n/(n^2+1)$ to be close to 0, so we try to prove that 0 is the greatest lower bound. Take any $\epsilon > 0$. We will have $n/(n^2+1) < 0+\epsilon$ if $n/n^2 < \epsilon$, i.e. if $1/n < \epsilon$. But this will be true if n is chosen greater than $1/\epsilon$. So if $n > 1/\epsilon$, then the corresponding term $n/(n^2+1)$ of the sequence is less than $0+\epsilon$. Consequently 0 is the greatest lower bound of the sequence.

2. If $A \in \mathbb{R}$ we can prove that A is not an upper bound of the sequence simply by choosing an even positive Integer n that exceeds $|A|$ because then $(-1)^n n = n > |A| \geq A$. Similarly we can prove that A is not a lower bound of the sequence by choosing an odd positive Integer n that exceeds $|A|$ because then $(-1)^n n = -n < -|A| \leq A$.

Exercises 3.3

1. Choose $\epsilon > 0$. Put $N = \frac{|a|+1}{\epsilon}$. Take any $n > N$ and consider $|\frac{a}{n} - 0| = \frac{|a|}{n} < \frac{|a|+1}{n} < \frac{|a|+1}{N} = \epsilon$. Hence $\frac{1}{n} \to 0$ as $n \to \infty$. [The reason for using $|a|$ rather than just a is that a could be negative. The reason for the additional $+1$ is that a could be 0 and without the +1 we'd get $N = 0$, a very undesirable denominator!]

2. Choose $\epsilon > 0$. Put $N = 2/\epsilon$. Take any $n > N$ and consider

$$\left|\frac{n^2}{n^2+n+1} - 1\right| = \left|\frac{-n-1}{n^2+n+1}\right| = \frac{n+1}{n^2+n+1} < \frac{n+n}{n^2} = \frac{2}{n} < \frac{2}{N} = \epsilon.$$

Hence $\frac{n^2}{n^2+n+1} \to 1$ as $n \to \infty$. [The choice of N is really made for you at the end of the previous line.]

3. Dividing top and bottom of $x_n = \frac{2n+7}{7n-3}$ gives $x_n = \frac{2+\frac{7}{n}}{7-\frac{3}{n}}$. For large n we expect $\frac{7}{n}$ and $\frac{3}{n}$ to be close to 0, so it looks likely that the limit is $2/7$. To prove this, choose any $\epsilon > 0$. Put $N = 2/\epsilon$. Take any $n > N$ and consider

$$\left|\frac{2n+7}{7n-3} - \frac{2}{7}\right| = \left|\frac{49+6}{7(7n-3)}\right| = \frac{55}{7(7n-3)} < \frac{8}{7n-3} \leq \frac{8}{7n-3n} = \frac{2}{n}$$

and $\frac{2}{n} < \frac{2}{N} = \epsilon$. Hence $\frac{2n+7}{7n-3} \to \frac{2}{7}$ as $n \to \infty$. [There are many possible choices for N. For example, we might argue that $7n-3$ is close to $7n$ if n is large - in fact $7n-3 > 6n$ if $n > 3$. So we might assume that $N \geq 3$. Then we have $\frac{55}{7(7n-3)} < \frac{55}{42n} < \frac{55}{42N} \leq \epsilon$ if we put $N = \max(3, \frac{55}{42\epsilon})$. Just do whatever is easiest!]

4. Informally, for large n the terms of the sequence are alternately close to 1 and -1. So, whatever the supposed limit, we can't always be close to it for large n. We can make this clear by choosing a measure of closeness (i.e. ϵ) that is well below 2. Here is a formal argument.
Suppose that $(-1)^n \frac{n}{n+1} \to l$ as $n \to \infty$. Take $\epsilon = 0.1$. Then $\exists N$ s.t. $\forall n > N$, $\left|(-1)^n \frac{n}{n+1} - l\right| < 0.1$. So take $n > \max(N, 10)$ even (so that $n+1$ is odd) and we have both

$$\Big|\frac{n}{n+1} - l\Big| < 0.1 \text{ and } \Big| - \frac{n+1}{n+2} - l\Big| < 0.1.$$

Hence

$$\begin{aligned}\Big|\frac{n}{n+1} - \Big(- \frac{n+1}{n+2}\Big)\Big| &= \Big|\Big(\frac{n}{n+1} - l\Big) + \Big(\frac{n+1}{n+2} + l\Big)\Big| \\ &\leq \Big|(\frac{n}{n+1} - l\Big| + \Big|\frac{n+1}{n+2} + l\Big| < 0.1 + 0.1 = 0.2,\end{aligned}$$

by the triangle inequality. But the left hand side is at least $\frac{10}{11} + \frac{11}{12} > 1$. So we have a contradiction: $1 < 0.2$. We deduce that the sequence $\left((-1)^n \frac{n}{n+1}\right)$ does not converge.

5. If $l > A$, put $\epsilon = l - A$ so that $\epsilon > 0$. Since $x_n \to l$ as $n \to \infty$, $\exists N$ s.t. $\forall n > N, |x_n - l| < \epsilon$. Take $n^* = \lfloor N \rfloor + 1$ so that $n^* > N$ and consequently $-\epsilon < x_{n^*} - l < \epsilon$, which gives $x_{n^*} > l - \epsilon = A$. But this contradicts the assumption that $x_n < A$ for every $n \in \mathbb{N}$. We deduce that $l \leq A$.
We cannot generally assert $l < A$ as shown by the sequence $\left(\frac{n}{n+1}\right)$. Here we have $\frac{n}{n+1} < A = 1$ for every $n \in \mathbb{N}$, and $\frac{n}{n+1} \to l = 1$ as $n \to \infty$, so $l = A$ in this case.

Exercises 3.4

1. The sequence $\left((-1)^n\right)$ is certainly bounded since $|(-1)^n| = 1$ for all $n \in \mathbb{N}$ (so the sequence is bounded above by 1 and below by -1). However, the sequence is not convergent (see Example 3.4).

2. Since $|\sin(n)| \leq 1$ for all $n \in \mathbb{N}$, we have

$$0 \leq \frac{|\sin(n)|}{n} \leq \frac{1}{n}.$$

So by the sandwich rule, $\frac{|\sin(n)|}{n} \to 0$ as $n \to \infty$. Consequently $\frac{\sin(n)}{n} \to 0$ as $n \to \infty$ (see the comments after Theorem 3.6).

3. Since $\dfrac{n!}{n^n} = \dfrac{1}{n} \cdot \dfrac{2}{n} \cdot \dfrac{3}{n} \cdots \dfrac{n}{n}$ we have $0 < \dfrac{n!}{n^n} \leq \dfrac{1}{n}$, and so by the sandwich rule $\dfrac{n!}{n^n} \to 0$ as $n \to \infty$.

Exercises 3.5

1. Divide the numerator and denominator by the dominant term n^2 to get

$$x_n = \frac{2n^2 + (-1)^n n + 7}{3n^2 - 7n + 1} = \frac{2 + \frac{(-1)^n}{n} + \frac{7}{n^2}}{3 - \frac{7}{n} + \frac{1}{n^2}}.$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$, by the combination rules we have $\frac{1}{n^2} = \frac{1}{n} \times \frac{1}{n} \to 0$ as $n \to \infty$, and consequently $\frac{7}{n^2} \to 0$ as $n \to \infty$. Since $(-1)^n$ forms a bounded sequence, $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$. Then, again applying the combination rules, $2 + \frac{(-1)^n}{n} + \frac{7}{n^2} \to 2$ as $n \to \infty$ and $3 - \frac{7}{n} + \frac{1}{n^2} \to 3$ as $n \to \infty$, and finally $x_n \to \frac{2}{3}$ as $n \to \infty$.

[Your instructor may or may not want you to specify every individual step in such a solution. You need to ask them what they want. My personal view is that once you've seen this sort of use of the combination rules, you could simply say that "by the combination rules $x_n = \dfrac{2 + \frac{(-1)^n}{n} + \frac{7}{n^2}}{3 - \frac{7}{n} + \frac{1}{n^2}} \to \dfrac{2+0+0}{3+0+0} = \dfrac{2}{3}$ as $n \to \infty$."]

2. Here the dominant term is n^3 so dividing numerator and denominator by this, and then using the combination rules, we get

$$\begin{aligned} x_n &= \frac{n^2 + 5n - 3}{2n^3 + 5n^2 - n + 3} \\ &= \frac{\frac{1}{n} + \frac{5}{n^2} - \frac{3}{n^3}}{2 + \frac{5}{n} - \frac{1}{n^2} + \frac{3}{n^3}} \\ &\to \frac{0+0+0}{2+0+0+0} = 0 \text{ as } n \to \infty. \end{aligned}$$

Exercises 3.6

1. We have

$$\begin{aligned}\frac{y_n}{y_{n+1}} &= \frac{n!a^n}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!a^{n+1}} \\ &= \frac{n!(n+1)^{n+1}}{an^n(n+1)n!}, \quad \text{since } (n+1)! = (n+1)n! \\ &= \frac{(n+1)^n}{an^n} = \frac{1}{a} \cdot \Big(1 + \frac{1}{n}\Big)^n\end{aligned}$$

But $\Big(1+\frac{1}{n}\Big)^n$ is strictly increasing, so $\Big(1+\frac{1}{n}\Big)^n \geq \Big(1+\frac{1}{1}\Big)^1 = 2$. Consequently, $\frac{y_n}{y_{n+1}} \geq \frac{2}{a} > 1$. Hence $y_{n+1} < y_n$ ($\forall n \in \mathbb{N}$), i.e. (y_n) is a strictly decreasing sequence. It is clearly bounded below (by 0) so it must converge. Suppose that the limit is l, so that $y_n \to l$ as $n \to \infty$. Of course this implies that $y_{n+1} \to l$ as $n \to \infty$ (it's the same sequence apart from the first term). However, we have

$$y_n = \frac{1}{a}\Big(1 + \frac{1}{n}\Big)^n y_{n+1}$$

so $l = \frac{e}{a}l$ by the combination rules and using the result of Example 3.6. This gives $l(1 - \frac{e}{a}) = 0$ but $\frac{e}{a} \neq 1$, so we must have $l = 0$.

2. Applying the hint in the question we get

$$\begin{aligned}\Big(\frac{2}{1}\Big)^1\Big(\frac{3}{2}\Big)^2\Big(\frac{4}{3}\Big)^3 \cdots \Big(\frac{n+1}{n}\Big)^n &< e^n, \text{ i.e.} \\ \frac{2^1 \cdot 3^2 \cdot 4^3 \cdots n^{n-1} \cdot (n+1)^n}{1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdots n^n} &< e^n, \text{ i.e.} \\ \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdots \frac{1}{n} \cdot (n+1)^n &< e^n\end{aligned}$$

Hence $n!e^n > (n+1)^n = n^n\Big(1+\dfrac{1}{n}\Big)^n > 2n^n$, and this gives $\dfrac{n!e^n}{n^n} > 2$.
[This and the previous exercise are related to Stirling's Theorem which states that $\dfrac{n!e^n}{n^n\sqrt{n}} \to \sqrt{2\pi}$ as $n \to \infty$. But we don't yet have enough tools at our disposal to prove that here. For large n, Stirling's Theorem provides a relatively good approximation to $n!$, namely $\sqrt{2\pi n}\Big(\dfrac{n}{e}\Big)^n$.]

Exercises 3.7

1. The dominant term is 3^n, so we divide the numerator and the denominator by 3^n to get

$$x_n = \frac{4\frac{n^2}{3^n} + 1}{5\frac{n^3}{3^n} + 2}.$$

Since $\frac{n^2}{3^n}$ and $\frac{n^3}{3^n}$ form basic null sequences, it follows from the combination rules that $x_n \to \frac{4\times0+1}{5\times0+2} = \frac{1}{2}$ as $n \to \infty$. So the sequence (x_n) converges with limit $\frac{1}{2}$.

2. The dominant term is $n!$, so we divide the numerator and the denominator by $n!$ to get

$$y_n = \frac{5\frac{n^2}{n!} + \frac{(-2)^n}{n!}}{\frac{6^n}{n!} + 5}.$$

Since $\frac{n^2}{n!}$, $\frac{(-2)^n}{n!}$ and $\frac{6^n}{n!}$ form basic null sequences, it follows from the combination rules that $y_n \to \frac{5\times0+0}{0+5} = 0$ as $n \to \infty$. So the sequence (y_n) converges with limit 0.

Exercises 3.8

1. Suppose first that $m < 1$. Then $x_1 < 1$. Assuming that $x_k < 1$, we have $x_{k+1} = \frac{1}{2}(1 + x_k) < \frac{1}{2}(1 + 1) = 1$. So, by induction, $x_n < 1$ for all n. Now consider $x_{n+1} - x_n = \frac{1}{2}(1 + x_n) - x_n = \frac{1}{2}(1 - x_n) > 0$ since $x_n < 1$. So (x_n) is strictly increasing if $m < 1$.
Next suppose that $m > 1$. Then $x_1 > 1$. Assuming that $x_k > 1$, we have $x_{k+1} = \frac{1}{2}(1 + x_k) > \frac{1}{2}(1 + 1) = 1$. So, by induction, $x_n > 1$ for all n. Now consider $x_{n+1} - x_n = \frac{1}{2}(1 + x_n) - x_n = \frac{1}{2}(1 - x_n) < 0$ since $x_n > 1$. So (x_n) is strictly decreasing if $m > 1$.
If $m = 1$ then $x_1 = 1$, and a very easy induction gives $x_n = 1$ for all n.
In each case the sequence (x_n) is convergent and the limit l satisfies the equation $l = \frac{1}{2}(1 + l)$, so that $l = 1$.

2. Since $x_1 > 0$ it should be clear that $x_n > 0$ for all n. Applying the hint, we have for $n \geq 1$

$$\begin{aligned} x_{n+1}^2 - m &= \frac{1}{4}\left(x_n^2 + 2m + \frac{m^2}{x_n^2}\right) - m \\ &= \frac{1}{4}\left(x_n^2 - 2m + \frac{m^2}{x_n^2}\right) \\ &= \frac{1}{4}\left(x_n - \frac{m}{x_n}\right)^2 \\ &\geq 0. \end{aligned}$$

We also have $x_1^2 = m^2 = m \times m > 1 \times m = m$, since $m > 1$. Hence $x_n \geq \sqrt{m}$ for all n.
To prove that (x_n) is monotonically decreasing, consider

$$\begin{aligned}
x_{n+1} - x_n &= \frac{1}{2}\Big(x_n + \frac{m}{x_n}\Big) - x_n \\
&= \frac{1}{2}\Big(\frac{m}{x_n} - x_n\Big) \\
&= \frac{1}{2}\Big(\frac{m - x_n^2}{x_n}\Big) \\
&\leq 0.
\end{aligned}$$

Combining results, we see that (x_n) is monotonically decreasing and bounded below by $\sqrt{m}$. Consequently it is a convergent sequence with some limit $l \geq \sqrt{m}$. The recurrence relation gives $l = \frac{1}{2}(l + \frac{m}{l})$, which reduces to $l^2 = m$. Since $l > 0$, it follows that $l = \sqrt{m}$.
[This recurrence relation provides a method for finding the square root of any number $m > 1$, and hence of any positive number. The initial value $x_1 = m$ is best replaced by a closer approximation to $\sqrt{m}$ to reduce the number of iterations required to get a good approximation.]

Exercises 3.9

1. (i) If $x_n = n^2 + (-1)^n n$, then (x_n) is eventually positive since $n^2 > n$ for $n > 1$. Also

$$\frac{1}{x_n} = \frac{1/n^2}{1 + (-1)^n/n} \to \frac{0}{1+0} = 0 \text{ as } n \to \infty.$$

Hence, by Theorem 3.14, $x_n \to +\infty$ as $n \to \infty$.
(ii) If $x_n = n + (-1)^n n^2$ then , arguing as in part (i), $|x_n| \to +\infty$ as $n \to \infty$. But for n even, $x_n > 0$, while for $n > 1$ odd, $x_n < 0$. Hence (x_n) oscillates infinitely.
(iii) With $x_n = 1 + (-1)^n$ we have $|x_n| \leq 2$, so the sequence is bounded. If $x_n \to l$ as $n \to \infty$ then $x_n - 1 = (-1)^n \to l - 1$ as $n \to \infty$. But as already shown in Example 3.4 the sequence $\big((-1)^n\big)$ does not converge. Thus (x_n) oscillates finitely.
(iv) Since $(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n}) = 1$, we have

$$0 < \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}.$$

But $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ (you can check this from the definition of convergence). Hence, by the sandwich rule, $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.
(v) Since $(\sqrt{n^2+n}+n)(\sqrt{n^2+n}-n) = n$, we have

$$\begin{aligned}\sqrt{n^2+n} - n &= \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\ &\to \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \text{ as } n \to \infty.\end{aligned}$$

If you are suspicious about $\sqrt{1+\frac{1}{n}} \to 1$ as $n \to \infty$, use the inequalities $1 < \sqrt{1+\frac{1}{n}} < 1 + \frac{1}{n}$ and the sandwich rule.
(vi) Using the same technique as before we have

$$x_n = \sqrt{n^3+n^2} - \sqrt{n^3} = \frac{n^2}{\sqrt{n^3+n^2}+\sqrt{n^3}}.$$

Very loosely speaking, the denominator in the last expression is of order $n^{3/2}$, making x_n roughly of size $n^{1/2}$. So it looks likely that (x_n) diverges to $+\infty$. We therefore consider

$$\frac{1}{x_n} = \frac{\sqrt{n^3+n^2}+\sqrt{n^3}}{n^2} = \frac{\sqrt{\frac{1}{n}+\frac{1}{n^2}}+\sqrt{\frac{1}{n}}}{1} \to \frac{0}{1} = 0 \text{ as } n \to \infty.$$

Since $x_n > 0$ for all n and $\left(\frac{1}{x_n}\right)$ is null, it follows from Theorem 3.14 that $x_n \to +\infty$ as $n \to \infty$.
[Comment: Parts (iv), (v) and (iv) of this question are closely related. The terms of the sequence in part (iv) are approximately equal to $1/(2\sqrt{n})$ for large n. The terms in part (v) are $\sqrt{n}$ times larger, and so approximately $1/2$ for large n. The terms in part (vi) are a factor $\sqrt{n}$ larger again, and so approximately $\sqrt{n}/2$ for large n.]

2. We have $\frac{1+x_n}{x_n} = \frac{1}{x_n} + 1 \to 0 + 1 = 1$ as $n \to \infty$ using Theorem 3.14.

3. We have

$$\begin{aligned}s_{n+1} - s_n &= \frac{1}{2^n+1} + \frac{1}{2^n+2} + \ldots + \frac{1}{2^{n+1}} \quad (2^n \text{ terms}) \\ &> \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{n+1}} \quad (2^n \text{ terms}) \\ &= \frac{2^n}{2^{n+1}} = \frac{1}{2}\end{aligned}$$

It follow that (s_n) is strictly increasing. The sequence cannot be bounded above because if A is the least upper bound, then there must be a positive integer m such that $s_m > A - \frac{1}{4}$, but then $s_{m+1} > A + \frac{1}{4}$, a contradiction. (Indeed, it is easy to show that $s_n > 1 + \frac{n}{2}$ for $n \geq 1$.) Hence $s_n \to +\infty$ as $n \to \infty$.

4. We have $t_1 = 1$ and when $n = 1$, $2 - \frac{1}{n} = 1$. Hence for $n = 1$ it is true that $t_n \leq 2 - \frac{1}{n}$. Now assume the inequality holds for a positive integer k, i.e. $t_k \leq 2 - \frac{1}{k}$. Then

$$\begin{aligned} t_{k+1} &= t_k + \frac{1}{(k+1)^2} \\ &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \quad \text{by the inductive hypothesis} \\ &\leq 2 - \frac{1}{k+1}. \end{aligned}$$

For the last step note $\frac{1}{k+1} + \frac{1}{(k+1)^2} = \frac{k+2}{(k+1)^2} < \frac{1}{k}$ since $k(k+2) < (k+1)^2$. It follows, by induction, that $t_n \leq 2 - \frac{1}{n}$ for every positive integer n. Consequently (t_n) is bounded above by 2. Clearly (t_n) is strictly increasing, so (t_n) converges to its least upper bound, which is at most 2. (In fact the limit is $\pi^2/6$, but we can't prove that here.)

