SECOND ORDER DIFFERENTIAL EQUATIONS

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Contents

1 Preamble

1.1 About this package

This package is for people who need to solve relatively easy types of second order differential equations. It doesn't contain a lot of theory. It isn't really designed for pure mathematicians who require a course discussing existence and uniqueness of solutions.

You will find that you need a background knowledge of differentiation, integration and first order differential equations in order to get the most out of this package. In particular, you need to be able to differentiate using the product, quotient and function-of-a-function rules. You also need to be able to carry out integrations by simple substitutions, by parts, and using partial fractions. You will need to be able to solve separable and linear first order differential equations. If you are a bit rusty, don't worry - but it would be sensible to do some revision either at the start or as the need arises. Reasonable revision texts are given in the bibliography (Section 11).

If you complete the whole package you should be able to

- recognise a **second order linear constant-coefficient differential equation,**
- understand what is meant by a **homogeneous** equation,
- obtain the **complementary function** using the **auxiliary equation,** in each of the three cases: (a) unequal real roots, (b) complex roots, (c) equal real roots,
- obtain a **particular integral** by one of the two methods described (**trial functions** or **D-operators**),
- obtain the **general solution** by adding the complementary function and a particular integral,
- obtain solutions satisfying given **boundary** or **initial** conditions by determining appropriate values for the constants in the general solution,
- understand the connection with mechanical and electrical vibrational problems, the physical significance of the individual terms, **damping** and **resonance,**
- solve simple **simultaneous** linear differential equations,
- understand how to reduce a second order equation to two simultaneous first order equations.

Depending on your own programme of study you may not need to cover everything in this package. Your tutor will advise you what, if anything, can be omitted. In particular, two methods are given for determining **particular integrals**; it is unlikely that you would be required to know both methods.

1.2 How to use this package

You MUST do examples! Doing lots of examples for yourself is generally the most effective way of learning the contents of this package and covering the objectives listed above. We recommend that you

- first read the theory - make your own notes where appropriate,
- then work through the worked examples - compare your solutions with the ones in the notes,
- finally do similar examples yourself in a workbook.

The original printing of these notes leaves every other page blank. Use the spare space for your own comments, notes and solutions. You will see certain symbols appearing in the right hand margin from time to time:

○ denotes the end of a worked example,

□ denotes the end of a proof,

V denotes a reference to videos (see below for details),

EX highlights a point in the notes where you should try examples.

By the time you have reached a package like this one you will probably have realised that learning mathematics rarely goes smoothly! When you get stuck, use your accumulated wisdom and cunning to get around the problem. You might try:

- re-reading the theory/worked examples,
- putting it down and coming back to it later,
- reading ahead to see if subsequent material sheds any light,
- talking to a fellow student,
- looking in a textbook (see the bibliography),
- watching the appropriate video (see the video summaries),
- raising the problem at a tutorial.

1.3 Videos, tutorials and self-help

The videos cover the main points in the notes. The areas covered are indicated in the notes, usually at the ends of sections and subsections. To resolve a particular difficulty you may not need to watch a whole video (they are each about 40 minutes long). They are broken up into sections prefaced with titles which can be read on fast scan. In addition, a summary of the videos associated with this package appears as an appendix to these notes.

Your tutor will tell you about the arrangements for viewing the videos. Try the worked examples **before** watching the solution unfold on the screen. Make notes of any points you cannot follow so that you can explain the difficulty in a subsequent tutorial session. If you are viewing a video individually, remember the rewind button! Unlike a lecture you can get instant and 100 percent accurate replay of what was said.

Your tutor will tell you about tutorial arrangements. These may be related to assessment arrangements. If attendance at tutorials is compulsory then make sure you know the details! The tutorials provide you with individual contact with a tutor. Use this time wisely - staff time is the most expensive of all our resources.

You should come to tutorials in a prepared state. This means that you should have read the notes and the worked examples. You should have tried appropriate examples for yourself. If you have had difficulty with a particular section then you should watch the corresponding video. If your tutor finds that you haven't done these things then s/he may refuse to help you. Your tutor will find it easier to assist you if you can make any queries as specific as possible.

Your fellow students are an excellent form of self-help. Discuss problems with one another and compare solutions. Just be careful that

1. any assessed coursework submitted by you is yours alone,
2. you yourself do really understand solutions worked out jointly with colleagues.