Exercises 3.10

1. $\left(\left(\frac{1}{(2n)!}\right)^{\frac{1}{2n}}\right)$ is a subsequence of $\left(\left(\frac{1}{n!}\right)^{\frac{1}{n}}\right)$, which is a basic null sequence. Hence

$$\left(\frac{1}{(2n)!}\right)^{\frac{1}{2n}} \to 0 \text{ as } n \to \infty.$$

But then by the product rule

$$\left(\frac{1}{(2n)!}\right)^{\frac{1}{n}} = \left[\left(\frac{1}{(2n)!}\right)^{\frac{1}{2n}}\right]^2 \to 0 \text{ as } n \to \infty.$$

2. Suppose that $x_{n_r} \to l$ as $n \to \infty$ is a convergent subsequence of (x_n). Then (x_{n_r}) is bounded above by some number A. But for each $r \in \mathbb{N}$ we have $r \leq n_r$. Since (x_n) is monotonically increasing, this gives $x_r \leq x_{n_r} \leq A$. Hence the entire sequence is bounded above by A. By Theorem 3.10 this establishes that (x_n) is a convergent sequence with some limit x. But every subsequence of a convergent sequence has the same limit (Theorem 3.15), so $x = l$.

3. The sequence $\left((1+\frac{2}{n})^n\right)$ has a subsequence $\left((1+\frac{2}{2n})^{2n}\right) = \left((1+\frac{1}{n})^{2n}\right)$. We know that $(1+\frac{1}{n})^n \to e$ as $n \to \infty$ so, by the product rule, $(1+\frac{1}{n})^{2n} \to$

e^2 as $n \to \infty$. It then follows from the previous question that $(1+\frac{2}{n})^n \to e^2$ as $n \to \infty$.

4. See Example 3.6 and modify what was done there to prove that $\left((1+\frac{2}{n})^n\right)$ is increasing. There is no need to prove boundedness.

5. We have $\left(1+\frac{1}{n-1}\right)^{n-1} \to e$ as $n \to \infty$. Therefore

$$\left(1+\frac{1}{n-1}\right)^n = \left(1+\frac{1}{n-1}\right)^{n-1}\left(1+\frac{1}{n-1}\right) \to e\times(1+0) = e \text{ as } n \to \infty.$$

Hence

$$\left(1-\frac{1}{n}\right)^n = \frac{1}{\left(1+\frac{1}{n-1}\right)^n} \to \frac{1}{e} \text{ as } n \to \infty.$$

6. In the light of the preceding questions it seems reasonable to guess that $(1+\frac{x}{n})^n \to e^x$ as $n \to \infty$, at least for integer values of x. We look at this in more detail when we consider the function $\exp(x)$ in a later chapter.

7. Since (x_n) is bounded, $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$ both exist. They have different values because otherwise (x_n) would be convergent (Theorem 3.18). Also, there are subsequences of (x_n) that converge to each of these values (Theorems 3.16 and 3.17).

8. Take x_{n_1} to be the first member of the sequence to exceed 1. Then define x_{n_r} for $r > 1$ to be the first member of the sequence to exceed $x_{n_{r-1}} + 1$. This is possible since (x_n) is unbounded above. Moreover, $x_{n_r} > r$ for each positive integer r. Therefore, given any real number A, there exists R (namely $R = |A| + 1$) such that if $r > R$ then $x_{n_r} > r > |A| + 1 > A$. Hence $x_{n_r} \to +\infty$ as $r \to \infty$.

9. In view of the previous two questions, if (x_n) does not converge to l it must have a subsequence (x_{n_r}) that converges to some other limit $l' \neq l$ or diverges to $\pm\infty$. But at least one of the subsequences $S_1, S_2, \ldots, S_k$, say S_i, must contain an infinite number of the terms of the subsequence (x_{n_r}) and so S_i cannot converge to l, a contradiction. Hence (x_n) must converge to l.

Exercises 3.11

1. First consider $|x_m - x_n|$ where (without loss of generality) $m \leq n$. By the triangle inequality we have

$$\begin{aligned}
|x_m - x_n| &\le |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + |x_{m+2} - x_{m+3}| + \dots \\
&\qquad \dots + |x_{n-1} - x_n| \\
&\le \frac{1}{2^m} + \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \dots + \frac{1}{2^{n-1}} \\
&= \frac{1}{2^m}\Big[1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-m-1}}\Big] \\
&= \frac{1}{2^m}\Big[\frac{1 - \frac{1}{2^{n-m}}}{1 - \frac{1}{2}}\Big] \qquad \text{by summing the geometric series} \\
&< \frac{1}{2^{m-1}}
\end{aligned}$$

Now choose $\epsilon > 0$. Take N to be the the first (i.e. least) integer such that $2^N > \frac{1}{\epsilon}$. Then if $m, n > N$ we have $|x_m - x_n| < \frac{1}{2^N} < \epsilon$. Hence (x_n) is a Cauchy sequence and therefore convergent.

Exercises 4.1

1. We have $\frac{1}{i(i+2)} = \frac{1}{2}\big(\frac{1}{i} - \frac{1}{i+2}\big)$. Hence

$$\begin{aligned}
S_n &= \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \dots + \frac{1}{n(n+2)} \\
&= \frac{1}{2}\Big[\Big(1 - \frac{1}{3}\Big) + \Big(\frac{1}{2} - \frac{1}{4}\Big) + \Big(\frac{1}{3} - \frac{1}{5}\Big) + \dots \\
&\qquad + \Big(\frac{1}{n-2} - \frac{1}{n}\Big) + \Big(\frac{1}{n-1} - \frac{1}{n+1}\Big) + \Big(\frac{1}{n} - \frac{1}{n+2}\Big)\Big] \\
&= \frac{1}{2}\Big[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\Big] \\
&\to \frac{1}{2}\Big[1 + \frac{1}{2}\Big] = \frac{3}{4} \text{ as } n \to \infty.
\end{aligned}$$

Hence the series converges with sum $3/4$.

2. The n^{th} partial sum of this series can be expressed as

$$\begin{aligned}
S_n &= \frac{1}{2}\Big[1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}\Big] \\
&= \frac{1}{2}\Big[\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\Big] \\
&= 1 - \frac{1}{2^n} \\
&\to 1 \text{ as } n \to \infty.
\end{aligned}$$

Hence the series converges with sum 1.

3. The n^{th} partial sum of this series can be expressed as

$$\begin{aligned} S_n &= \frac{1}{2}\Big[1 - \frac{1}{2} + \frac{1}{2^2} + \ldots + \Big(-\frac{1}{2}\Big)^{n-1}\Big] \\ &= \frac{1}{2}\Big[\frac{1 - \big(-\frac{1}{2}\big)^n}{1 + \frac{1}{2}}\Big] \\ &= \frac{1}{3}\Big[1 - \Big(-\frac{1}{2}\Big)^n\Big] \\ &\to \frac{1}{3} \text{ as } n \to \infty. \end{aligned}$$

Hence the series converges with sum $1/3$.

Exercises 4.2

1. a) (Thinks:

$$\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdots \frac{2}{n} \cdot \frac{1}{n} < \frac{2}{n} \cdot \frac{1}{n} = \frac{2}{n^2} \text{ for } n > 2;$$

try non-limit version of the comparison test with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.)
If $a_n = \frac{n!}{n^n}$ then for $n > 2$, $a_n < \frac{2}{n^2}$ and so by the comparison test the series converges.

b) (Thinks: $n/(n^3+1)$ behaves like $1/n^2$ and so the square root behaves like $1/n$, suggesting divergence; try comparison test with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$.)
If $a_n = \sqrt{\left(\frac{n}{n^3+1}\right)}$ then $a_n/(\frac{1}{n}) = \sqrt{\left(\frac{n^3}{n^3+1}\right)} \to 1$ as $n \to \infty$, and so the series diverges.

c) (Thinks: $\frac{n}{\sqrt{4n^5+1}}$ behaves like $\frac{1}{\sqrt{4n^3}} = \frac{1}{2n^{3/2}}$, suggesting convergence; try comparison test with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.)
If $a_n = \frac{n}{\sqrt{4n^5+1}}$ then $a_n/\frac{1}{n^{3/2}} = \frac{n^{5/2}}{\sqrt{4n^5+1}} = \sqrt{\frac{n^5}{4n^5+1}} \to \frac{1}{2}$ as $n \to \infty$, and so the series converges.

d) (Thinks: $\sqrt{n}$ is small compared with n^2, so $\frac{1}{\sqrt{n^2+\sqrt{n}}}$ will probably behave like $\frac{1}{\sqrt{n^2}} = 1/n$; try comparison test with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$.)
If $a_n = \frac{1}{\sqrt{n^2+\sqrt{n}}}$ then $a_n/(\frac{1}{n}) = \frac{n}{\sqrt{n^2+\sqrt{n}}} = \frac{1}{\sqrt{1+n^{-3/2}}} \to 1$ as $n \to \infty$, and so the series diverges.

e) (Thinks: each term roughly $1/3$ of its predecessor; try the ratio test.) If $a_n = n/3^n$, then $a_{n+1}/a_n = \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} = \frac{n+1}{n} \cdot \frac{1}{3} \to \frac{1}{3}$ as $n \to \infty$, and so the series converges.

f) (Thinks: n^{th} power suggests n^{th} root test.) If $a_n = \left(\frac{n}{2n+1}\right)^n$ then $(a_n)^{\frac{1}{n}} = \frac{n}{2n+1} \to \frac{1}{2}$ as $n \to \infty$, and so the series converges.

2. Put $a_n = (\sqrt{n+1} - \sqrt{n})$ and note that $(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = (n+1) - n = 1$. Hence

$$a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{\sqrt{4n} + \sqrt{n}} = \frac{1}{3\sqrt{n}}.$$

And because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, it follows from the comparison test that $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

From the previous working we see that $a_n = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$ and so (a_n) is monotonically decreasing and tends to zero. It follows from Leibniz' alternating series test that $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

3. Since $\frac{a_n}{n} \geq 0$ for all $n \in \mathbb{N}$, the partial sums of the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ are monotonically increasing. We prove that these partial sums are bounded above and it then follows that the series must be convergent.

There are 9 1-digit numbers (1 to 9), all of these have no zero in their decimal representation, and the smallest of these numbers is 1.

So $\sum_{n=1}^{9} \frac{a_n}{n} \leq \frac{9}{1}$.

There are 90 2-digit numbers (10 to 99). To avoid a zero in the decimal representation, there are 9 choices for each digit, so 9^2 2-digit numbers have no zero, and the smallest 2-digit number is 10.

Hence $\sum_{n=10}^{99} \frac{a_n}{n} \leq \frac{9^2}{10}$.

There are 900 3-digit numbers (100 to 999). To avoid a zero in the decimal representation, there are 9 choices for each digit, so 9^3 3-digit numbers have no zero, and the smallest 3-digit number is $100 = 10^2$.

Hence $\sum_{n=100}^{999} \frac{a_n}{n} \leq \frac{9^3}{10^2}$.

Proceeding in this way, there are $9 \times 10^{m-1}$ m-digit numbers (10^{m-1} to $10^m - 1$). To avoid a zero in the decimal representation, there are 9 choices for each digit, so 9^m m-digit numbers have no zero, and the smallest m-digit number is 10^{m-1}.

Hence $\sum_{n=10^{m-1}}^{10^m-1} \frac{a_n}{n} \leq \frac{9^m}{10^{m-1}}$.

It follows that the partial sums of $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ are bounded above by the sum of the convergent geometric series

$$\frac{9}{1} + \frac{9^2}{10} + \frac{9^3}{10^2} + \ldots = 9\sum_{n=0}^{\infty}\Big(\frac{9}{10}\Big)^n = 9\Big(\frac{1}{1-\frac{9}{10}}\Big) = 90.$$

Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

It is perhaps surprising that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges even though $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. The reason is that almost every large n has a zero somewhere in its decimal representation - think of decimals that are 100 digits long - the chance of one or more zeros in the decimal expansion is $1 - (0.9)^{100} = 0.99997$ to 5 decimal places. This might get you thinking of other ways to thin-out the harmonic series and get convergence. An obvious thing to try is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{p_n}$, where p_n is the n^{th} prime number. In fact this series still diverges, but the proof is not easy.

Exercises 4.3

1. a) The expression $\cos(n)$ does not alternate in sign: $\cos(1) > 0, \cos(2) < 0, \cos(3) < 0$, and so on. This series is not alternating.

b) The expression $\cos(n\pi/2)$ does not alternate in sign: $\cos(\pi/2) = 0, \cos(\pi) = -1, \cos(3\pi/2) = 0, \cos(2\pi) = 1$, and so on. This series is not alternating.

c) The expression $\sin((2n+1)\pi/2)$ does alternate in sign: $\sin(\pi/2) = 1, \sin(3\pi/2) = -1, \sin(5\pi/2) = 1$, and so on. This series is alternating.

d) The function $\tan(x)$ is undefined for $x = \pi/2, 3\pi/2, 5\pi/2, \ldots$ so this series is undefined and certainly not alternating.

2. a) Since $\displaystyle\left|\frac{\cos(n)}{n^2}\right| \leq \frac{1}{n^2}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, this series is absolutely convergent by the comparison test.

b) By the same argument as in part a), this series is also absolutely convergent.

c) The terms of this series alternate in sign, the absolute value of the n^{th} term is $\frac{1}{n}$, and $\frac{1}{n}$ forms a decreasing null sequence. By Leibniz' alternating series test the series converges, but since $\sum_{n=1}^{\infty} \frac{1}{n}$ (the harmonic series) is divergent, the convergence of the given series is not absolute, i.e. the series converges conditionally.

d) This series is ill-defined and questions of convergence do not arise.

3. a) For any $x \in \mathbb{R}$, $\left|\dfrac{(-1)^n}{n^2+x^2}\right| \leq \dfrac{1}{n^2}$. So the series converges absolutely by the comparison test (with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$) for every Real Number x.

b) If $|x| \leq 1$ then $1+x^{2n} \leq 2$ and so $\dfrac{n}{1+x^{2n}} \geq \dfrac{n}{2} \to \infty$ as $n \to \infty$. So the terms of the series are not null, and the series therefore diverges if $|x| \leq 1$.

If $|x| > 1$ we can employ the ratio test with $a_n = \dfrac{n}{1+x^{2n}}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n}\,\frac{1+x^{2n}}{1+x^{2n+2}} = \Big(1+\frac{1}{n}\Big)\Big(\frac{x^{-2n}+1}{x^{-2n}+x^2}\Big) \to \frac{1}{x^2} \text{ as } n \to \infty.$$

But $\dfrac{1}{x^2} < 1$, so the series converges by the limit form of the ratio test if $|x| > 1$.

c) If $|x| \geq 1$ then $|x^{n!}| \geq 1$ and the series diverges because the terms do not tend to zero. If $|x| < 1$ then $|x^{n!}| \leq |x|^n$, and so the series converges (absolutely) by comparison with the geometric series $\displaystyle\sum_{n=1}^{\infty} |x|^n$.

d) The ratio test with $a_n = n^2 \dfrac{x^{3n}}{2^n}$ and $x \neq 0$ gives

$$\left|\frac{a_{n+1}}{a_n}\right| = \Big(\frac{n+1}{n}\Big)^2 \frac{|x|^3}{2} \to \frac{|x|^3}{2} \text{ as } n \to \infty.$$

Hence the series converges (absolutely) if $|x| < \sqrt[3]{2}$ and diverges if $|x| > \sqrt[3]{2}$. (The radius of convergence of this series is $\sqrt[3]{2}$.) In the cases $|x| = \sqrt[3]{2}$, $|a_n| = n^2$, so that (a_n) is not a null sequence and the series therefore diverges.

4. Taking the absolute value of each term, the resulting series is the geometric series $\sum_{n=0}^{\infty}(\frac{1}{2})^n$, with common ratio $\frac{1}{2}$, which is convergent. So the series given in the question is absolutely convergent.

Taking the series formed from the positive terms we have $1 + \frac{1}{8} + \frac{1}{64} + \ldots$. This is a geometric series with common ratio $\frac{1}{8}$ which converges to $\frac{1}{1-\frac{1}{8}} = \frac{8}{7}$. But then $\frac{1}{2}+\frac{1}{16}+\frac{1}{128}+\ldots = \frac{1}{2}\times\frac{8}{7} = \frac{4}{7}$ and $\frac{1}{4}+\frac{1}{32}+\frac{1}{256}+\ldots = \frac{1}{4}\times\frac{8}{7} = \frac{2}{7}$. So the given series converges to the sum $\frac{8}{7} - \frac{4}{7} - \frac{2}{7} = \frac{2}{7}$.

5. For $x \neq 0$, we have $\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = \left(\frac{n+2}{n+1}\right)|x| \to |x|$ as $n \to \infty$. So, by the ratio test, the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $|x| = 1$, the terms of the series do not form a null sequence, and so the series diverges if $|x| = 1$. To determine the sum of the series we calculate $S_n - 2xS_n + x^2S_n$ as follows.

$$\begin{aligned} S_n &= 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} \\ -2xS_n &= \quad - 2x - 4x^2 - 6x^3 - \ldots - 2(n-1)x^{n-1} - 2nx^n \\ +x^2S_n &= \qquad\qquad + x^2 + 2x^3 + \ldots + (n-2)x^{n-1} + (n-1)x^n + nx^{n+1} \end{aligned}$$

Now add the three lines, combining like powers of x to get

$$(1 - 2x + x^2)S_n = 1 + (n-3)x^n + nx^{n+1}.$$

If $|x| < 1$ we deduce that $(1 - 2x + x^2)S_n \to 1$ as $n \to \infty$, and so $S_n \to \frac{1}{1 - 2x + x^2} = \frac{1}{(1-x)^2}$ as $n \to \infty$. Hence, when convergent, the sum of the series is $\frac{1}{(1-x)^2}$.

6. We dealt with $\exp(x)$ in Example 4.12.
For $\sin(x)$ put $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Then for $x \neq 0$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{(2n+3)(2n+2)} \to 0 \text{ as } n \to \infty.$$

By D'Alembert's ratio test, it follows that the series converges for all x, i.e. the radius of convergence is infinite.
For $\cos(x)$ put $a_n = \frac{(-1)^n x^{2n}}{(2n)!}$. Then for $x \neq 0$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{(2n+2)(2n+1)} \to 0 \text{ as } n \to \infty.$$

By D'Alembert's ratio test, it follows that the series converges for all x, i.e. the radius of convergence is infinite.

7. Use the ratio test for $x \neq 0$ with $a_n = \dfrac{n!x^n}{n^n}$. We have

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot |x| \\ &= (n+1) \cdot \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} \cdot |x| \\ &= \frac{1}{(1+\frac{1}{n})^n} \cdot |x|\end{aligned}$$

But, from example 3.6 of Chapter 3, $\left(1 + \dfrac{1}{n}\right)^n \to e$ as $n \to \infty$, so $\left|\dfrac{a_{n+1}}{a_n}\right| \to \dfrac{|x|}{e}$ as $n \to \infty$. Hence the series converges if $|x| < e$ and diverges if $|x| > e$, so the radius of convergence is e (the value of e is approximately 2.718).

If you think carefully about this result, you will see that the series in the question has the same radius of convergence as the geometric series $\sum_{n=1}^{\infty}(\frac{x}{e})^n$. So, speaking very loosely, $n!/n^n$ behaves a bit like $1/e^n$ for large n. In fact it can be shown that $n!e^n/(\sqrt{2\pi n})n^n \to 1$ as $n \to \infty$, so that $(\sqrt{2\pi n})\left(\frac{n}{e}\right)^n$ is a reasonable approximation to $n!$ for large n. This result is known as Stirling's Theorem or Stirling's Approximation.

8. We have (with decimals to 6dp):
$1 > 0.875$, $1 - \frac{1}{2} < 0.875$, $1 - \frac{1}{2} + \frac{1}{3} = 0.833333 < 0.875$,
$1-\frac{1}{2}+\frac{1}{3}+\frac{1}{5} = 1.033333 > 0.875$, $1-\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-\frac{1}{4} = 0.783333 < 0.875$,
$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} = 0.926190 > 0.875$,
$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} = 0.759524 < 0.875$,
$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \frac{1}{9} = 0.870635 < 0.875$,
$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \frac{1}{9} + \frac{1}{11} = 0.961544 > 0.875$,
$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8} = 0.836544 < 0.875$.
So the first ten terms of the rearrangement in the correct order are

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8}.$$

Exercises 4.4

1. The x^n term in the Cauchy product is

$$1 \cdot \frac{x^n}{n!} + \frac{x}{1!} \cdot \frac{x^{n-1}}{(n-1)!} + \ldots \frac{x^i}{i!} \cdot \frac{x^{n-i}}{(n-i)!} + \ldots + \frac{x^n}{n!} \cdot 1.$$

This can be written as

$$\frac{x^n}{n!}\Big(1+\frac{n!}{1!(n-1)!}+\ldots+\frac{n!}{i!(n-i)!}\ \ldots+1\Big),$$

or more suggestively as

$$\frac{x^n}{n!}\Big(\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{i}+\ldots+\binom{n}{n}\Big),$$

where $\binom{n}{i}$ denotes a binomial coefficient. But the sum

$$\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{i}+\ldots+\binom{n}{n}$$

is just the binomial expansion of $(1+1)^n$, and so this sum comes to 2^n. It follows that the Cauchy product

$$(\exp(x))^2=\sum_{n=0}^{\infty}\Big(\frac{x^n}{n!}\cdot 2^n\Big)=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}=\exp(2x).$$

2. The general term in the Cauchy product is

$$1\cdot\frac{y^n}{n!}+\frac{x}{1!}\cdot\frac{y^{n-1}}{(n-1)!}+\ldots\frac{x^i}{i!}\cdot\frac{y^{n-i}}{(n-i)!}+\ldots+\frac{x^n}{n!}\cdot 1.$$

This can be written as

$$\frac{1}{n!}\Big(y^n+y^{n-1}x\frac{n!}{1!(n-1)!}+\ldots+y^{n-i}x^i\frac{n!}{i!(n-i)!}\ \ldots+x^n\Big),$$

or more suggestively as

$$\frac{1}{n!}\Big(y^nx^0\binom{n}{0}+y^{n-1}x^1\binom{n}{1}+\ldots+y^{n-i}x^i\binom{n}{i}+\ldots+y^0x^n\binom{n}{n}\Big).$$

But the sum

$$y^nx^0\binom{n}{0}+y^{n-1}x^1\binom{n}{1}+\ldots+y^{n-i}x^i\binom{n}{i}+\ldots+y^0x^n\binom{n}{n}$$

is just the binomial expansion of $(y+x)^n$. It follows that the Cauchy product

$$\exp(x)\cdot\exp(y)=\sum_{n=0}^{\infty}\frac{(y+x)^n}{n!}=\exp(x+y).$$

3. Denote the n^{th} term in the Cauchy product by c_n. The first few terms are $c_1 = 1 \cdot 1 = 1, \quad c_2 = -1 \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot 1,$
$c_3 = 1 \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot 1.$
Observe that $|c_1| = 1$, $|c_2| = \frac{2}{\sqrt{2}} = \sqrt{2} > 1$, and $|c_3| = 2 \cdot \frac{1}{\sqrt{3}} + \frac{1}{2} > \frac{3}{2} > 1$. With these in mind we try to prove that (c_n) is not a null sequence. It will suffice to consider only odd-numbered terms. We have

$$c_{2n+1} = 1 \cdot \frac{1}{\sqrt{2n+1}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2n}} + \ldots + \frac{1}{\sqrt{k}} \cdot \frac{1}{\sqrt{2n+2-k}} + \ldots + \frac{1}{\sqrt{2n+1}} \cdot 1.$$

We need to estimate the general term $\frac{1}{\sqrt{k}} \cdot \frac{1}{\sqrt{2n+2-k}}$ in this sum. This is likely to be smallest for the middle term, i.e. when $k = n+1$. So we try to prove that for $k = 1, 2, \ldots, 2n+1$ we have

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{\sqrt{2n+2-k}} \geq \frac{1}{\sqrt{n+1}} \cdot \frac{1}{\sqrt{n+1}} = \frac{1}{n+1}.$$

It is certainly true that $(n+1-k)^2 \geq 0$, which gives $(n+1)^2 - 2k(n+1) + k^2 \geq 0$, and hence $(n+1)^2 \geq k(2n+2-k)$. Taking roots and then reciprocals gives what we wanted, namely

$$\frac{1}{\sqrt{k(2n+2-k)}} \geq \frac{1}{n+1}.$$

Consequently each term in the expression for c_{2n+1} is at least $\frac{1}{n+1}$, and there are precisely $2n+1$ such terms, so $c_{2n+1} \geq \frac{2n+1}{n+1}$. It follows that (c_n) is not a null sequence and so $\sum_{n=1}^{\infty} c_n$ is divergent.

Exercises 5.1

1.

a) This mapping is not surjective (d is not in the image set), and it is not a function because 2 has two images, c and e.

b) This mapping is not surjective (c is not in the image set), but it is a many-one function.

c) This mapping is surjective, but it is a many-one function.

d) This mapping is surjective, and it is an injective function, and so bijective. The inverse function is $\phi^{-1} = \{(a,2),(b,1),(c,4),(d,3)\}$ with domain $\{a,b,c,d\}$ and co-domain $\{1,2,3,4\}$.

e) This is a bijective function with inverse $\phi^{-1}(x) = \sqrt[3]{x}$, domain $\mathbb{R}$, and co-domain $\mathbb{R}$.

f) This mapping is not surjective (-1 is not in the image set). It is a many-one function (both -1 and 1 have the same image).

g) This mapping is not a function (1 has two images, namely -1 and 1). However, it is surjective.

h) This mapping is surjective, but it is a many-one function (both -1 and 1 have the same image.

i) This mapping is a bijective function. The inverse function is $\phi^{-1}(x) = -\sqrt{x}$ with domain $\{x : x \geq 0\}$ and co-domain $\{x : x \leq 0\}$.

2. Put $y = x^2 - 3x + 2$. Then $x^2 - 3x + (2 - y) = 0$ and using the quadratic formula we get

$$x = \frac{3 \pm \sqrt{9 - 4(2 - y)}}{2} = \frac{3 \pm \sqrt{1 + 4y}}{2}.$$

From the domain of f we see that we must have $x \geq \frac{3}{2}$, so we are obliged to only accept the positive root, giving $x = (3 + \sqrt{1 + 4y})/2$. This has a solution for x in the domain of f provided that $1 + 4y \geq 0$, i.e. provided $y \geq -\frac{1}{4}$. Hence $f^{-1}(y) = (3 + \sqrt{1 + 4y})/2$, with domain $\{y : y \geq -\frac{1}{4}\}$ and co-domain (= the domain of f) $\{x : x \geq \frac{3}{2}\}$. [Of course y in this expression is a dummy variable and we can replace it by the more conventional x if we feel an urge to do so: $f^{-1}(x) = (3 + \sqrt{1 + 4x})/2$, etc.]

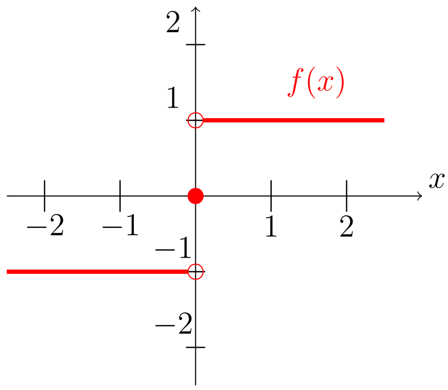
3. We have $(f \circ g)(x) = 2(x + 1)$ and $(g \circ f)(x) = 2x + 1$, both with domain and co-domain $\mathbb{R}$.

4. Suppose that for a given $x \geq 6$ we have $f(x, y) = f(x, z) = 0$. Then $y^3 - y - x = z^3 - z - x$, so $y^3 - z^3 = y - z$. But $y^3 - z^3 = (y - z)(y^2 + yz + z^2)$ and we have $(y - z)(y^2 + yz + z^2) = y - z$. It follows that either $y = z$ or $y^2 + yz + z^2 = 1$. In the latter case we have $y = \dfrac{-z \pm \sqrt{z^2 - 4(z^2 - 1)}}{2}$, and there is no solution to this if $4 - 3z^2 < 0$. Therefore to establish the result, we must show that if $x \geq 6$ and $z^3 - z - x = 0$ then $4 - 3z^2 < 0$. So assume that $x \geq 6$.
If $z < -1$ then $z^3 < z$ and so $z^3 - z < 0$, giving $z^3 - z - x \neq 0$. If $-1 \leq z \leq 1$ then $|z^3 - z| \leq |z^3| + |z| \leq 2$, again giving $z^3 - z - x \neq 0$. If $1 < z < 2$ then $z^3 - z = z(z^2 - 1) < 2 \times 3 = 6$, again giving $z^3 - z - x \neq 0$. It follows that if $x \geq 6$ and if $z^3 - z - x = 0$, we must have $z \geq 2$, which implies that $4 - 3z^2 \leq -8 < 0$.

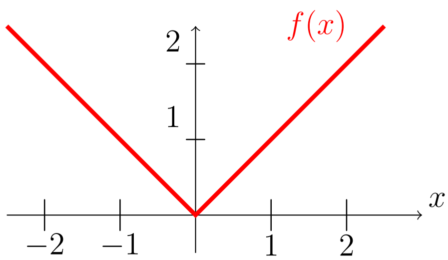
Exercises 5.2

1.

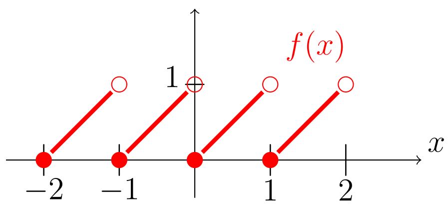
a) This is an odd function.



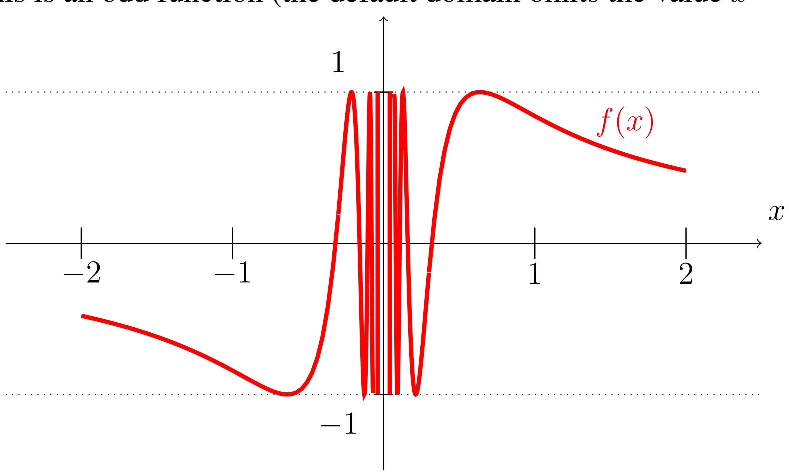
b) This is an even function.



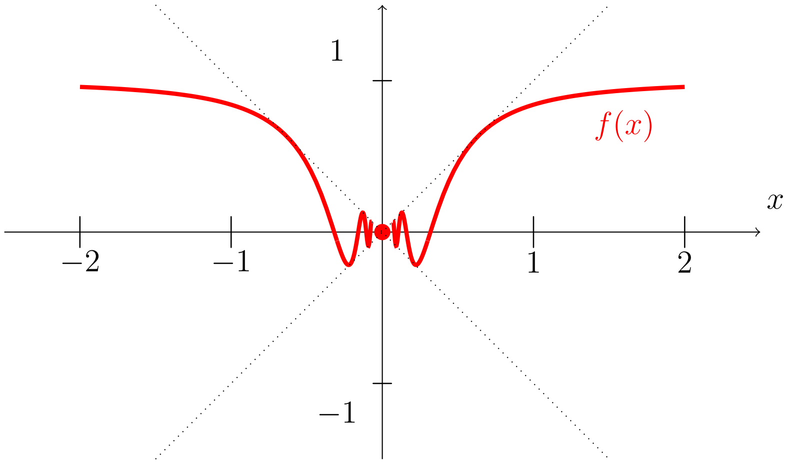
c) This function is neither even nor odd.



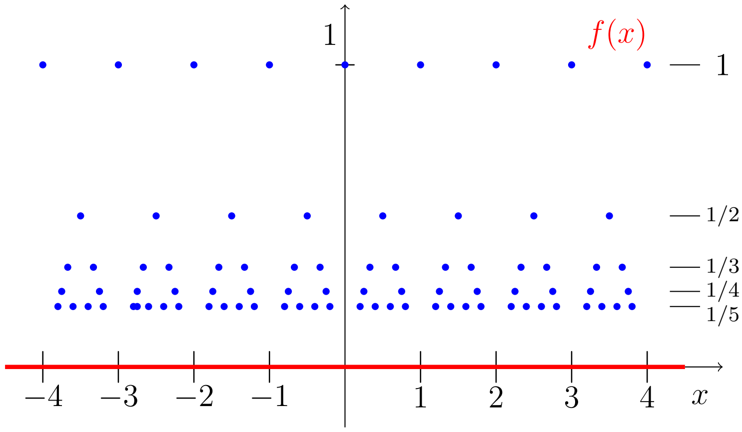
d) This is an odd function (the default domain omits the value $x = 0$).



e) This is an even function.



f) This is an even function.



This diagram is potentially misleading. There are different scales used on the horizontal and vertical axes. The red line along the horizontal axis represents all the irrational points x, and it omits all the rational points. The blue dots up above correspond to rational values of x. There is a vertical scale at the right-hand side showing dots at heights corresponding to rational numbers $x = p/q$ (expressed in lowest terms, with $q > 0$) for $q = 1, 2, 3, 4, 5$. There should be further lower levels of blue dots corresponding to $q = 6, 7, 8, \ldots$ but it isn't feasible to show these without cluttering the diagram to the extent it becomes unreadable.

Exercises 5.3.2

1. Choose $\epsilon > 0$. Put $X = \sqrt{1/\epsilon}$. Take any $x > X$ and consider

$$|f(x) - 0| = \frac{1}{1+x^2} < \frac{1}{X^2} = \epsilon.$$

2. Choose A. Put $X = \sqrt{|A|}$ so that $X \geq 0$. Take any $x > X$ and consider

$$f(x) = x^2 > X^2 = |A| \geq A.$$

3. We assume that $|\sin(x)| \leq 1$ for all x.
Method (a) First Principles. Choose $\epsilon > 0$. Put $\delta = \epsilon$. Take any x satisfying $0 < x < \delta$ and consider

$$\left| x\sin\left(\frac{1}{x}\right) - 0 \right| = \left| x\sin\left(\frac{1}{x}\right) \right| \leq |x| = x < \delta = \epsilon.$$

Method (b) One-sided Sandwich Rule. We have $-x \leq x\sin\left(\frac{1}{x}\right) \leq x$ for $x > 0$. Both x and $-x$ are (very simple) polynomials and so as x tends to zero (from above) both tend to their value (namely 0) at $x = 0$. It follows that $x\sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0+$.

4. We can use results about polynomials. For $x \neq a$ we have $f(x) = (x^2 - a^2)/(x - a) = x + a = g(x)$, say. But $g(x) = x + a$ is a (very simple) polynomial. So $f(x) = g(x) \to g(a) = 2a$ as $x \to a$.

5. (a) Suppose that $f(x) \to l_1$ as $x \to a$ and $f(x) \to l_2$ as $x \to a$, where $l_1 \neq l_2$. Without loss of generality we can assume that $l_1 < l_2$. Put $\epsilon = (l_2 - l_1)/2$ then $\exists \delta_1 > 0$ s.t. $\forall x$ satisfying $0 < |x - a| < \delta_1$, $|f(x) - l_1| < \epsilon$. Similarly $\exists \delta_2 > 0$ s.t. $\forall x$ satisfying $0 < |x - a| < \delta_2$, $|f(x) - l_2| < \epsilon$. Choose x satisfying $0 < |x - a| < \min(\delta_1, \delta_2)$. Then we have

$$|l_2 - l_1| = |(l_2 - x) - (l_1 - x)| \leq |l_2 - x| + |l_1 - x| < 2\epsilon = l_2 - l_1,$$

which is a contradiction. So we conclude that we must have $l_1 = l_2$.

(b) Choose $\epsilon > 0$. Put $\delta = 1$ (or any positive number you like). Then if $0 < |x - a| < \delta$, we have $|f(x) - l| = |l - l| = 0 < \epsilon$. [A very rare example where δ does not depend on ϵ.]

(c) Take $\epsilon = 1$. Then $\exists \delta_1 > 0$ s.t. $\forall x$ satisfying $0 < |x - a| < \delta_1$, $|f(x) - l| < 1$. But if $|f(x) - l| < 1$ then $l - 1 < f(x) < l + 1$, so the set $S = \{y : y = f(x) \text{ for some } x \text{ satisfying } 0 < |x - a| < \delta_1\}$ is a bounded set of Real Numbers, bounded below by $l - 1$ and above by $l + 1$.

(d) (i) If $k = 0$ then $kf(x)$ is the constant function with value 0, which converges to $0 = kl$. So suppose that $k \neq 0$. Choose $\epsilon > 0$. Put $\epsilon' = \epsilon/(|k|)$. Since $f(x) \to l$ as $x \to a$, there exists $\delta > 0$ such that for all x satisfying $0 < |x-a| < \delta$, $|f(x)-l| < \epsilon'$. In other words, if x satisfies $0 < |x-a| < \delta$, then $|f(x)-l| < \epsilon/(|k|)$, giving $|kf(x) - kl| < \epsilon$. Hence $kf(x) \to kl$ as $x \to a$.

(ii) Choose $\epsilon > 0$. Put $\epsilon' = \epsilon/2$. Then $\exists\delta_1, \delta_2$ such that if $0 < |x-a| < \delta_1$ then $|f(x)-l| < \epsilon'$, while if $0 < |x-a| < \delta_2$ then $|g(x)-m| < \epsilon'$. Put $\delta = \min(\delta_1, \delta_2)$. Then if $0 < |x-a| < \delta$,

$$\begin{aligned}|f(x)+g(x)-(l+m)| &\leq |f(x)-l| + |g(x)-m| \\ &\leq \epsilon' + \epsilon' = \epsilon\end{aligned}$$

Hence $f(x) + g(x) \to l + m$ as $x \to a$.

(iii) Since $g(x)$ converges as x tends to a it is locally bounded, and so $\exists\delta_0$ and a corresponding value $A > 0$ such that $|g(x)| < A$ for all x satisfying $0 < |x-a| < \delta_0$.
Choose $\epsilon > 0$. Put $\epsilon' = \epsilon/(A + |l|)$. Then $\exists\delta_1, \delta_2$ such that if $0 < |x-a| < \delta_1$ then $|f(x)-l| < \epsilon'$, while if $0 < |x-a| < \delta_2$ then $|g(x)-m| < \epsilon'$. Put $\delta = \min(\delta_0, \delta_1, \delta_2)$. Then if $0 < |x-a| < \delta$,

$$\begin{aligned}|f(x)g(x) - lm| &= |f(x)g(x) - lg(x) + lg(x) - lm| \\ &\leq |g(x)||f(x)-l| + |l||g(x)-m| \\ &< A\epsilon' + |l|\epsilon' \\ &= (A + |l|)\epsilon' = \epsilon.\end{aligned}$$

Hence $f(x)g(x) \to lm$ as $x \to a$.

(iv) In view of part (iii) it is only necessary to show that if $g(x) \to m$ as $x \to a$ and $m \neq 0$, then $1/g(x) \to 1/m$ as $x \to a$. First we obtain a positive lower bound for $|g(x)|$ in the vicinity of $x = a$. Taking $\epsilon = |m|/2$ in the definition, we obtain:

$$\exists\delta_1 \text{ s.t. } \forall x \text{ satisfying } 0 < |x-a| < \delta_1, |g(x)-m| < |m|/2.$$

By the triangle inequality $|m| - |g(x)| \leq |g(x) - m|$, and so for $0 < |x-a| < \delta_1$, $|m| - |g(x)| < |m|/2$. Hence $|g(x)| > |m|/2$ if $0 < |x-a| < \delta_1$.
Next choose $\epsilon > 0$. Put $\epsilon' = m^2\epsilon/2$. Then $\exists\delta_2$ such that if $0 < |x-a| < \delta_2$ then $|g(x)-m| < \epsilon'$. Put $\delta = \min(\delta_1, \delta_2)$.