Familiarize yourself with the layout and contents of these notes; scan them before reading them more carefully. The contents page will help you find your way about - use it. The bibliography will point you to textbooks covering the same material as these notes.

When you graduate, your future employer will be just as interested in your capacity for learning as in what you already know. If you can learn mathematics from this package and from textbooks then you will not only have learnt a particular mathematical topic. You will also (and more importantly) have learnt **how to learn** mathematics.

2 Introduction

We shall be mainly concerned with a very restricted class of second order differential equations, namely those which are linear and have constant coefficients. In fact the solution methods described apply also to n^{th} order linear constant-coefficient differential equations; it is only for ease of calculations that we shall concentrate on the second order variety. In practice it is second order ones which occur most frequently.

The general n^{th} order linear constant-coefficient differential equation has the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x) \tag{1}$$

where n is a positive integer, the quantities $a_n, a_{n-1}, \cdots, a_1, a_0$ are all constants (the constant coefficients) and $a_n \neq 0$ (so that the equation is genuinely of n^{th} order). The function $f(x)$ is a function of x (alone) or is a constant. An example of a second order one is

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x$$

Here $a_2 = 1, a_1 = 3, a_0 = 2$ and $f(x) = \sin x$.

The general form (1) is described as linear (in y) because it does not contain terms like $y^2, \sin y, \sqrt{1 + (\frac{dy}{dx})^2}$; it only contains y as itself or in $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, etc.

If $f(x)$ is identically zero ($f(x) \equiv 0$) then equation (1) is said to be homogeneous because all the non-zero terms then contain y (as itself or in $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, etc.). Unfortunately this is a different use of the word "homogeneous" to that which is employed in connection with first order equations - so beware! If $f(x)$ is not identically zero, the equation is said to be inhomogeneous or non-homogeneous.

A function $y(x)$ which, when substituted into the differential equation (1), makes it an identity is called a solution of the differential equation.

Example 2.1 Show that $y = e^{-x}$ is a solution of the differential equation

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

Solution If $y = e^{-x}$ then $\frac{dy}{dx} = -e^{-x}$ and $\frac{d^2y}{dx^2} = e^{-x}$. Hence

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x} - 3e^{-x} + 2e^{-x} = 0$$

as required. $\bigcirc$

In rough and ready terms, for an n^{th} order equation, we would expect to have to perform n integrations to eliminate $\frac{d^ny}{dx^n}$ (i.e. to convert it to y). Each such integration would introduce an integration constant. Consequently we would expect the most general solution of an n^{th} order differential equation to contain n (independent) arbitrary constants. For equations of the form (1) this is indeed the case. Furthermore any solution of (1) which contains n independent arbitrary constants is a form of the general solution. We shall not prove this last remark, but we shall find it extremely useful. We summarize the position in the following theorem which we state without proof.

Theorem 2.1. *The general solution of an n^{th} order linear constant - coefficient differential equation contains n independent arbitrary constants. Conversely, any solution of such an equation which does contain n independent arbitrary constants is the general solution.*

A solution of (1) which is not the general solution is called a **particular solution** or a **particular integral**. In the previous example, e^{-x} is a particular integral of the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

Any particular integral can be obtained from the general solution by assigning appropriate values to the n independent arbitrary constants.

In an equation such as (1) we sometimes refer to the variable x as the independent variable and the function y as the dependent variable.

(The video discusses the general form, notation and terminology.) V

3 D-Operator Notation

In equations such as (1) we shall often use D for the differential operator $\frac{d}{dx}$. Thus D^2 means $\frac{d^2y}{dx^2}$, and so on. We can then, for example, write the differential equation

$$5\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = \sin x$$

in the form

$$5D^2y + 6Dy + y = \sin x$$

or even as

$$(5D^2 + 6D + 1)y = \sin x$$

We can even factorise:

$$(5D + 1)(D + 1)y = \sin x$$

the implication being that we first compute what $(D+1)$ does to y [it produces $\frac{dy}{dx} + y$] and then we compute what $(5D + 1)$ does to this result [it produces $(5\frac{d^2y}{dx^2} + 5\frac{dy}{dx}) + (\frac{dy}{dx} + y)$ which equals $5\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y$, as expected]. Clearly the order of the factors is unimportant.