Then if $0 < |x - a| < \delta$,

$$\begin{aligned}\Big|\frac{1}{g(x)} - \frac{1}{m}\Big| &= \Big|\frac{g(x) - m}{mg(x)}\Big| \\ &= \frac{|g(x) - m|}{|m||g(x)|} \\ &< \frac{2|g(x) - m|}{|m|^2} \\ &< \frac{2\epsilon'}{m^2} = \epsilon.\end{aligned}$$

Hence $1/g(x) \to 1/m$ as $x \to a$ and part (iv), the quotient rule, follows.

6. Choose A. Put $\delta = \frac{1}{|A|+1}$. Take any x satisfying $2 - \delta < x < 2$ so that $-\delta < x - 2 < 0$. Then consider

$$\frac{1}{x-2} < -\frac{1}{\delta} = -(|A| + 1) < A.$$

Hence $f(x) = \frac{1}{x-2} \to -\infty$ as $x \to 2-$.

7. Choose $\xi \in [a, b)$. Define the set $S = \{f(x) : x \in (\xi, b)\}$. Then S is non-empty since $\xi < b$, and S is bounded below by $f(\xi)$. Hence S has a greatest lower bound (infimum), say $l = \inf S$. Then if $x \in (\xi, b)$ we have $f(x) \geq l$ since l is a lower bound of S. Next choose $\epsilon > 0$. Since l is the greatest lower bound of S, there exists $\delta > 0$ such that $\xi + \delta < b$ and $f(\xi + \delta) < l + \epsilon$. But $f(x)$ is monotonically increasing, so if $x \in (\xi, \xi + \delta)$, then $f(x) \leq f(\xi + \delta) < l + \epsilon$. So, for $x \in (\xi, \xi + \delta)$ we have $l \leq f(x) < l + \epsilon$, and consequently $|f(x) - l| < \epsilon$. Hence $f(x) \to l$ as $x \to \xi+$.
The solution for $\lim_{x\to\xi-} f(x)$ is similar with $\xi \in (a, b]$, $S = \{f(x) : x \in (a, \xi)\}$, and $\inf$ replaced by $\sup$.

Exercises 5.4

1. Choose $\epsilon > 0$. Since f is continuous at a, there exists $\delta > 0$ such that for any x satisfying $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. But if $x_n \to a$ as $n \to \infty$, there exists N such that for all $n > N$, $|x_n - a| < \delta$. Consequently if $n > N$, $|f(x_n) - f(a)| < \epsilon$. Hence $f(x_n) \to f(a)$ as $n \to \infty$.

2. The definition of continuity of f at a can be expressed as statement P:

$$P : \forall\epsilon > 0, \exists\delta > 0 \text{ s.t. } \forall x \text{ satisfying } |x - a| < \delta,\ |f(x) - f(a)| < \epsilon.$$

If f is *not* continuous at a it must satisfy the negation of this, namely

$$\neg P : \exists \epsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \text{ satisfying } |x - a| < \delta \text{ s.t. } |f(x) - f(a)| \geq \epsilon.$$

In this negation, ϵ is fixed, but we have free choice for δ. Now suppose this negation $\neg P$ is true. For each positive integer n take x_n satisfying $|x_n - a| < \frac{1}{n}$ such that $|f(x_n) - f(a)| \geq \epsilon$. We have $x_n \to a$ as $n \to \infty$ (since $|x_n - a| \to 0$ as $n \to \infty$), but $f(x_n) \not\to f(a)$ as $n \to \infty$ (since $|f(x_n) - f(a)| \not\to 0$ as $n \to \infty$). But this contradicts the property of f given in the question, so $\neg P$ is false and therefore P must be true, i.e f is continuous at the point a.

3. The function f is a polynomial and therefore continuous on any interval. We draw up a table of values and apply the Intermediate Value Theorem to intervals where there is a change of sign of $f(x)$.

x	-2	-1	0	1	2	3	4
$f(x)$	23	-4	3	8	-1	-12	11

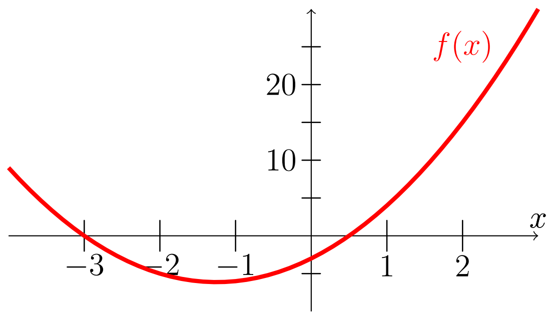
It follows that $f(x)$ has zeros in $(-2, -1)$, $(-1, 0)$, $(1, 2)$ and $(3, 4)$. Examining the interval $(-1, 0)$ we get (to 4 decimal places) $f(-\frac{1}{2}) = -1.9375$, $f(-\frac{1}{4}) = 0.4414$, $f(-\frac{3}{8}) = -0.8005$, $f(-\frac{5}{16}) = -0.1887$, $f(-\frac{9}{32}) = 0.1245$, $f(-\frac{19}{64}) = -0.0326$. So there is a zero of f in the interval $(-\frac{19}{64}, -\frac{9}{32})$, which has length $\frac{1}{64}$.

4. For each of the following functions determine where it is continuous and the image set. Justify your answers. [There is no shame in sketching the graph to find the answer before proving it is the answer.]

a) $2x^2 + 5x - 3$, b) $\frac{x^2+1}{x^2-1}$,

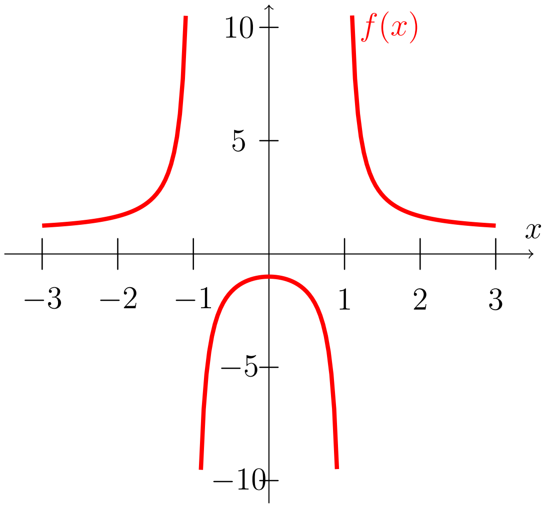
c) $\sqrt{x^2 - 1}$, d) $\frac{1}{\sqrt{x^2-1}}$.

a) The graph of $f(x) = 2x^2 + 5x - 3$ is sketched below.



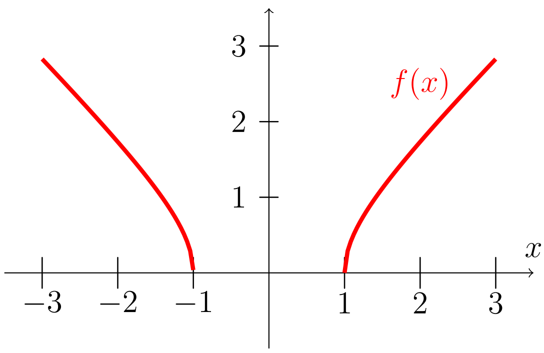
This function is a polynomial and therefore continuous on $\mathbb{R}$. Rearranging the formula gives $2x^2 + 5x - 3 = 2\big(x^2 + \frac{5}{2}x - \frac{3}{2}\big) = 2\big((x + \frac{5}{4})^2 - \frac{49}{16}\big)$. So $f(x)$ has minimum value $-\frac{49}{8}$ (at $x = -\frac{5}{4}$). Moreover, $f(x) \to +\infty$ as $x \to +\infty$. So by the Intermediate Value Theorem, $f(x)$ has image set $[-\frac{49}{8}, \infty)$.

b) The graph of $f(x) = \frac{x^2+1}{x^2-1}$ is sketched below.



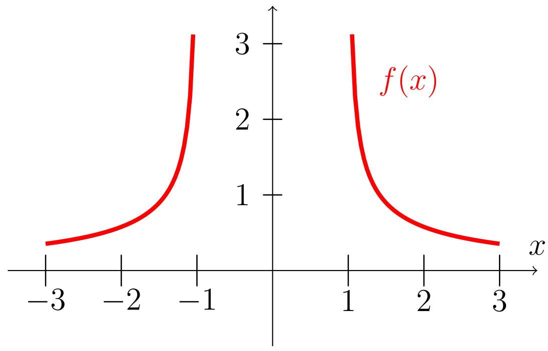
The function $f(x)$ is a rational function and so continuous except at points where the denominator vanishes, namely $x = \pm 1$ in this case. To determine the image set, put $y = \frac{x^2+1}{x^2-1}$. Then $x^2(y-1) = y+1$, so that $x^2 = \frac{y+1}{y-1} = 1 + \frac{2}{y-1}$. There will be a solution for x provided that $1 + \frac{2}{y-1} \geq 0$, i.e. $\frac{2}{y-1} \geq -1$. This last inequality clearly holds if $y > 1$, and if $y < 1$ it is equivalent to $2 \leq 1 - y$, i.e $y \leq -1$. So the image set is $(-\infty, -1] \cup (1, \infty)$.

c) The graph of $f(x) = \sqrt{x^2 - 1}$ is sketched below.



The polynomial $x^2 - 1$ is continuous on $\mathbb{R}$ and non-negative for $|x| \geq 1$. The function $\sqrt{x} = x^{\frac{1}{2}}$ is continuous on $[0, \infty)$. So by the composition rule for continuous functions $f(x) = \sqrt{x^2-1}$ is continuous on $(-\infty, -1] \cup [1, \infty)$. Since $f(1) = 0$ is the minimum value of $f(x)$, and $f(x) \to +\infty$ as $x \to +\infty$. by the Intermediate Value Theorem, the image set is $[0, \infty)$.

d) The graph of $f(x) = \frac{1}{\sqrt{x^2-1}}$ is sketched below.



From part c), the function $\sqrt{x^2-1}$ is continuous on $(-\infty, -1] \cup [1, \infty)$ and only takes the value 0 at $x = \pm 1$. So by the reciprocal rule $f(x)$ is continuous on $(-\infty, -1) \cup (1, \infty)$. Again by considering the previous answer, we see that the image set is $(0, \infty)$.

5. Put $M = \max(|a_i|, i = 0, 1, \ldots, n-1)$. Then if $|x| \geq 1$,

$$|a_{n-1}x^{n-1} + \ldots + a_1x + a_0| \leq nM|x|^{n-1}.$$

So $x^n - nM|x|^{n-1} \leq f(x) \leq x^n + nM|x|^{n-1}$.
Choose A and put $X = 1 + |A| + nM$. Then if $x > X$ we have

$$\begin{aligned} f(x) &\geq x^n - nM|x|^{n-1} \\ &= x^n - nMx^{n-1} \\ &= x^{n-1}[x - nM] \\ &\geq x - nM \quad (\text{note } \; x - nM > 0) \\ &> X - nM = 1 + |A| \geq A. \end{aligned}$$

So $f(x) \to +\infty$ as $x \to +\infty$.
But if $x < -X$ we have

$$\begin{aligned} f(x) &\leq x^n + nM|x|^{n-1} \\ &= x^n + nMx^{n-1} \\ &= x^{n-1}[x + nM] \\ &\leq x + nM \quad (\text{note } \; x + nM < 0) \\ &< -X + nM = -1 - |A| \leq A. \end{aligned}$$

So $f(x) \to -\infty$ as $x \to -\infty$.

It follows that there exist a and b, with $a < b$, such that $f(a) < 0$ and $f(b) > 0$. But f is a polynomial and so continuous on $[a, b]$. Then the Intermediate Value Theorem asserts that f has a zero in the interval $[a, b]$.

6. Taking $x_1 = x_2 = 1$ gives $f(1)+f(1) = f(1)$, so $f(1) = 0$. Of course this gives $f(2^0) = 0$ and we are told that $f(2^1) = 1$. Now suppose that $q > 1$ is a positive integer, and consider the sum of q terms: $f(2^{\frac{1}{q}}) + f(2^{\frac{1}{q}}) + \ldots + f(2^{\frac{1}{q}}) = f((2^{\frac{1}{q}})^q)$, using property (a) repeatedly. The left hand side of this is $qf(2^{\frac{1}{q}})$ and the right hand side is $f(2) = 1$. This gives $f(2^{\frac{1}{q}}) = \frac{1}{q}$.

Now suppose that p is any positive integer and consider the sum of p terms: $f(2^{\frac{1}{q}}) + f(2^{\frac{1}{q}}) + \ldots + f(2^{\frac{1}{q}}) = f((2^{\frac{1}{q}})^p)$, again using property (a). The left hand side is $pf(2^{\frac{1}{q}}) = p \times \frac{1}{q} = \frac{p}{q}$ and the right hand side is $f(2^{\frac{p}{q}})$. This gives $f(2^{\frac{p}{q}}) = \frac{p}{q}$, so we have $f(2^r) = r$ for any positive rational number r.

To deal with negative rational numbers consider the identity $f(2^r)+f(2^{-r}) = f(1) = 0$. For negative rational r this gives $f(2^r) = -f(2^{-r}) = r$.

To establish continuity at any point $x \in (0, \infty)$ we consider $f(x + h) - f(x) = f(x + h) + f(\frac{1}{x}) = f(1 + \frac{h}{x})$ Since f is continuous at 1, we have $f(1 + \frac{h}{x}) \to f(1) = 0$ as $h \to 0$. Hence $f(x + h) - f(x) \to 0$ as $h \to 0$, and consequently $f(x + h) \to f(x)$ as $h \to 0$, i.e. f is continuous at x.

[You might begin to suspect that $f(x)$ is actually $\log_2(x)$, logarithm to base 2.]

Exercises 6.1

1.

a) The values to 4dp for $h = 0.1, 0.01, 0.001$ are respectively $-0.2381, -0.2488, -0.2499$. For $h = -0.1, -0.01, -0.001$ the values are $-0.2632, -0.2513, -0.2501$. It looks fairly obvious that as h tends to zero, the limiting value is $-\frac{1}{4}$.

b) Again to 4dp for $h = 0.1, 0.01, 0.001$ the values are respectively $-0.4609, -0.4207. - 0.4166$. For $h = -0.1, -0.01, -0.001$ the values are $-0.3700, -0.4116, -0.4157$. It looks like there is a limiting value around -0.416, which compares with $\cos(2) = -0.4161$ (4dp).

2. For $x \neq 0$, $h \neq 0$ and $x+h \neq 0$ we have

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right] = \frac{x-(x+h)}{hx(x+h)}\\ &= \frac{-1}{x(x+h)} \to -\frac{1}{x^2} \text{ as } h \to 0.\end{aligned}$$

Hence if $x \neq 0$, f is differentiable at x with derivative $-\dfrac{1}{x^2}$. The function is not differentiable at $x=0$ since it is not defined at $x=0$.

3. If $h \in (-1,0)$ then $f(1+h) = 1+h-0 = 1+h$. If $h \in (0,1)$ then $f(1+h) = 1+h-1 = h$, and $f(1) = 0$.
So for $h \in (-1,0)$ we have $[f(1+h)-f(1)]/h = (1+h)/h \to -\infty$ as $h \to 0-$. Consequently f has no left-derivative at $x=1$. For $h \in (0,1)$ we have $[f(1+h)-f(1)]/h = h/h = 1 \to 1$ as $h \to 0+$. Consequently f has a right-derivative at $x=1$ with value 1.

Exercises 6.2

1.

a) $2x+1$,

b) $4x/(x^2+1)^2$,

c) $[\cos^2(x)+\sin^2(x)]/\cos^2(x) = 1/\cos^2(x), x \neq \pm\pi/2+2n\pi$, where n is any integer,

d) $2x\cos(x^2)$,

e) $-\frac{1}{2\sqrt{x}}\sin(\sqrt{x}), x > 0$,

f) $-2x\sin(x^2)\exp(\cos(x^2))$,

g) $-\frac{2}{x^3}\exp(1/x^2), x \neq 0$,

h) $(2x^2+1)\exp(x^2)\cos(x\exp(x^2))$,

i) $-(\exp(x)+2x)\sin(\exp(x)+x^2)$.

2. By Corollary 6.5.2, the function $g(z) = -z^{\frac{1}{n}}$ is differentiable at $z \in (0,\infty)$ with derivative $-\frac{1}{n}z^{\frac{1}{n}-1}$. The function $z(x) = -x$ is differentiable at each x with derivative -1 and maps the interval $(-\infty,0)$ to $(0,\infty)$, so the composite function $g(z(x)) = f(x)$ is differentiable on $(-\infty,0)$ with derivative $-\frac{1}{n}(-x)^{\frac{1}{n}-1} \times (-1)$. We can write the derivative as

$$\frac{1}{n}(-x)^{\frac{1}{n}-1} = \frac{1}{n}(-x)^{\frac{1}{n}} \times (-x)^{-1} = -\frac{1}{n}(x^{\frac{1}{n}}) \times (-1)(x^{-1}) = \frac{1}{n}(x^{\frac{1}{n}-1}).$$

3. By the product and composite rules, f is differentiable at every $x \neq 0$ with derivative $f'(x) = x^2\left(\frac{-1}{x^2}\right)\cos(\frac{1}{x}) + 2x\sin(\frac{1}{x}) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$. To determine differentiability at 0, consider for $h \neq 0$

$$\frac{f(0+h)-f(0)}{h} = \frac{h^2\sin(\frac{1}{h})}{h} = h\sin\left(\frac{1}{h}\right).$$

But $|\sin(x)| \leq 1$ for any x, so $h\sin(\frac{1}{h}) \to 0$ as $h \to 0$. Hence f is differentiable at 0 with derivative 0.

Exercises 6.3

1. The function f is differentiable on $\mathbb{R}$ and $f'(x) = 3x^2 - 6x + 1$, so f has stationary points at $x = \dfrac{6 \pm \sqrt{36-12}}{6} = 1 \pm \dfrac{2\sqrt{6}}{6} = 1 \pm \dfrac{\sqrt{6}}{3} = \alpha$ and β, say, ($\alpha < \beta$) and we have $f'(x) = 3(x-\alpha)(x-\beta)$. These values are approximately $\alpha \approx 0.18$ and $\beta \approx 1.82$. If $x < \alpha$ then $f'(x) > 0$, if $\alpha < x < \beta$ then $f'(x) < 0$, and if $x > \beta$ then $f'(x) > 0$. It follows from the first derivative test that α is a local maximum of f and β is a local minimum of f.

Alternatively, you can use the second derivative test since f'' exists and is given by $f''(x) = 6x - 6$. This is negative in the interval $(0,1)$ and positive in the interval $(1,2)$. It follows that $f''(\alpha) < 0$ and $f''(\beta) > 0$ and so confirms that α is a local maximum of f and β is a local minimum of f.

By considering the sign of $f'(x)$ we see that f is strictly increasing on $(-\infty, \alpha]$, strictly decreasing on $[\alpha, \beta]$, and strictly increasing on $[\beta, \infty)$. We have $f(-1) = -4$ and $f(0) = 1$, so by the Intermediate Value Theorem, there is a zero of f in $(-1, 0)$, and since f is strictly increasing on $(-\infty, \alpha]$, this is the only zero in that interval. Similarly $f(0.5) = 0.875$ and $f(1.5) = -0.875$, so there is precisely one zero of f in $[\alpha, \beta]$. Finally $f(2) = -1$ and $f(3) = 19$, so there is precisely one zero of f in $[\beta, \infty)$.