The general form (1) can then be written as

$$(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = f(x) \tag{2}$$

We can shorten this to $P(D)y = f(x)$ where $P(D)$ is understood to be the differential operator $a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$. Note that $P(D)$ is an n^{th} order polynomial in D.

(The video describes the use of the D-operator notation.) V

4 Method of Solution

We wish to solve the general form (2). The solution is formed from two components. We shall look at methods for determining each of these components in due course. In this section we shall simply describe the two components and show how they are combined.

Definition 4.1 Given the equation

$$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)y = f(x), \qquad (a_n \neq 0)$$

the general solution of the corresponding homogeneous equation is called the **complementary function**. We often denote this function by y_c. By "the corresponding homogeneous equation" we mean the equation obtained by deleting $f(x)$ and replacing it by 0. Thus y_c satisfies the equation

$$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)y_c = 0 \tag{3}$$

Moreover, since y_c is the general solution of equation (3) it contains n independent arbitrary constants.

Definition 4.2 (We've already met this.) Given the equation

$$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)y = f(x), \qquad (a_n \neq 0)$$

any solution which satisfies the equation and which is not the general solution is called a **particular integral**. Should such a solution contain any arbitrary constants we may set these to arbitrary values (for example we could set them all to zero). We shall assume this has been done, so that a particular integral contains no arbitrary constants. We often denote such a particular integral by y_p. We reiterate that y_p contains no arbitrary constants and that it satisfies the equation

$$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)y_p = f(x) \tag{4}$$

The functions y_c and y_p are the two components needed to solve equation (2).

Theorem 4.1. *The general solution of the equation*

$$(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = f(x), \qquad (a_n \neq 0)$$

is given by $y = y_c + y_p$.

Proof Firstly we note that $y_c + y_p$ will contain n independent arbitrary constants because y_c does and y_p has none. Secondly we verify that $y_c + y_p$ is a solution:

$$\begin{aligned}
&(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)(y_c + y_p) \\
&\qquad = (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y_c \\
&\qquad\quad + (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y_p \\
&\qquad = 0 + f(x) \qquad \text{(using equations (3) and (4))} \\
&\qquad = f(x)
\end{aligned}$$

as required. This completes the proof because by theorem 2.1 any solution with n arbitrary constants is the general solution. □

It is, perhaps, worth remarking that a given differential equation can have several different particular integrals. Any one of these can be employed in the previous result. This sometimes confuses students who compare their results with those of their colleagues; two different-looking answers may be equivalent. To investigate this briefly, suppose that y_{p_1} and y_{p_2} are two different particular integrals of (2). Then if $y = y_{p_1} - y_{p_2}$ we have

$$\begin{aligned}
&(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y \\
&\qquad = (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y_{p_1} \\
&\qquad\quad - (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y_{p_2} \\
&\qquad = f(x) - f(x) \\
&\qquad = 0
\end{aligned}$$

Thus y is a solution of the corresponding homogeneous equation. It is therefore obtainable from the general solution of the homogeneous equation (i.e. from y_c) by assigning appropriate values to the constants. It follows that the difference between two particular integrals must be contained within the complementary function. If Fred obtains a general solution $y_{c_1} + y_{p_1}$ and Mary a general solution $y_{c_2} + y_{p_2}$ then their different-looking answers can both be correct provided that y_{c_1} and y_{c_2} can be obtained from one another by renaming the arbitrary constants and provided that $(y_{p_1} - y_{p_2})$ can be obtained from these by giving the constants appropriate values. (An example is given at the end of section 6.2.1.)

We shall now consider each of our two ingredients y_c and y_p separately. The simpler of the two turns out to be y_c. We shall therefore start with this by trying to obtain the general solution of the homogeneous equation

$$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)y = 0$$

(The video deals with the form of the general solution and the definition of complementary functions and particular integrals.) V

5 Solution of Homogeneous Equations (Complementary Functions)

Consider initially the first order equation

$$(D + a)y = 0$$

where a is a constant. This equation can be written as

$$\frac{dy}{dx} = -ay$$

Hence

$$\int \frac{dy}{y} = -a \int 1.dx + c \qquad \text{(where c is an arbitrary constant)}$$

i.e.

$$\log_e y = -ax + c$$

therefore

$$y = e^{-ax+c} = e^{-ax}e^c = Ae^{-ax}$$

where $A = e^c$ is a constant. Thus any first order linear constant-coefficient equation will have a solution which is an exponential function of x.