2. By the composite rule, $\frac{d}{dx}\big(\exp(x^2)\big) = 2x\exp(x^2)$. Hence by the product rule,

$$\frac{d}{dx}\big(y\exp(x^2)\big) = \frac{dy}{dx}\exp(x^2) + 2xy\exp(x^2) = \exp(x^2)\Big(\frac{dy}{dx} + 2xy\Big).$$

But from the original equation

$$\exp(x^2)\Big(\frac{dy}{dx} + 2xy\Big) = x\exp(x^2) = \frac{d}{dx}\big(\frac{1}{2}\exp(x^2)\big).$$

So the original equation gives $\frac{d}{dx}\big(y\exp(x^2)\big) = \frac{d}{dx}\big(\frac{1}{2}\exp(x^2)\big)$. The argument is reversible since $\exp(x^2) \neq 0$ for any value of x. Hence the new form of the equation is equivalent to the original form.

We now have two expressions with equal derivatives. By Corollary 6.8.2 it follows that they differ by some constant A, i.e. $y\exp(x^2) = \frac{1}{2}\exp(x^2) + A$. Multiplying both sides by $\exp(-x^2)$ gives $y = \frac{1}{2} + A\exp(-x^2)$. If $y = 1$ when $x = 0$ then $1 = \frac{1}{2} + A$, so $A = \frac{1}{2}$.

3. By the combination rules and composite rule, f is differentiable on $\mathbb{R}$ with the possible exception of zero. To deal with differentiability at zero, take $h \neq 0$ and consider

$$\frac{f(0+h) - f(0)}{h} = h\Big(2 - \sin\big(\frac{1}{h}\big)\Big) \to 0 \text{ as } h \to 0,$$

since $1 \leq 2 - \sin(\frac{1}{h}) \leq 3$. So f is differentiable at 0 and $f'(0) = 0$.

For $x \neq 0$,

$$f'(x) = x^2\frac{1}{x^2}\cos\big(\frac{1}{x}\big) + 2x\Big(2 - \sin\big(\frac{1}{x}\big)\Big) = \cos\big(\frac{1}{x}\big) + 2x\Big(2 - \sin\big(\frac{1}{x}\big)\Big).$$

If $|x| < \frac{1}{6}$ then $\Big|2x\Big(2 - \sin\big(\frac{1}{x}\big)\Big)\Big| < 1$. Now take any positive or negative integer n $(n \neq 0)$. Then $\cos(2n\pi) = 1$ and $\cos((2n+1)\pi) = -1$, so for $x = \frac{1}{2n\pi}$ we have $f'(x) > 1 - 1 = 0$, while for $x = \frac{1}{(2n+1)\pi}$ (except possibly for $n = -1$) we have $f'(x) < -1 + 1 = 0$. But there are infinitely many numbers of the form $x = \frac{1}{2n\pi}$ and of the form $x = \frac{1}{(2n+1)\pi}$ in any intervals immediately to the left of 0 and immediately to the right of 0. It follows that the first derivative test cannot be applied to determine the nature of the stationary point at zero.

Examining the definition of $f(x)$ we see that $f(x) > 0$ if $x \neq 0$, while $f(0) = 0$. Hence the function has a local minimum at 0.

4. All the functions forming the numerators and denominators in this question have derivatives of all orders on $\mathbb{R}$ We consider each case in turn.

a) With $f(x) = x^3 - 3x^2 + x + 1$ and $g(x) = \sin(\pi x)$, we have $f(1) = g(1) = 0$, $f'(x) = 3x^2 - 6x + 1$ and $g'(x) = \pi\cos(\pi x)$. By L'Hôpital's Rule

$$\lim_{x\to 1}\frac{x^3 - 3x^2 + x + 1}{\sin(\pi x)} = \lim_{x\to 1}\frac{3x^2 - 6x + 1}{\pi\cos(\pi x)},$$

provided that the latter limit exists. This latter limit does exist because the numerator and denominator are continuous and the denominator is non-zero at $x = 1$. The value of the limit is $f'(1)/g'(1) = (-2)/(-\pi) = 2/\pi$.

b) With $f(x) = \exp(x) - 1$ and $g(x) = \exp(2x) - 1$, we have $f(0) = g(0) = 0$, $f'(x) = \exp(x)$ and $g'(x) = 2\exp(2x)$. By L'Hôpital's Rule
$$\lim_{x\to 0} \frac{\exp(x) - 1}{\exp(2x) - 1} = \lim_{x\to 0} \frac{\exp(x)}{2\exp(2x)},$$
provided that the latter limit exists. This latter limit does exist because the numerator and denominator are continuous and the denominator is non-zero at $x = 0$. The value of the limit is $f'(0)/g'(0) = 1/2$.

c) With $f(x) = (\exp(x) - 1)^3$ and $g(x) = \sin(x) - x$, we have $f(0) = g(0) = 0$, $f'(x) = 3\exp(x)(\exp(x) - 1)^2$ and $g'(x) = \cos(x) - 1$. By L'Hôpital's Rule
$$\lim_{x\to 0} \frac{(\exp(x) - 1)^3}{\sin(x) - x} = \lim_{x\to 0} \frac{3\exp(x)(\exp(x) - 1)^2}{\cos(x) - 1},$$
provided that the latter limit exists. Unfortunately the numerator and denominator are both zero at $x = 0$, so we apply L'Hôpital's Rule again. But we can simplify the reapplication by noting that $3\exp(x) \to 3$ as $x \to 0$, so it suffices to consider
$$\lim_{x\to 0} \frac{(\exp(x) - 1)^2}{\cos(x) - 1}.$$
In a similar manner to the first step we get
$$\lim_{x\to 0} \frac{(\exp(x) - 1)^2}{\cos(x) - 1} = \lim_{x\to 0} \frac{2\exp(x)(\exp(x) - 1)}{-\sin(x)},$$
provided that the latter limit exists. Unfortunately the numerator and denominator are again both zero at $x = 0$, so we apply L'Hôpital's Rule a third time. Again we can remove the factor $2\exp(x)$ and consider
$$\lim_{x\to 0} \frac{\exp(x) - 1}{-\sin(x)} = \lim_{x\to 0} \frac{\exp(x)}{-\cos(x)},$$
provided this limit exists, which it does and has the value $1/(-1) = -1$. Putting back the discarded factors 3 and 2, we find that
$$\lim_{x\to 0} \frac{(\exp(x) - 1)^3}{\sin(x) - x} = -6.$$

d) Here there is no need to apply L'Hôpital's Rule; in fact you must not apply it since $g(x) = \sin(x) + 1$ is non-zero at $x = 0$. Instead just use the fact that both $f(x) = x + \cos(x) - 1$ and $g(x)$ are continuous at $x = 0$ to get

$$\lim_{x\to 0} \frac{x + \cos(x) - 1}{\sin(x) + 1} = \frac{f(0)}{g(0)} = \frac{0}{1} = 0.$$

If you do use L'Hôpital's Rule you will get the incorrect answer 1.

5. We use Leibniz' Theorem with $f(x) = \exp(x)$ and $g(x) = \sin(x)$. Bearing in mind that $f'(x) = \exp(x) = f(x)$, while $g'(x) = \cos(x)$ and $g''(x) = -\sin(x) = -g(x)$, this gives

$$D^{10}\big(\exp(x)\sin(x)\big) = A\exp(x)\sin(x) + B\exp(x)\cos(x),$$

where

$$\begin{aligned} A &= \left(1 - \binom{10}{2} + \binom{10}{4} - \binom{10}{6} + \binom{10}{8} - 1\right) = 0, \quad \text{and} \\ B &= \left(\binom{10}{1} - \binom{10}{3} + \binom{10}{5} - \binom{10}{7} + \binom{10}{9}\right) \\ &= 10 - 120 + 252 - 120 + 10 = 32. \end{aligned}$$

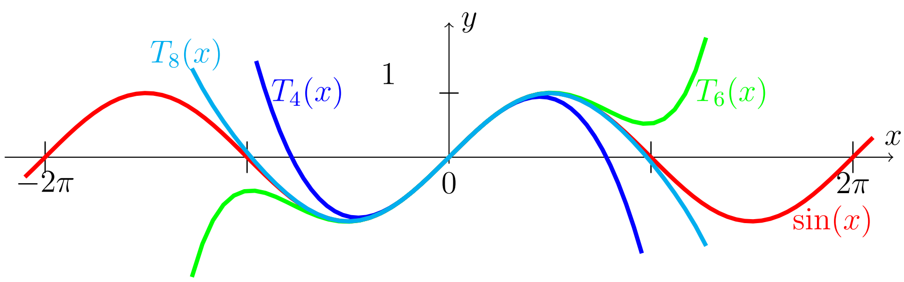
Hence the tenth derivative of $\exp(x)\sin(x)$ is $32\exp(x)\cos(x)$.

Exercises 6.4

1. The Taylor polynomials are

$$T_4(x) = x - \frac{x^3}{3!}, \quad T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \quad T_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

The graphs are sketched below.



Taylor series approximations to $\sin(x)$.

2. Evaluating, $T_8(0.1) = 0.099\,833\,416\,646\,825\,4$ to 16 decimal places. Then $R_8(0.1) = \frac{(0.1)^8}{8!}\sin(\xi)$, where $0 < \xi < 0.1$. Use the fact that $\sin(\xi)$ lies between 0 and 1 to show that $R_8(0.1)$ lies between between 0 and -2.481×10^{-13}. Hence $T_8(0.1)$ should give $\sin(0.1)$ correct to 12 decimal places as $0.099\,833\,416\,647$ (and it does).

3. First note that $\sin(45^{\mathrm{o}}) = \cos(45^{\mathrm{o}}) = \frac{1}{\sqrt{2}}$. Expanding about $\pi/4$ gives

$$\begin{aligned}\sin\Big(\frac{\pi}{4}+h\Big) &= \sin\Big(\frac{\pi}{4}\Big) + h\cos\Big(\frac{\pi}{4}\Big) - \frac{h^2}{2!}\sin\Big(\frac{\pi}{4}\Big) - \frac{h^3}{3!}\cos\Big(\frac{\pi}{4}\Big)\\ &\quad + \frac{h^4}{4!}\sin\Big(\frac{\pi}{4}\Big) + R_5(h),\end{aligned}$$

where $R_5(h) = -\frac{h^5}{5!}\cos(\xi)$ and ξ lies between $\frac{\pi}{4}$ and $\frac{\pi}{4}+h$. We are interested in $50^{\mathrm{o}} = 45^{\mathrm{o}} + 5^{\mathrm{o}}$, and $5^{\mathrm{o}} = 5\pi/180 = 0.087\,266\,463$ (to 9 decimal places), so we take $h = 0.087\,266\,463$. For our approximation we get

$$\sin(50^{\mathrm{o}}) \approx \frac{1}{\sqrt{2}}\Big[1 + h - \frac{h^2}{2} - \frac{h^3}{6} + \frac{h^4}{24}\Big] = 0.766\,044\,413 \text{ to 9 d.p.,}$$

with an error between $-\frac{h^5}{5!}$ and 0. This error is between -4.22×10^{-8} and 0. So we can confidently say that $\sin(50^{\mathrm{o}}) = 0.766\,044$, correct to 6 decimal places.

4. Observe that $(1+x)^\alpha = x^\alpha\Big(1+\frac{1}{x}\Big)^\alpha$. If $|x|>1$ then $|\frac{1}{x}|<1$ and so we may expand $\Big(1+\frac{1}{x}\Big)^\alpha$ by the binomial theorem to give

$$(1+x)^\alpha = x^\alpha \sum_{r=0}^{\infty}\binom{\alpha}{r}\Big(\frac{1}{x}\Big)^r.$$

If we put $x=4$ and $\alpha=\frac{1}{2}$, then we obtain an expression for $\sqrt{5}$, and taking the first 5 terms (i.e. up to the term with $(\frac{1}{x})^4$) we get

$$\sqrt{5} \approx 2\Big(1 + \frac{1}{8} - \frac{1}{128} + \frac{1}{2^{10}} - \frac{5}{2^{15}}\Big) \approx 2.236\,023.$$

This compares with the "correct" value (to 6 decimal places) $2.236\,068$.

5. (a) For $x>0$ we have $g_0(x) = \exp(x) - 1 > 0$. Assume that for $k\ge 0$ we have $g_k(x) > 0$ and consider $g_{k+1}(x) = \exp(x) - \frac{x^{k+1}}{(k+1)!}$. Differentiating gives $g'_{k+1}(x) = g_k(x) > 0$. Hence g_{k+1} is strictly increasing on $[0,\infty)$. But $g_{k+1}(0) = \exp(0) = 1$, so $g_{k+1}(x) > 0$ for $x>0$. It follows, by induction, that if $x>0$ and if n is a non-negative integer then $g_n(x) = \exp(x) - \frac{x^n}{n!}$ is strictly positive.

(b) We have $g_{2n}(u)$ is strictly positive, so $\frac{\exp(u)}{u^n} > \frac{u^n}{(2n)!}$. For $n>0$, $\frac{u^n}{(2n)!}\to\infty$ as $u\to\infty$ and so $\frac{\exp(u)}{u^n}\to\infty$ as $u\to\infty$. Since we may write $\exp(u) = u\cdot\frac{\exp(u)}{u}$, it also follows that $\frac{\exp(u)}{u^n}\to\infty$ as $u\to\infty$ in the case $n=0$.

(c) Write $u = 1/x^2$ so that $u \to \infty$ as $x \to 0$. Then part (b) gives $\frac{\exp(1/x^2)}{x^{-2n}} \to \infty$ as $x \to 0$. Taking the reciprocal gives $\frac{\exp(-1/x^2)}{x^{2n}} \to 0$ as $x \to 0$.

(d) Part (c) gives the result for even values of r. To obtain the result for odd values, write $\frac{\exp(-1/x^2)}{x^{2n-1}} = x \cdot \frac{\exp(-1/x^2)}{x^{2n}}$ $(n \geq 1)$.

(e) First note that for $x \neq 0$, $\exp(-1/x^2)$ is differentiable with derivative $2x^{-3}\exp(-1/x^2)$ (by the composite rule). To determine differentiability at 0, take $h \neq 0$ and consider

$$\frac{f(0+h) - f(0)}{h} = \frac{\exp(-1/h^2)}{h} \to 0 \text{ as } h \to 0,$$

by part (d). So f is differentiable at 0 with derivative $f'(0) = 0$. Then again for $x \neq 0$, using the product rule, we obtain

$$\begin{aligned} f''(x) &= 2x^{-3} \cdot 2x^{-3}\exp(-1/x^2) - 6x^{-4}\exp(-1/x^2) \\ &= \phi_2(x)\exp(-1/x^2), \end{aligned}$$

where $\phi_2(x) = 4x^{-6} - 6x^{-4}$. To determine differentiability at 0, take $h \neq 0$ and consider

$$\frac{f'(0+h) - f'(0)}{h} = \frac{2}{h^3}\exp(-1/h^2) \to 0 \text{ as } h \to 0,$$

again by part (d). So f is twice differentiable at 0 and $f''(0) = 0$. If, for $x \neq 0$, $f^{(k)}(x) = \phi_k(x)\exp(-1/x^2)$ then $f^{(k+1)}(x) = [2x^{-3}\phi_k(x) + \phi_k'(x)]\exp(-1/x^2)] = \phi_{k+1}(x)\exp(-1/x^2)$, where $\phi_{k+1}(x) = 2x^{-3}\phi_k(x) + \phi_k'(x)$. If $\phi_k(x)$ is a finite sum of multiples of negative powers of x, then $\phi_{k+1}(x)$ will be likewise. It follows by induction that for $x \neq 0$, f is n times differentiable on $\mathbb{R}$ with a derivative of the form

$$f^{(n)}(x) = \phi_n(x)\exp(-1/x^2).$$

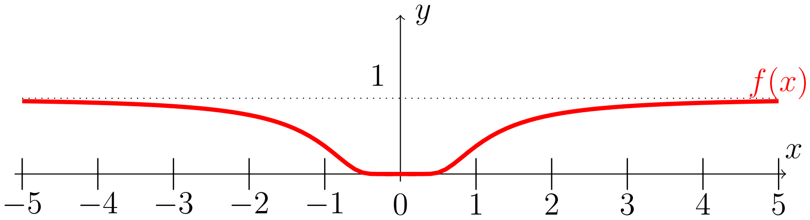
To determine differentiability at 0, take $h \neq 0$, assume that $f^{(k-1)}(0) = 0$, and consider

$$\frac{f^{(k-1)}(0+h) - f^{(k-1)}(0)}{h} = \frac{\phi_{n-1}(x)}{h}\exp(-1/h^2) \to 0 \text{ as } h \to 0,$$

again by part (d). Since $f'(0) = 0$, it follows by induction that $f^{(n)}(0) = 0$ for every positive integer n.

(f) The Maclaurin series of f is $\sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0) = 0$, since $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$. But $f(x)$ is non-zero unless $x = 0$. So (with the exception of this single value) the Maclaurin series of $f(x)$ converges, but not to $f(x)$. [Incidentally, this also shows that the remainder term $R_n(x)$ must equal $f(x)$ for every positive integer n and every $x \in \mathbb{R}$.]

(g)



The graph of $f(x)$ is very "flat" around $x = 0$.

6. The Maclaurin series of $s(x)$ is $\sum_{r=0}^{\infty} \frac{x^r}{r!} s^{(r)}(0)$. Note that

$$s'(x) = c(x),\ s''(x) = c'(x) = -s(x),\ s^{(3)}(x) = -c(x),\ s^{(4)}(x) = s(x),$$

etc. Hence

$$s(0) = 0,\ s'(0)) = 1,\ s''(0) = 0,\ s^{(3)}(0) = -1,\ s^{(4)}(0) = 0,\ \text{ etc.}$$

The Lagrange form of the remainder term is $R_n(x) = \frac{x^n}{n!} s^{(n)}(\xi)$, where ξ lies between 0 and x. But $s^{(n)}$ is one of $c, s, -c, -s$, all of which are differentiable and therefore continuous and bounded on the finite closed interval I with end points 0 and x. If M is a bound for all four then $|s^{(n)}(z)| < M$ for $z \in I$. Consequently

$$|R_n(x)| \le \frac{|x|^n}{n!} M \to 0 \text{ as } n \to \infty.$$

Hence for every $x \in \mathbb{R}$,

$$s(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} s^{(r)}(0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots.$$

This can be expressed as

$$s(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!}.$$

An almost identical argument gives

$$c(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r)!}.$$

Exercises 6.5

1. For $x \neq 0$, put $a_n = x^n/n!$. Then $|a_{n+1}/a_n| = |x|/(n+1) \to 0$ as $n \to \infty$. So, by D'Alembert's ratio test, the series converges for all x. Applying term-by-term differentiation gives

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x).$$

2. As in the previous question, both series converge for all x. Applying term-by-term differentiation gives

$$s'(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = c(x), \quad \text{and}$$
$$c'(x) = \sum_{n=1}^{\infty}(-1)^n \frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} = -s(x).$$

3. For $x \neq 0$, put $a_n = (-1)^{n-1}\frac{x^n}{n}$. Then $|a_{n+1}/a_n| = \frac{n}{n+1}|x| \to |x|$ as $n \to \infty$. So, by D'Alembert's ratio test, the series converges for $|x| < 1$ and diverges for $|x| > 1$, hence the radius of convergence is $R = 1$. Applying term-by-term differentiation for $|x| < 1$ gives

$$l'(x) = \sum_{n=1}^{\infty}(-1)^{n-1} \frac{nx^{n-1}}{n} = \sum_{n=0}^{\infty}(-1)^n x^n.$$

This latter series is the binomial series for $(1+x)^{-1}$, alternatively it can be recognised as a geometric series with common ratio $-x$, which converges for $|x| < 1$ to $\frac{1}{1-(-x)} = \frac{1}{1+x}$.

Exercises 7.1

1. If we go up to the term $\frac{1}{7!}$ we should be OK, because the remaining terms are positive, and

$$\begin{aligned}\frac{1}{8!} + \frac{1}{9!} + \frac{1}{10!} + \frac{1}{11!} + \ldots &< \frac{1}{8!}\Big(1 + \frac{1}{9} + \big(\frac{1}{9}\big)^2 + \big(\frac{1}{9}\big)^3 + \ldots\Big) \\ &= \frac{1}{8!}\frac{1}{1-\frac{1}{9}} = \frac{9}{8(8!)} \\ &< 2.8 \times 10^{-5}.\end{aligned}$$

So the error in omitting terms after $\frac{1}{7!}$ lies between 0 and 2.8×10^{-5}. We have $1+1+\frac{1}{2!}+\ldots+\frac{1}{7!} = 2.718\,254$ to 6 decimal places, and we conclude that $e = 2.7183$ to 4 decimal places.

Exercises 7.2

1. Put $x = \log_a(b)$, so that $b = a^x = \left(b^{\log_b(a)}\right)^x = b^{x\log_b(a)}$. Hence $x\log_b(a) = 1$, so that $x = \frac{1}{\log_b(a)}$, i.e. $\log_a(b) = \frac{1}{\log_b(a)}$.

2. For $|x| < 1$ we have

$$\begin{aligned}\log_e\left(\frac{1+x}{1-x}\right) &= \log_e(1+x) - \log_e(1-x)\\ &= (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots)\\ &\qquad - (-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots)\\ &= 2\Big(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\Big)\\ &= 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}.\end{aligned}$$

For $y > 0$ define $x = (y-1)/(y+1) = 1-2/(y+1)$, so that $|x| < 1$. Also $x(y+1) = y-1$ so that $y(1-x) = 1+x$, which gives $y = (1+x)/(1-x)$. It follows that

$$\log_e(y) = 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\cdot\Big(\frac{y-1}{y+1}\Big)^{2n+1}.$$

With the same notation, if $y = 2$ then $x = \frac{y-1}{y+1} = \frac{1}{3}$, giving

$$\log_e(2) = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)3^{2n+1}}.$$

If we truncate this series at $n = k$ then the remaining terms are all positive and their sum is

$$\begin{aligned}R_k &= 2\sum_{n=k+1}^{\infty}\frac{1}{(2n+1)3^{2n+1}}\\ &< \frac{2}{(2k+3)3^{2k+3}}\Big(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \dots\Big)\\ &= \frac{2}{(2k+3)3^{2k+3}}\cdot\frac{9}{8}\\ &= \frac{1}{4(2k+3)3^{2k+1}}.\end{aligned}$$

Taking $k = 4$ gives $R_k < 1.2 \times 10^{-6} < 10^{-5}$, so we have the value of $\log_e(2)$ correct to within 10^{-5} given by

$$2\Big(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9}\Big) = 0.69315$$

If we used the alternating harmonic series we might need to take terms as far as $\frac{1}{k}$ where k is chosen so that $\frac{1}{k} < 10^{-5}$, i.e. $k = 10^5 + 1$. It is possible that we could do better than this, but that would depend on getting an estimate for the tail of the series, i.e for $\sum_{n=k+1}^{\infty}(-1)^{n-1}\frac{1}{n}$, not a trivial task.

3. If we look at the partial sum of the first $3n$ terms (say, S_{3n}) and bracket each positive term with the succeeding negative term we get

$$\begin{aligned} S_{3n} &= \Big(1 - \frac{1}{2}\Big) - \frac{1}{4} + \Big(\frac{1}{3} - \frac{1}{6}\Big) - \frac{1}{8} + \Big(\frac{1}{5} - \frac{1}{10}\Big) - \frac{1}{12} + \dots \\ &\qquad + \Big(\frac{1}{2n-1} - \frac{1}{4n-2}\Big) - \frac{1}{4n} \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots + \frac{1}{4n-2} - \frac{1}{4n} \\ &= \frac{1}{2}\Big(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{2n-1} - \frac{1}{2n}\Big). \end{aligned}$$

This last expression is a half of a partial sum of the alternating harmonic series, and it follows that $S_{3n} \to \frac{1}{2}\log_e(2)$ as $n \to \infty$. Since the terms of the series tend to zero as $n \to \infty$, it follows that S_{3n+1} and S_{3n+2} also converge to $\frac{1}{2}\log_e(2)$ as $n \to \infty$. Hence the rearranged series converges with sum $\frac{1}{2}\log_e(2)$.

Exercises 7.3

1. Parts (a) and (b) are easily proven by replacing y in the addition formulae of Theorem 7.9 by $-y$. Note that $\cos(-y) = \cos(y)$ and $\sin(-y) = -\sin(y)$.
For part (c) we have

$$\begin{aligned} \tan(x + y) &= \frac{\sin(x + y)}{\cos(x + y)} \\ &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)} \\ &= \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}, \end{aligned}$$

where the last step comes from dividing numerator and denominator by $\cos(x)\cos(y)$.
For part (d) we use part (c), replacing y by $-y$ and noting that $\tan(-y) = -\tan(y)$.

2. From Theorem 7.13 we have $\sin(x + \pi/2) = \cos(x)$. Replace x by $-x$ to get $\sin(\pi/2 - x) = \cos(-x) = \cos(x)$. Similarly $\cos(x+\pi/2) = -\sin(x)$ gives $\cos(\pi/2 - x) = -\sin(-x) = \sin(x)$.

3. We have $a = (a + b)/2 + (a - b)/2$, so

$$\sin(a) = \sin((a + b)/2)\cos((a - b)/2) + \cos((a + b)/2)\sin((a - b)/2).$$

Similarly $b = (a + b)/2 - (a - b)/2$, so

$$\sin(b) = \sin((a + b)/2)\cos((a - b)/2) - \cos((a + b)/2)\sin((a - b)/2).$$

Now subtract these expressions to get

$$\sin(a) - \sin(b) = 2\cos((a + b)/2)\sin((a - b)/2).$$

In the same way we get

$$\cos(a) = \cos((a+b)/2)\cos((a-b)/2) - \sin((a+b)/2)\sin((a-b)/2), \quad \text{and}$$

$$\cos(b) = \cos((a + b)/2)\cos((a - b)/2) + \sin((a + b)/2)\sin((a - b)/2).$$

Subtraction gives

$$\cos(a) - \cos(b) = -2\sin((a + b)/2)\sin((a - b)/2).$$

4. Since $\tan'(x) = \sec^2(x) > 0$ we see that $\tan(x)$ is strictly increasing on $(-\pi/2, \pi/2)$. From Theorem 7.13 we have $\sin(\pi/4) = \cos(\pi/4)$, so $\tan(\pi/4) = 1$. From the same Theorem, $\sin(x + \pi/2) = \cos(x)$ and $\cos(x + \pi/2) = -\sin(x)$, and dividing these gives $\tan(x + \pi/2) = -\cos(x)/\sin(x) = -\cot(x)$. Likewise from $\sin(x + \pi) = -\sin(x)$ and $\cos(x + \pi) = -\cos(x)$ we obtain $\tan(x + \pi) = \tan(x)$.
If $0 < a < \pi$ then $0 < a/2 < \pi/2$, so $\tan(a/2)$ is well-defined and strictly positive (since $\tan(0) = 0$ and $\tan$ is strictly increasing). Hence

$$\tan(a) = \frac{2\tan(a/2)}{1 - \tan^2(a/2)} \neq 0.$$

So $\tan(0 + a) \neq \tan(0)$ and consequently a cannot be a period of $\tan(x)$.

Exercises 7.4

1. We take the same approach as we did for $\arctan(x)$. Using the binomial theorem, for $|x| < 1$ we have

$$\begin{aligned}\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n.\end{aligned}$$

So we put $F(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^{2n+1}}{2n+1}$. The series defining $F(x)$ has radius of convergence $R = 1$ and so we may differentiate term-by-term to obtain $F'(x) = \arcsin'(x)$ for every $x \in (-1, 1)$. Thus both $F(x)$ and $\arcsin(x)$ are primitives for $\arcsin'(x)$ on the interval $(-1, 1)$. They must therefore differ by a constant on this interval. But $F(0) = \arcsin(0) = 0$, so the value of this constant is zero. Hence for $|x| < 1$,

$$\arcsin(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

2. Looking back to the proof that $\tan(\pi/6) = \frac{1}{\sqrt{3}}$, in the penultimate step it was shown that $3\sin^2(\pi/6) = \cos^2(\pi/6)$. But $\cos^2(\pi/6) = 1 - \sin^2(\pi/6)$, so we get $4\sin^2(\pi/6) = 1$. Noting that $\sin(x) > 0$ for $x \in [0, 2]$, we take the positive square root and get $\sin(\pi/6) = \frac{1}{2}$.

3. From the previous questions

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \frac{(\frac{1}{2})^{2n+1}}{2n+1}.$$

Hence

$$\pi = 3 \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(-1)^n}{(2n+1) \cdot 2^{2n}}.$$

But

$$\begin{aligned}\binom{-\frac{1}{2}}{n} &= \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2}) \ldots (-\frac{2n-1}{2})}{n!} \\ &= (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{n! 2^n} \\ &= (-1)^n \frac{(2n)!}{n! 2^n (2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n)} \\ &= (-1)^n \frac{(2n)!}{(n!)^2 2^{2n}}.\end{aligned}$$

It follows that

$$\pi = 3\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2(2n+1)2^{4n}}.$$

The first six terms of this series sum to 3.141577 (to 6 decimal places). To estimate the error, note first that all the terms are positive, so this will be an underestimate. We have

$$\frac{(2n!)}{(n!)^2 2^{2n}} = \frac{(2n!)}{(2^n n!)\cdot(2^n n!)} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot \ldots \cdot (2n-1)\cdot(2n)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \ldots\cdot (2n)\cdot(2n)} < 1$$

If we denote the $(n+1)^{\text{th}}$ term by a_n, then $0 < a_n < \frac{1}{(2n+1)\cdot 2^{2n}}$. Hence the error in the estimate is less than

$$\begin{aligned}
3\sum_{n=6}^{\infty}\frac{1}{(2n+1)\cdot 2^{2n}} &= 3\Big[\frac{1}{13\cdot 2^{12}} + \frac{1}{15\cdot 2^{14}} + \frac{1}{17\cdot 2^{16}}\ldots\Big] \\
&< \frac{3}{13\cdot 2^{12}}\Big[1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\Big] \\
&= \frac{3}{13\cdot 2^{12}}\Big[\frac{1}{1-\frac{1}{4}}\Big] \\
&= \frac{4}{13\cdot 2^{12}} < 0.000076.
\end{aligned}$$

4. We have

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)} = \frac{2/5}{1-(1/25)} = 5/12.$$

Then

$$\tan(4\theta) = \frac{2\tan(2\theta)}{1-\tan^2(2\theta)} = \frac{10/12}{1-(25/144)} = 120/119.$$

Hence

$$\tan(4\theta-\phi) = \frac{\tan(4\theta)-\tan(\phi)}{1+\tan(4\theta)\tan(\phi)} = \frac{120/119 - 1/239}{1+(120)/(119\times 239)} = 1.$$

From this it is a short step to argue that $4\theta - \phi = \arctan(1) = \pi/4$. The only slight difficulty is that we must check that $4\theta-\phi$ lies in the interval $(-\pi/2, \pi/2)$. A crude argument will do this job. Note firstly that $\tan(\theta) = \frac{1}{5} < \frac{1}{\sqrt{3}}$ and so $0 < \theta < \pi/6$, hence $0 < 2\theta < \pi/3$. But it is also true that $\tan(2\theta) = \frac{5}{12} < \frac{1}{\sqrt{3}}$, so by the same argument again $0 < 2\theta < \pi/6$, which gives $0 < 4\theta < \pi/3$. Finally ϕ also lies between 0 and $\pi/6$, so $4\theta-\phi$ lies between $-\pi/6$ and $\pi/3$.
It follows that $\pi/4 = 4\theta - \phi = 4\arctan(1/5) - \arctan(1/239)$.

5. We have

$$\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right).$$

But

$$\arctan\left(\frac{1}{5}\right) = \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \cdot 5^{2n+1}}$$

A bit of experimenting suggests taking the partial sum up to and including $1/(17 \cdot 5^{17})$ should be enough to get accuracy within 10^{-10}. For the other term

$$\arctan\left(\frac{1}{239}\right) = \frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \cdot 239^{2n+1}}$$

Here, taking terms up to up to and including $1/(5 \cdot 239^5)$ looks sufficient. We obtain the approximate value (to 12 decimal places)

$$16 \sum_{n=0}^{8} \frac{(-1)^n}{(2n+1) \cdot 5^{2n+1}} - 4 \sum_{n=0}^{2} \frac{(-1)^n}{(2n+1) \cdot 239^{2n+1}} = 3.141\,592\,653\,590.$$

To check this is correct within 10^{-10} you need to estimate the error term. The terms of both series alternate in sign and their absolute values are strictly decreasing. So the error cannot exceed the weighted sum of the next two terms, i.e $16\Big(1/(19 \cdot 5^{19})\Big) + 4\Big(1/(7 \cdot 239^7)\Big) < 5 \times 10^{-14}$.

Exercises 7.5

1.

a)

$$\begin{aligned} \cosh^2(x) - \sinh^2(x) &= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} \\ &= \frac{4}{4} = 1 \end{aligned}$$

b)

$$\begin{aligned}
&\sinh(a)\cosh(b)+\cosh(a)\sinh(b)\\
&=\frac{e^a-e^{-a}}{2}\cdot\frac{e^b+e^{-b}}{2}+\frac{e^a+e^{-a}}{2}\cdot\frac{e^b-e^{-b}}{2}\\
&=\frac{e^{a+b}+e^{a-b}-e^{b-a}-e^{-(a+b)}+e^{a+b}-e^{a-b}+e^{b-a}-e^{-(a+b)}}{4}\\
&=\frac{2e^{a+b}-2e^{-(a+b)}}{4}=\frac{e^{a+b}-e^{-(a+b)}}{2}=\sinh(a+b).
\end{aligned}$$

c)

$$\begin{aligned}
&\cosh(a)\cosh(b)+\sinh(a)\sinh(b)\\
&=\frac{e^a+e^{-a}}{2}\cdot\frac{e^b+e^{-b}}{2}+\frac{e^a-e^{-a}}{2}\cdot\frac{e^b-e^{-b}}{2}\\
&=\frac{e^{a+b}+e^{a-b}+e^{b-a}+e^{-(a+b)}+e^{a+b}-e^{a-b}-e^{b-a}+e^{-(a+b)}}{4}\\
&=\frac{2e^{a+b}+2e^{-(a+b)}}{4}=\frac{e^{a+b}+e^{-(a+b)}}{2}=\cosh(a+b).
\end{aligned}$$

2. If $x>1$ then $-1>-2x+1$, so $x^2-1>x^2-2x+1=(x-1)^2$. Taking positive square roots gives $\sqrt{x^2-1}>x-1$. Hence $x-\sqrt{x^2-1}<1$. An alternative approach is to observe that for $x>1,\ x+\sqrt{x^2-1}\geq x>1$, while $(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})=1$, and so $x-\sqrt{x^2-1}<1$.[This result was used to justify rejecting $\exp(y)=x-\sqrt{x^2-1}$ in the derivation of the logarithmic formula for $\operatorname{argcosh}(x)$.]

3. (a) For $\operatorname{argsinh}(x)$ we have $\operatorname{argsinh}'(x)=\dfrac{1}{\sqrt{1+x^2}}$. So $\operatorname{argsinh}(x)$ is a primitive for $\dfrac{1}{\sqrt{1+x^2}}$. The binomial expansion gives

$$\frac{1}{\sqrt{1+x^2}}=(1+x^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}x^{2n}\ \text{ for }\ |x|<1.$$

By Chapter 6, Corollary 6.14.2, the series

$$f(x)=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{x^{2n+1}}{2n+1}$$

also converges for $|x| < 1$. This series is differentiable with derivative $\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n} = \dfrac{1}{\sqrt{1+x^2}}$. Hence $f(x)$ is also a primitive for $\dfrac{1}{\sqrt{1+x^2}}$ on the interval $(-1, 1)$. It follows from Chapter 6, Corollary 6.8.2 that $\operatorname{argsinh}(x) - f(x)$ takes a constant value on the interval $(-1, 1)$. However, $\operatorname{argsinh}(0) = f(0) = 0$. so $\operatorname{argsinh}(x) = f(x)$ for $|x| < 1$. In other words

$$\operatorname{argsinh}(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1} \quad \forall x \in (-1, 1).$$

(b) For $\operatorname{argtanh}(x)$ we can use the same approach as for $\operatorname{argsinh}(x)$. However, it is probably easier to use the logarithmic expression for $\operatorname{argtanh}(x)$ as follows.

$$\begin{aligned}
\operatorname{argtanh}(x) &= \frac{1}{2} \log_e \left(\frac{1+x}{1-x} \right) \\
&= \frac{1}{2} \Big[\log_e(1+x) - \log_e(1-x) \Big] \\
&= \frac{1}{2} \Big[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots - \Big(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \Big) \Big] \\
&= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.
\end{aligned}$$

As regards getting a similar series for $\operatorname{argcosh}(x)$, this is clearly impossible because the function is only defined for $x > 1$ and so we cannot get a power series in x valid in any open interval containing 0.
However, the situation is not entirely hopeless. We have

$$\operatorname{argcosh}'(x) = \frac{1}{\sqrt{x^2-1}} = \frac{1}{x} \cdot \frac{1}{\sqrt{1 - (\frac{1}{x})^2}} = y \frac{1}{\sqrt{1-y^2}} = y(1-y^2)^{-\frac{1}{2}},$$

where $y = \frac{1}{x}$. The binomial theorem can be used to expand this latter expression as a power series in y. Without giving all the details this eventually gives a series for $\operatorname{argcosh}(x)$ for $x > 1$ that involves ascending powers of $\frac{1}{x}$, a $\log_e(x)$ term, and a non-zero constant (not nice).

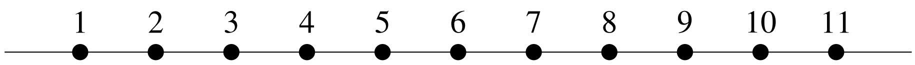
Appendix A

Construction of the Real Numbers

The construction is in stages, starting from the set of Natural Numbers $\mathbb{N}$. We give an outline of the process. There isn't room here to give all the proofs. A few sample proofs are given to indicate what is involved.

A.1 The Natural Numbers, $\mathbb{N}$

The Natural Numbers can be envisaged marked out along a straight line:



From the picture, you can see that each Natural Number n has an immediate successor that can be denoted as $S(n)$. For example, the successor of 3 is 4, i.e. $S(3) = 4$. And every Natural Number except 1 is the successor of some other Natural Number. We can *define* 2 as the successor of 1: $2 = S(1)$, and likewise define 3 as the successor of 2: $3 = S(2)$, and this could be written as $3 = S(S(1))$, read as "the successor of the successor of 1". We can go on like this defining 4, 5, 6, etc. We might describe the whole collection $2, 3, 4, \ldots$ as the repeated successors of 1, reserving the term "the successor" to mean the *immediate* successor, namely the number 2.

Formally, all the properties of the Natural Numbers can be deduced using the idea of successors. All that is required to do this is the following five properties, collectively known as the Peano Postulates.

1. 1 is a Natural Number.

2. Each Natural Number n has a successor $S(n)$ which is also a Natural Number.

3. 1 is not the successor of any Natural Number.

4. If m and n are Natural Numbers and $S(m) = S(n)$, then $m = n$.

5. If we have a collection C of Natural Numbers that contains the Natural Number 1 and also contains the successor of every Natural Number in the collection, then the collection C contains all the Natural Numbers.

The construction of $\mathbb{R}$ assumes the existence of a set of Natural Numbers $\mathbb{N}$ that satisfies the Peano Postulates.

The Peano Postulates may look a bit abstract, but they closely follow the earliest known form of recorded counting - the use of tally marks. So in terms of tally marks these properties might be expressed roughly as follows.

1. Here is the first tally mark representing "one": /

2. If //.../ is a tally mark representing a number, then we get the next number (the successor) by adding another tally mark: //...//

3. There is no tally mark for nothing.

4. If two people each have the same number of tally marks and they each rub out one tally mark, then they will still each have the same number of tally marks.

5. The only numbers we recognize are those given by the tally mark for "'one" and the tally marks for its repeated successors.

The last Postulate deserves some extra explanation. If it wasn't there we might have additional objects in our collection, such as an additional "starting point", call it $1'$. So the last property ensures that the Natural Numbers comprise just the number 1 along with its repeated successors, $S(1),\ S(S(1)),\ S(S(S(1))),\ \ldots$. This last postulate also provides the basis for the method of induction. We will have some more to say about induction after we have shown how to identify the Natural Numbers within the axiomatic description of the Real Numbers in Appendix B.

Addition of Natural Numbers is defined using the notion of successors, in effect by repeatedly adding 1. If m and n are any Natural Numbers, we can define $m + 1 = S(m)$ and $m + S(n) = S(m + n)$. For example, once you know the value of $5 + 3$ (which we call 8), then the value of $5 + S(3)$ (that is to say the value of $5 + 4$) must be $S(5 + 3)$, which is the successor of 8, $S(8)$, known more commonly by us as 9.

A consequence of this definition is that addition has all the properties you would expect. We won't prove all the properties but, by way of example, we will prove the commutative law of addition.

Theorem A.1. $\forall m, n \in \mathbb{N}, m + n = n + m$.

Proof. First, we prove that $\forall m \in \mathbb{N}, m + 1 = 1 + m$. We do this by induction. It is obviously true for $m = 1$ since it reduces to $1 + 1 = 1 + 1$. So suppose that for some $k \in \mathbb{N}, k + 1 = 1 + k$. Then $(k + 1) + 1 = S(k + 1) = S(1 + k) = 1 + S(k) = 1 + (k + 1)$. So, by induction, $m + 1 = 1 + m$ for every $m \in \mathbb{N}$.

Second, with $m \in \mathbb{N}$ fixed, we prove that $m + S(n) = S(m) + n$ for every $n \in \mathbb{N}$. Again we use induction. For $n = 1$,

$$m + S(n) = m + S(1) = S(m + 1) = S(S(m)) = S(m) + 1 = S(m) + n.$$

Next suppose that for some $k \in \mathbb{N}, m + S(k) = S(m) + k$. Then

$$\begin{aligned} m + S(k + 1) &= m + S(S(k)) = S(m + S(k)) = S(S(m) + k) = S(m) + S(k) \\ &= S(m) + (k + 1). \end{aligned}$$

By induction it follows that $\forall m, n \in \mathbb{N}, m + S(n) = S(m) + n$.

Using the previous two parts of the proof, we can prove that $\forall m, n \in \mathbb{N}, m + n = n + m$. Again we use induction. The result is certainly true for $n = 1$, by the first part. Now suppose that for some $k \in \mathbb{N}, m + k = k + m$. Then

$$\begin{aligned} m + (k + 1) &= m + S(k) = S(m + k) = S(k + m) = k + S(m) = S(k) + m \\ &= (k + 1) + m. \end{aligned}$$

Consequently $m + n = n + m$ for every pair $m, n \in \mathbb{N}$. $\square$

A further consequence of the definition of addition is the associative law: $(m + n) + p = m + (n + p)$, which can be proved in a similar way to the commutative law.

Multiplication of Natural Numbers can be defined as repeated addition. For example, 3×4 means " 3 lots of 4", in symbols $4+4+4 = 12$. Formally, $1 \times n = n$, and $S(m) \times n = (m \times n) + n$. It turns out that multiplication is commutative: $m \times n = n \times m$. And the associative law also holds: $(m \times n) \times p = m \times (n \times p)$. A useful fact connecting $\times$ and $+$ is the distributive law: $m \times (n+p) = m \times n + m \times p$.

The idea of successors also allows us to define what we mean by saying that one number is less than (i.e. lies to the left of) another. So we write $m < n$ (and $n > m$) if and only if there exists $p \in \mathbb{N}$ such that $m + p = n$. This definition enables us to see that the Natural Numbers come in a definite order: $1 < 2 < 3 < \ldots$.

In summary, *assuming the Peano Postulates*, we have a set of Natural Numbers $\mathbb{N}$ such that if m, n, p are any members of $\mathbb{N}$ then

1. $m+n$ and $m \times n$ both lie in $\mathbb{N}$ ($\mathbb{N}$ is *closed* under addition and multiplication),
2. $m+n = n+m$ and $m \times n = n \times m$ (*commutative* laws),
3. $(m+n)+p = m+(n+p)$ and $(m \times n) \times p = m \times (n \times p)$ (*associative* laws),
4. $m \times (n+p) = m \times n + m \times p$ (multiplication is *distributive* over addition),
5. $1 \times m = m$ (the number 1 is a *multiplicative identity*),
6. if $m \neq n$ then either $m > n$ or $n > m$ (we say that $\mathbb{N}$ is *ordered* by $>$).

Before moving on to look at Integers and Rational Numbers we remark that it is possible to define limited forms of subtraction $(-)$ and division $(\div)$ in $\mathbb{N}$. The equation $3 + x = 5$ has solution $x = 2$ because $5 - 3 = 2$. In general we can define $m - n$ for Natural Numbers m and n to be the Natural Number x for which $n + x = m$ **when such a Natural Number x exists**. When there is no Natural Number x for which $n + x = m$ we leave the subtraction of n from m undefined until we have defined the Integers. Similarly, the equation $3 \times x = 6$ has solution $x = 2$ because $6 \div 3 = 2$. In general we can define $m \div n$ for Natural Numbers m and n to be the Natural Number x for which $n \times x = m$ **when such a Natural Number x exists**. When there is no Natural Number x for which $n \times x = m$ we leave division of m by n undefined until we have defined the Rational Numbers.

A.2 The Integers, $\mathbb{Z}$

The set of Integers $\mathbb{Z}$ is constructed from the set of Natural Numbers $\mathbb{N}$ by identifying an Integer such as -2 with all pairs of Natural Numbers of the form $(a, a+2)$ [**think of this as a minus** $(a+2)$]. Because I want you to think of the ordered pair $(a, a+2)$ as a minus $(a+2)$, I will write this pair as $(a \ominus (a+2))$, so the symbol $\ominus$ just replaces the comma. (I won't use the minus sign itself because it carries too much baggage.) For a while I will use red ink to distinguish the new Integers from the old Natural Numbers. If you want a word for the invented symbol $\ominus$ you could call it "oh-minus".

For example, we can represent -2 as $(1 \ominus 3)$ or $(2 \ominus 4)$ or $(3 \ominus 5)$ or even (just to be obscure) as $(165 \ominus 167)$. Similarly 0 is identified with all pairs $(a \ominus a)$, the simplest of which is probably $(1 \ominus 1)$, but we could equally well represent it (for example) as $(5 \ominus 5)$. Rather awkwardly we then have to identify (e.g.) $2 \in \mathbb{Z}$ with all pairs $((a+2) \ominus a)$ (where $a \in \mathbb{N}$). Thus the Integer 2 is represented by a pair of Natural Numbers, and so it is something ever so slightly different from the Natural Number 2.

This way of presenting Integers may well be unfamiliar to you but it is similar to what we do with fractions where (e.g.) a half can be written as $\frac{1}{2}$ or $\frac{2}{4}$ or $\frac{3}{6}$ or, more generally as $\frac{a}{2a}(a \in \mathbb{N})$. Here too there is an awkwardness about identifying $2 \in \mathbb{N}$ with all "pairs" $\frac{2a}{a}$, where $a \in \mathbb{N}$, and the fraction $\frac{2}{1}$ is something ever so slightly different from the Natural Number 2. We'll look at fractions in the next section.

In general, the pair of Natural Numbers $(m \ominus n)$ represents an Integer. If $m > n$, we write it as $m - n$, if $n > m$ we write it as $-(n - m)$, and if $m = n$ we write it as 0. For example, $(7 \ominus 4) = 3$, $(4 \ominus 7) = -3$ and $(6 \ominus 6) = 0$. Note that the subtraction involved in $(4 \ominus 7)$ is actually the subtraction $7 - 4$ of the Natural Number 4 from the Natural Number 7, and we then put a minus sign in front of the resulting Natural Number 3 to give the Integer -3.

An important question is how to identify if two apparently different looking Integers are in fact equal. Is the Integer $(36 \ominus 921)$ the same as $(245 \ominus 1130)$? There is an easy solution once we realize that we are really asking if $36 - 921 = 245 - 1130$, and we can answer this question without using subtraction or minus signs by adding $921 + 1130$ to both sides to give the equivalent question: does $36 + 1130$ equal $245 + 921$? The general question whether $(m \ominus n)$ represents the same Integer as $(p \ominus q)$ really asks if $m - n = p - q$, or equivalently (by adding $n + q$ to both sides) if $m + q = p + n$. Consequently we *define* equality between these ordered pairs of Natural Numbers by saying

$$(m \ominus n) = (p \ominus q) \text{ if and only if } m + q = p + n.$$

The important thing to note here is that the right-hand side only contains Natural Numbers ($m + q$ and $p + n$), so this definition of equality between Integers only involves Natural Numbers. Thus equality of Integers is defined in terms of something we already understand: equality of Natural Numbers. So for our example, we compare $36+1130$ with $245+921$, and since both come to 1166, they are equal and consequently both $(36 \ominus 921)$ and $(245 \ominus 1130)$ represent the same Integer.

In general if n is a Natural Number, the pair $(1 \ominus (n + 1))$ represents the negative Integer $-n$, and the pair $((n + 1) \ominus 1)$ represents the positive Integer n. The Integer 0 has the simplest form $(1 \ominus 1)$.

Once we've got hold of $\mathbb{Z}$ we can define the operations $+$ and $\times$. Addition is very easy:

$$(m \ominus n) + (p \ominus q) = ((m + p) \ominus (n + q)).$$

For example, $(-3) + 2 = (1 \ominus 4) + (3 \ominus 1) = (4 \ominus 5) = -1$. You might like to think what happens if you add $(m \ominus n)$ and $(n \ominus m)$. It is also fairly obvious from this definition of $+$ that addition of Integers is commutative and associative because addition of Natural Numbers has these properties.

Multiplication is just a shade more complicated and it helps to remember that we think of $(m \ominus n)$ as $m - n$, so we would expect $(m \ominus n) \times (p \ominus q)$ to come out as a representative of $(m - n) \times (p - q) = mp - mq - np + nq$. So our definition is

$$(m \ominus n) \times (p \ominus q) = ((mp + nq) \ominus (mq + np)).$$

As an example, let's try $(-2) \times 3$. First write -2 as the pair $(1 \ominus 3)$ and 3 as the pair $(4 \ominus 1)$. Then we get

$$(-2) \times 3 = (1 \ominus 3) \times (4 \ominus 1) = ((4 + 3) \ominus (1 + 12)) = (7 \ominus 13) = -6.$$

So we get $(-2) \times 3 = -6$. Now you try $(-2) \times (-3)$. You should get 6.

Again commutativity and associativity of multiplication in $\mathbb{Z}$ follow immediately from the corresponding properties of $\mathbb{N}$. Likewise, multiplication is distributive over addition and the Integer 1 is a multiplicative identity.

We should check that these definitions of $+$ and $\times$ do not depend on the particular representations of the Integers involved, i.e. if we have $(m \ominus n) = (r \ominus s)$ and $(p \ominus q) = (t \ominus u)$ then we find that $(m \ominus n) + (p \ominus q) = (r \ominus s) + (t \ominus u)$ and $(m \ominus n) \times (p \ominus q) = (r \ominus s) \times (t \ominus u)$. I will check the first of these (the addition) and I invite you to check the second (the multiplication).

Let us give names X and Y to the two Integers produced by the addition, i.e. $X = (m \ominus n) + (p \ominus q)$ and $Y = (r \ominus s) + (t \ominus u)$. According to the definition of addition, $X = ((m + p) \ominus (n + q))$ and $Y = ((r + t) \ominus (s + u))$. So according to the definition of equality of Integers, $X = Y$ if and only if

$$(m + p) + (s + u) = (r + t) + (n + q),$$

i.e (by rearranging slightly) if and only if

$$(m + s) + (p + u) = (r + n) + (t + q).$$

But $(m \ominus n) = (r \ominus s)$ gives $m + s = r + n$, and $(p \ominus q) = (t \ominus u)$ gives $p + u = t + q$. Consequently $(m + s) + (p + u)$ does equal $(r + n) + (t + q)$, and so $X = Y$.

The Integers have some interesting additional properties compared to the Natural Numbers (and now I will drop the red ink).

(a) For each $a \in \mathbb{Z}$, $0 + a = a$. We say that the number 0 is an *additive identity*.

(b) For each $a \in \mathbb{Z}$ there is a number $(-a) \in \mathbb{Z}$ such that $a + (-a) = 0$. The number $(-a)$ is called the *additive inverse* or *negative* of a.
If a is represented by the pair of Natural Numbers $(m \ominus n)$, then $(-a)$ is given by $(n \ominus m)$ because $(m \ominus n) + (n \ominus m) = ((m + n) \ominus (n + m)) = 0.$

Subtraction can now be defined by putting

$$a - b = a + (-b) \text{ for } a, b \in \mathbb{Z}.$$

As an example, consider the equation $5 + x = 3$. If we subtract 5 from both sides then on the left-hand side we get $5 + x - 5 = 5 + (-5) + x = 0 + x = x$ by properties (a) and (b) above. On the right-hand side we get $3 - 5 = 3 + (-5) = (4 \ominus 1) + (1 \ominus 6) = (5 \ominus 7) = -2$. So we conclude that $x = -2$.

As with the Natural Numbers, order among the Integers can be pictured using a number line. We can define order more precisely by looking at what is meant by saying that $(m \ominus n)$ is greater than zero (positive) or less than zero (negative). As already mentioned, an Integer $(m \ominus n)$ is called positive if the Natural Numbers m and n have $m > n$ and it is called negative if $m < n$. So if x and y are Integers, we will say that x is less than y ($x < y$) if and only if $x - y$ is negative. If x is represented by $(m \ominus n)$ and y is represented by $(p \ominus q)$ then $x - y = x + (-y)$ is represented by $(m \ominus n) + (q \ominus p) = ((m + q) \ominus (p + n))$, and this is negative if and only if $m + q < p + n$. So

$$(m \ominus n) < (p \ominus q) \text{ if and only if } m + q < p + n.$$

The important thing to note here is that the right-hand side only contains Natural Numbers ($m + q$ and $p + n$), so our definition of order between Integers only involves Natural Numbers.

There are three alternatives when comparing the Integers x and y: (i) $x > y$, (ii) $x = y$, (iii) $y > x$. Of course we may write $x > y$ as $y < x$, etc.

Here is a summary of the properties we have discussed. If a, b, c are any members of $\mathbb{Z}$ then

1. $a + b$ and ab both lie in $\mathbb{Z}$ ($\mathbb{Z}$ is closed under addition and multiplication),
2. $a + b = b + a$ and $ab = ba$ (the commutative laws),
3. $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ (the associative laws),
4. $a(b + c) = ab + ac$ (multiplication is distributive over addition),
5. $0 + a = a$ (0 is an additive identity),
6. $1a = a$ (1 is a multiplicative identity),
7. for each $a \in \mathbb{Z}$ there is an additive inverse $(-a) \in \mathbb{Z}$ such that $a + (-a) = 0$,

8. for each $a \in \mathbb{Z}$ precisely one of the following three alternatives holds
   i) $a > 0$, ii) $a = 0$, iii) $0 > a$,
   and in case i) we say a is positive, in case iii) we say a is negative.

9. If $a, b \in \mathbb{Z}$ and $a > 0, b > 0$ (i.e. a and b are positive), then $a + b > 0$ and $ab > 0$.

We can then define $a > b$ to mean $a - b > 0$, define $b < a$ to mean $a > b$, define $a \geq b$ to mean that either $a > b$ or $a = b$, and define $b \leq a$ to mean $a \geq b$.

A.3 The Rational Numbers, $\mathbb{Q}$

You are probably familiar with fractions such as $\frac{1}{2}, \frac{-1}{3}, \frac{2}{3}, \frac{3}{-4}$, etc. It is clear just by looking at these fractions that each is formed from two Integers, one on the top of the fraction (the numerator) and one on the bottom (the denominator). A fraction is occasionally called a *quotient*, so we might call $\frac{3}{4}$ the quotient of 3 and 4. This explains the use of the symbol $\mathbb{Q}$ to denote the set of fractions. The only denominator that is prohibited is the integer zero.

Formally, each Rational Number is **constructed from** (you might say represented by) an *ordered pair* of Integers (m, n), where $n \neq 0$, that we choose to write as $\frac{m}{n}$. In other words we just replace the comma in (m, n) by the fraction bar - and drop the brackets ( ). Each fraction has many alternative representations, for example, $\frac{1}{2} = \frac{2}{4}$. In general, two Rational Numbers $\frac{m}{n}$ and $\frac{p}{q}$ are said to be equal if and only if $m \times q = p \times n$. The important thing to note is that this definition of equality between fractions only involves Integers. Thus equality of fractions is defined in terms of something we already understand: equality of Integers.

Addition of Rational Numbers $f = \frac{m}{n}$ and $g = \frac{r}{s}$ is defined as follows:

$$f + g = \frac{m}{n} + \frac{r}{s} = \frac{m \times s}{n \times s} + \frac{r \times n}{s \times n} = \frac{m \times s + r \times n}{n \times s}.$$

It is necessary to check that we get the same answer for $f + g$ irrespective of the representation of f and g, so if we have $f = \frac{m}{n} = \frac{p}{q}$ and $g = \frac{r}{s} = \frac{t}{u}$, we get the same answer whether we use $f = \frac{m}{n}$ or $f = \frac{p}{q}$, and whether we use $g = \frac{r}{s}$ or $g = \frac{t}{u}$. This is indeed the case, so this is a *good* definition!

Multiplication of $f = \frac{m}{n}$ and $g = \frac{r}{s}$ is defined as follows:

$$f \times g = \frac{m \times r}{n \times s}.$$

Again this turns out to be a good definition, meaning that it does not depend on how the fractions f and g are represented.

Each Integer m can be identified with the Rational Number $\frac{m}{1}$ so we may write $m = \frac{m}{1}$. In particular, $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$.

The properties already mentioned for the Integers $\mathbb{Z}$ are inherited by the Rational Numbers $\mathbb{Q}$. We can add and multiply Rational Numbers and the result is always a Rational Number. The commutative and associative laws for addition and multiplication hold, and multiplication is distributive over addition. There is an additive identity (0) and a multiplicative identity (1), and every Rational Number f has an additive inverse $-f$.

But now there is a further interesting property. For each Rational Number f (except for $f = 0$) there is another Rational Number f^{-1}, the *multiplicative inverse* or *reciprocal* of f with the property that $f \times f^{-1} = 1$. If $f = \frac{m}{n}$ then $f^{-1} = \frac{n}{m}$ because $\frac{m}{n} \times \frac{n}{m} = \frac{m \times n}{n \times m} = \frac{1}{1} = 1$. For example, the multiplicative inverse (the reciprocal) of $\frac{5}{7}$ is $\frac{7}{5}$.

By using multiplicative inverses (reciprocals) we can *define* division as follows. If f and g are Rational Numbers, then $f \div g$ is defined as $f \times g^{-1}$, and we can write this in alternative forms as $f \times \frac{1}{g}$ or more briefly as f/g. For example, if we want $6 \div 3$ we take $f = 6 = \frac{6}{1}$ and $g = 3 = \frac{3}{1}$, so that $g^{-1} = \frac{1}{3}$. Then $6 \div 3 = \frac{6}{1} \times \frac{1}{3} = \frac{6 \times 1}{3 \times 1} = \frac{6}{3} = \frac{2}{1} = 2$. I'm not suggesting you should actually do divisions like this in practice, but just illustrating how division is really just multiplication in disguise.

We can extend the picture of a number line mentioned in the case of $\mathbb{Z}$ to represent Rational Numbers $\mathbb{Q}$. As examples, $\frac{3}{2}$ lies halfway between 0 and 3 while $\frac{2}{3}$ lies one third of the way between 0 and 2. This picture enables us to put Rational Numbers into *order*. We write $a > b$ if a is to the right of b and we write $a < b$ if a is to the left of b. The only other alternative is that $a = b$. The formal definition of this ordering is that for Rational Numbers $\frac{m}{n}$ and $\frac{p}{q}$ with n and q both positive, we say $\frac{m}{n} < \frac{p}{q}$ if and only if $m \times q < p \times n$.

Of course Rational Numbers *never* have 0 as a denominator. This is not because mathematicians have an unreasonable hatred of dividing by zero (our hatred of this is perfectly reasonable) and are spitefully stopping people from doing this, but because it just doesn't make sense. You can cut an object into, say, 10 parts each of which is a tenth of the whole, or 3 parts each of which is a third of the whole, or even (stretching English usage a bit) into 1 part comprising the whole of the whole, but you simply can't cut an object into 0 parts.

Here is a list summarising the properties of $\mathbb{Q}$. If f, g, h are any members of $\mathbb{Q}$ then

1. $f + g$ and fg both lie in $\mathbb{Q}$ ($\mathbb{Q}$ is closed under addition and multiplication),

2. $f+g=g+f$ and $fg=gf$ (the commutative laws),

3. $(f+g)+h=f+(g+h)$ and $(fg)h=f(gh)$ (the associative laws),

4. $f(g+h)=fg+fh$ (multiplication is distributive over addition),

5. $0+f=f$ (0 is an additive identity),

6. $1f=f$ (1 is a multiplicative identity),

7. for each $f \in \mathbb{Q}$ there is an additive inverse $(-f) \in \mathbb{Q}$ such that $f+(-f)=0$,

8. for each $f \in \mathbb{Q}$, except for $f=0$, there is a multiplicative inverse (reciprocal) $f^{-1} \in \mathbb{Q}$ such that $f(f^{-1})=1$.

9. for each $f \in \mathbb{Q}$ precisely one of the following three alternatives holds
   i) $f>0$, ii) $f=0$, iii) $0>f$,
   and in case i) we say f is positive, in case iii) we say f is negative.

10. If $f,g \in \mathbb{Q}$ and $f>0, g>0$ (i.e. f and g are positive), then $f+g>0$ and $fg>0$.

We can then define $f>g$ to mean $f-g>0$, define $g<f$ to mean $f>g$, define $f \geq g$ to mean that either $f>g$ or $f=g$, and define $g \leq f$ to mean $f \geq g$.

A.4 The Real Numbers, $\mathbb{R}$

We have seen how to construct $\mathbb{Z}$ from $\mathbb{N}$, and how to construct $\mathbb{Q}$ from $\mathbb{Z}$. In each case the extended system was formed from its precursor as a collection of ordered pairs with a rule for determining when two of the pairs represent the same number (as in $\frac{1}{2}=\frac{2}{4}$). There is such a method for constructing $\mathbb{R}$, but rather than pairs of Rational Numbers, it uses pairs of *sets* of Rational Numbers. This method is due to Richard Dedekind c.1858, and the pairs of sets are called *Dedekind cuts*. Using pairs of sets looks like a big complication, but it does have the merit that every Real Number has one and only one representation, unlike in $\mathbb{Q}$ where each number has infinitely many representations such as $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\ldots$.

The idea is to represent each $x \in \mathbb{R}$ by the two "sections" of $\mathbb{Q}$ which it produces. For example, $\sqrt{2}$ produces the collection of all Rationals below $\sqrt{2}$ (L, say) and the collection of all Rationals above $\sqrt{2}$ (R, say). In the ordered pair (L,R), L is called the left section and R is called the right section. The two sections L and R determine the position of $\sqrt{2}$ on the number line, so knowing L and R is as good as knowing the value of $\sqrt{2}$. So we *define* $\sqrt{2}=(L,R)$. There is no circularity in this definition because we can test elements of $\mathbb{Q}$ to see if they lie in L or R simply by squaring them. So if $x \in \mathbb{Q}$ is negative (or zero), we place it in L. If $x \in \mathbb{Q}$ is positive and $x^2 < 2$, we also place it in L, but if $x \in \mathbb{Q}$ is positive and $x^2 > 2$, we place it in R.

In general we can envisage the Rational Numbers $\mathbb{Q}$ distributed along the number line. We "cut" the number line by forming two sets L and R *of Rational Numbers* such that

1. every Rational Number lies in either L or R,
2. every Rational Number in L is less than every Rational Number in R,
3. neither L nor R is empty.
4. there is no least element in R

The pair (L, R) may then be used to define a Real Number x corresponding to the point where the cut is made. Figure A.1 illustrates the line of Rational Numbers partitioned into two sections defining a Real Number x. The left section is denoted by L and shown in green, the right section is denoted by R and shown in red.

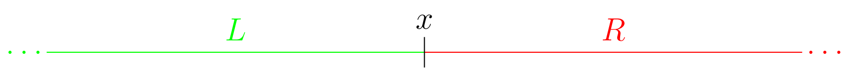


Figure A.1: Dedekind cut at $x = (L, R)$.

Requirement 4 above deserves some explanation. If we decide to cut the Rationals at $\frac{1}{2}$, then L contains all the Rationals less than $\frac{1}{2}$ and R contains all the Rationals greater than $\frac{1}{2}$, but where to put $\frac{1}{2}$ itself? Requirement 4 tells us not to put it into R because it would then be the least element in R, so we must put it into L. This leaves R containing every Rational greater than $\frac{1}{2}$, and L containing all the remaining Rationals, namely those less than $\frac{1}{2}$, together with $\frac{1}{2}$ itself. So requirement 4 is really just there to tell us which set to use if we decide to cut the Rationals at a Rational Number. If we cut the Rationals at $\sqrt{2}$, there isn't a problem since $\sqrt{2}$ isn't a candidate for either L or R because it isn't a Rational Number ($\sqrt{2} \notin \mathbb{Q}$).

For any Real Number x defined by a cut (L, R), if we are told what R contains, then we can deduce what L contains since L contains all the Rationals that are not in R. So x is actually defined once we know the right section R. (Of course it is also defined once we know L.) Consequently we may identify each Real Number x with just its right section R and omit mention of its left section L. For example, the right section for $\sqrt{2}$ consists of all the positive Rational Numbers whose squares exceed 2. So we get

$$\sqrt{2} = \{f \in \mathbb{Q} : f > 0 \text{ and } f^2 > 2\}.$$

And here I am using red ink to distinguish the new numbers ($\mathbb{R}$) from the old numbers ($\mathbb{Q}$).

When we constructed the Rational Numbers from the Integers, there was the issue that the Rational Number $\frac{2}{1}$ is identified with the Integer 2. This didn't cause a problem but, logically speaking, they are slightly different since one lies in $\mathbb{Q}$ while the other lies in $\mathbb{N}$. We encountered the same sort of issue when discussing the formation of the set of Integers $\mathbb{Z}$ from the Natural Numbers $\mathbb{N}$. In constructing the Real Numbers we again have this slightly irritating logical difference between $0.5 \in \mathbb{R}$ and $\frac{1}{2} \in \mathbb{Q}$. Although we choose to identify them as the same number, strictly speaking in terms of a right section

$$0.5 = \{f \in \mathbb{Q} : f > \tfrac{1}{2}\}.$$

If this looks a bit bizarre, remember that the number on the left is regarded as a Real Number, while the f and $\frac{1}{2}$ inside the curly brackets are Rational Numbers. I wrote 0.5 rather than $\frac{1}{2}$ just to emphasize this distinction. But having made the point, I'll now drop using the red ink.

Of course we have to explain how to add and multiply cuts (i.e. right sections), how to define the order relation ($<$), and then show that our new numbers have all the required properties. All this can be done. We will just indicate how to begin. Suppose that R_x and R_y are the two right sections defining Real Numbers x and y respectively.

First we deal with the order relation by defining $x < y$ to mean that R_y is a (proper) subset of R_x. ["proper" means that we exclude the case $R_y = R_x$, which corresponds to $x = y$.] Figure A.2 illustrates this definition with R_x shown in red and R_y shown in blue. We can then define "positive" and "negative" for Real Numbers.

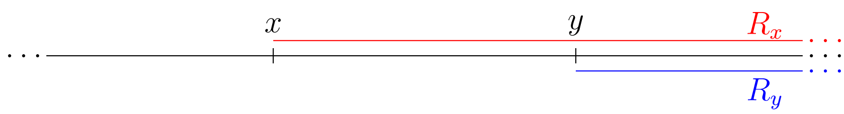


Figure A.2: $x < y$ means that R_y is a proper subset of R_x.

Next we deal with addition by defining $x + y$ as the (new) right section that contains the sums of all the pairs (s, t) of Rationals for $s \in R_x$ and $t \in R_y$:

$$x + y = \{r : r = s + t \text{ for some } s \in R_x \text{ and } t \in R_y\}.$$

For example, if the Real Numbers 1 and 2 are to be added then $1 + 2$ has the right section R containing the sum of each Rational Number greater than 1 and each Rational Number greater than 2. So R certainly contains $1.001 + 2.001 = 3.002$, and it is then easy to see that R contains every Rational Number greater than 3, and no Rational Numbers less than 3. Consequently for Real Numbers, $1 + 2 = 3$ (reassuring).

Multiplication is defined in a similar way although there is a complication caused by the fact that the product of two negative Rationals is positive. Using the same terminology as for addition, we would really like to define

$$x \times y = \{r : r = s \times t \text{ for some } s \in R_x \text{ and } t \in R_y\}.$$

Although this works fine when x and y are both positive (or zero), it is not good enough when negative numbers are involved. For example, if $x = (-2)$ and $y = (-3)$ then we'd like the right section corresponding to the product to be $R = \{f \in \mathbb{Q} : f > 6\}$. However we have $(-1) \in R_x$ and $(-2) \in R_y$, but $(-1) \times (-2) = 2$, which does not lie in R. So some technical adjustments are needed to deal with this problem. We won't do that here. It is actually easier to stick with the definition above for $x \times y$ when $x \geq 0$ and $y \geq 0$, and then define $(-x) \times y$ and $x \times (-y)$ as $-(x \times y)$, and define $(-x) \times (-y)$ as $x \times y$.

It is now relatively easy, if somewhat tedious, to check that the Real Numbers $\mathbb{R}$, as defined by Dedekind cuts, satisfy the properties listed for $\mathbb{Q}$ in the previous subsection. We will omit these checks. But we will prove that $\mathbb{R}$ has the completeness property.

Suppose that S is a (non-empty) set of Real Numbers that is bounded above. Then each $x \in S$ is defined by a cut of $\mathbb{Q}$ that we will denote as (L_x, R_x). Define

$$\bar{R} = \bigcap_{x \in S} R_x = \{f : f \in R_x \text{ for every } x \in S\}.$$

Then $\bar{R}$ is a subset of $\mathbb{Q}$ and we define $\bar{L} = \mathbb{Q} \setminus \bar{R}$, so that $\bar{L}$ contains all the remaining elements of $\mathbb{Q}$.

We check that neither $\bar{R}$, nor $\bar{L}$ is empty. Let $M \in \mathbb{R}$ denote an upper bound of S and suppose that $M = (L_M, R_M)$. Then $R_M \subseteq \bar{R}$, so $\bar{R}$ is non-empty. If x is any element of S and f is any Rational Number in L_x, then $f \in \bar{L}$, so $\bar{L}$ is non-empty.

We also check that every $f \in \bar{L}$ is less than every $g \in \bar{R}$. If $f \in \bar{L}$ then there exists $y \in S$ such that $f \notin R_y$, so $f \in L_y$. If $g \in \bar{R}$ then $g \in R_x$ for every $x \in S$ and, in particular, $g \in R_y$. Since (L_y, R_y) is a cut of $\mathbb{Q}$ it follows that $f < g$.

Thus $(\bar{L}, \bar{R})$ satisfies three of the four properties required for a Dedekind cut of $\mathbb{Q}$. Let us assume initially that $\bar{R}$ has no least element so that $(\bar{L}, \bar{R})$ defines a cut, and put $b = (\bar{L}, \bar{R}) \in \mathbb{R}$. For each $x \in S$, $\bar{R} \subseteq R_x$, so $x \leq b$. Hence b is an upper bound of S. Next suppose that $a = (L_a, R_a)$ is any upper bound of S. Then for each $x \in S$, $R_a \subseteq R_x$, and consequently $R_a \subseteq \bar{R}$. Hence $b \leq a$, and this proves that b is the least upper bound of S.

The same argument, with minor modifications, deals with the case when $\bar{R}$ has a least element b. In such a case, remove b from $\bar{R}$ to form R^* and add it to $\bar{L}$ to form L^*. Then (L^*, R^*) forms a cut of $\mathbb{Q}$ and L^* has maximum element $b \in \mathbb{Q}$. The cut defines the equivalent Real Number $b = (L^*, R^*)$ (To avoid confusion, I am using red ink to distinguish $b \in \mathbb{Q}$ and $b \in \mathbb{R}$.) For each $x \in S$, $R^* \subseteq \bar{R} \subseteq R_x$, so $x \leq b$. Hence b is an upper bound of S. Next suppose that $a = (L_a, R_a)$ is any upper bound of S. Then for each $x \in S$, $R_a \subseteq R_x$, and consequently $R_a \subseteq \bar{R}$. If we had $b \in R_a$ then R_a would have b as a least element, but this is not possible since R_a is a right section. Hence $R_a \subseteq R^*$, giving $b \leq a$, and this proves that b is the least upper bound of S.

Appendix B

Identifying $\mathbb{N}$ in $\mathbb{R}$

In Appendix A we showed how the set of Real Numbers $\mathbb{R}$ may be constructed from the set of Natural Numbers $\mathbb{N}$, whose properties we assume to be correct as set out in the Peano Postulates. But that is not how we defined $\mathbb{R}$ originally in this book. We assumed that there is a set of numbers called $\mathbb{R}$ that obeys the Axioms A, B and C as described in Section 2.3 of Chapter 2. These axioms make no specific mention of Natural Numbers, Integers or Rational Numbers. Our purpose here is to show how these may be identified as subsets of $\mathbb{R}$ as specified by the axioms. The only tricky identification is that of $\mathbb{N}$ (not that tricky). Once we have $\mathbb{N}$ it is easy to get $\mathbb{Z}$ (Integers) and $\mathbb{Q}$ (Rational Numbers). We will also give a fuller explanation of proof by induction.

First define an *inductive set* to be any subset S of $\mathbb{R}$ with the properties that (a) $1 \in S$ and (b) for each $x \in S$, $x + 1 \in S$. There are many inductive sets. For examples, $\mathbb{R}$ is itself an inductive set, the set of positive Real Numbers is an inductive set, and $\{x \in \mathbb{R} : x > \frac{1}{2}\}$ is an inductive set. Of course some subsets of $\mathbb{R}$ are *not* inductive sets, examples are the set of negative Real Numbers, the set $\{1, 2, 3\}$, and $\{x \in \mathbb{R} : x > 1\}$. Now ask yourself the question: "what is the smallest inductive set?" and you will probably understand what we are trying to do (of course, "smallest" is rather imprecise).

By the very definition of an inductive set, 1 is an element of every inductive set, as is $1+1 = 2$, $2+1 = 3$, and so on. We therefore identify $\mathbb{N}$ as the set of Real Numbers that lie in every inductive set. It follows that $\mathbb{N}$ is itself an inductive set and it is contained in every inductive set. It is unique because if there was another inductive set $\mathbb{N}'$ contained in every inductive set, then we would have $\mathbb{N} \subseteq \mathbb{N}'$ and $\mathbb{N}' \subseteq \mathbb{N}$, and so $\mathbb{N}' = \mathbb{N}$.

It should also be reasonably clear that this set $\mathbb{N}$ satisfies the Peano Postulates with the successor of n defined as $n + 1$ using the addition operation on $\mathbb{R}$ guaranteed by the axioms. In particular, if $C \subseteq \mathbb{N}$ is an inductive set, then $\mathbb{N} \subseteq C$ and so $C = \mathbb{N}$, and this deals with the fifth postulate.

Having identified $\mathbb{N}$ as a subset of $\mathbb{R}$ we define the set of Integers

$$\mathbb{Z} = \{x : x \in \mathbb{N} \text{ or } -x \in \mathbb{N} \text{ or } x = 0\}.$$

Then we define the set of Rational Numbers

$$\mathbb{Q} = \{x : x = pq^{-1} \text{ for some } p, q \in \mathbb{Z} \text{ (with } q \neq 0)\}.$$

The inductive nature of $\mathbb{N}$ puts the method of proof by induction onto a firm footing. Suppose that for each Natural Number n, $P(n)$ is a statement about n. Let C be the set of Natural Numbers n for which $P(n)$ is true. Then $C \subseteq \mathbb{N}$. Suppose that we can show (a) $1 \in C$ and (b) for each $k \in C$, $k + 1 \in C$. Then C is inductive and so $\mathbb{N} \subseteq C$. It follows that $C = \mathbb{N}$, in other words $P(n)$ is true for every Natural Number n.

To be absolutely precise we should specify what constitutes a *statement* about n and whether this leads to a *set* of numbers for which the statement is true. We are not going to do that as it would takes us into a long discussion of Mathematical Logic and Set Theory. The sort of statements that we will use are eminently respectable, as are the resulting sets.

Appendix C

Integer roots in $\mathbb{R}$

Here we show that the completeness axiom ensures that every positive Real Number has a positive n^{th} root for each positive integer n. This proof is from first principles. A better (shorter) method of proof is to use the intermediate value theorem applied to the continuous function $f(x) = x^n$, which is strictly increasing on $[0, \infty)$ and satisfies $f(0) = 0$, $f(x) \to \infty$ as $x \to \infty$, and therefore takes every value $a > 0$. However, we introduced n^{th} roots before discussing continuity, so a longer proof from first principles is given here.

Theorem C.1. If n is a positive integer and $a > 0$ then there exists a number $b > 0$ such that $b^n = a$. In other words a has a n^{th} root $b = a^{\frac{1}{n}}$.

Proof. Put $S = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^n \leq a\}$. Then S is non-empty since $0 \in S$. Actually S contains some strictly positive numbers because if $a \leq 1$ then $a \in S$, while if $a > 1$ then $1 \in S$. The set S is also bounded above because if $a \leq 1$ then 1 is an upper bound, while if $a > 1$ then a is an upper bound. Hence S has a least upper bound $b > 0$. We will prove that $b^n = a$.

Suppose that $b^n < a$ and consider $(b + \epsilon)^n$ where $1 > \epsilon > 0$. By the binomial theorem

$$
\begin{aligned}
(b+\epsilon)^n &= b^n + \epsilon n b^{n-1} + \epsilon^2 \frac{n(n-1)}{2!} b^{n-2} + \ldots + \epsilon^n \\
&= b^n + \epsilon \Big[n b^{n-1} + \epsilon \frac{n(n-1)}{2!} b^{n-2} + \ldots + \epsilon^{n-1} \Big] \\
&< b^n + \epsilon \Big[n b^{n-1} + \frac{n(n-1)}{2!} b^{n-2} + \ldots + 1 \Big] \qquad (\text{ since } \epsilon < 1) \\
&= b^n + \epsilon \Big[(b+1)^n - b^n \Big].
\end{aligned}
$$

So if we now fix $\epsilon = \min\left(\frac{1}{2}, \frac{a - b^n}{(b+1)^n - b^n}\right)$ then $1 > \epsilon > 0$ and $(b + \epsilon)^n < a$.

Thus $b + \epsilon \in S$, contradicting the fact that b is an upper bound of S. We conclude that $b^n \geq a$.

Now suppose that $b^n > a$ and consider $(b-\epsilon)^n$ where now $\min(1, b) > \epsilon > 0$. Arguing as before,

$$\begin{aligned}
(b-\epsilon)^n &= b^n - \epsilon\Big[nb^{n-1} - \epsilon\frac{n(n-1)}{2!}b^{n-2} + \ldots + (-1)^{n-1}\epsilon^{n-1}\Big] \\
&> b^n - \epsilon\Big[nb^{n-1} + \frac{n(n-1)}{2!}b^{n-2} + \ldots + 1\Big] \text{ (by the triangle inequality)} \\
&= b^n - \epsilon\Big[(b+1)^n - b^n\Big].
\end{aligned}$$

So if we now fix $\epsilon = \min\left(\frac{1}{2}, \frac{b}{2}, \frac{b^n - a}{(b+1)^n - b^n}\right)$ then $\min(1, b) > \epsilon > 0$ and $(b-\epsilon)^n > a$. Thus $b - \epsilon$ is an upper bound of S, contradicting the fact that b is the least upper bound of S. We conclude that $b^n \leq a$.

So we now have $a \leq b^n \leq a$, and so we must have $b^n = a$. $\square$

It is easy to see that the positive n^{th} root b of the positive number a is unique. If $0 < c < b$ then $c^n < b^n = a$, so c is not an n^{th} root of a, and if $c > b$ then $c^n > b^n = a$, so again c is not an n^{th} root of a.

References

[1] TO BE ADDED

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