Accordingly, for the second order equation

$$(a_2D^2 + a_1D + a_0)y = 0$$

we try $y = e^{mx}$ to see if, for suitable values of m, this represents a solution. This will be the case if, and only if

$$(a_2D^2 + a_1D + a_0)e^{mx} = 0 \qquad \text{(for all } x\text{)}$$

$$\text{i.e.} \qquad a_2D^2e^{mx} + a_1De^{mx} + a_0e^{mx} = 0$$

$$\text{i.e.} \qquad a_2m^2e^{mx} + a_1me^{mx} + a_0e^{mx} = 0$$

(since $De^{mx} = \frac{d}{dx}(e^{mx}) = me^{mx}$ and likewise $D^2e^{mx} = m^2e^{mx}$). Hence $y = e^{mx}$ is a solution if, and only if

$$(a_2m^2 + a_1m + a_0)e^{mx} = 0$$

and since e^{mx} can never be zero we deduce

$$a_2m^2 + a_1m + a_0 = 0 \qquad \text{(the \textbf{auxiliary equation})}$$

Leaving aside the problems of equal and complex roots, suppose this auxiliary equation has the roots $m = m_1$ and $m = m_2$. Then $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ will both be solutions of the differential equation

$$(a_2 D^2 + a_1 D + a_0)y = 0$$

i.e.

$$(a_2 D^2 + a_1 D + a_0)e^{m_1 x} = 0 \quad \text{and} \quad (a_2 D^2 + a_1 D + a_0)e^{m_2 x} = 0$$

It then follows that for any two arbitrary constants A and B that the function

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

represents a solution of the equation since

$$\begin{aligned}
(a_2 D^2 + a_1 D + a_0)\{Ae^{m_1 x} + Be^{m_2 x}\} \\
&= A(a_2 D^2 + a_1 D + a_0)e^{m_1 x} + B(a_2 D^2 + a_1 D + a_0)e^{m_2 x} \\
&= A.0 + B.0 \\
&= 0
\end{aligned}$$

But this function y is not only a solution of the differential equation, it contains two arbitrary and independent constants. Hence, by theorem 2.1 this function y is the general solution of the equation.

Example 5.1 Find the general solution of

$$D^2 y - 5Dy + 6y = 0$$

Solution The auxiliary equation is $m^2 - 5m + 6 = 0$ which gives $(m - 3)(m - 2) = 0$, and so $m = 2$ or 3. Hence the general solution is $y = Ae^{2x} + Be^{3x}$. ○

The method described and illustrated above works fine provided that the auxiliary equation has distinct real roots. For equations with real coefficients (the only kind we shall consider) there are two other possibilities:

(i) a pair of complex conjugate roots $\alpha \pm i\beta$, and

(ii) equal real roots.

We examine each of these cases in turn.

(i) Suppose that the auxiliary equation has a pair of complex conjugate roots $\alpha \pm i\beta$. The above method gives the general solution as

$$y = Ae^{(\alpha + i\beta)x} + Be^{(\alpha - i\beta)x}$$

Although this is correct, a little re-arrangement gives a form which is preferable in that it contains no explicit reference to $i = \sqrt{-1}$ (one would hardly expect the solution to a real equation to contain i). Remember that $e^{i\theta} = \cos\theta + i\sin\theta$. We have

$$\begin{aligned} y &= Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} \\ &= Ae^{\alpha x}e^{i\beta x} + Be^{\alpha x}e^{-i\beta x} \\ &= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}) \\ &= e^{\alpha x}(A[\cos\beta x + i\sin\beta x] + B[\cos\beta x - i\sin\beta x] \\ &= e^{\alpha x}[(A+B)\cos\beta x + (Ai - Bi)\sin\beta x] \\ &= e^{\alpha x}[E\cos\beta x + F\sin\beta x] \end{aligned}$$

where E, F are arbitrary independant constants (in fact $E = A + B, F = Ai - Bi$). As promised, this latter form contains no explicit reference to i . In practice, we do NOT go through the above deduction on each occasion; we simply move from the auxiliary equation solutions $\alpha \pm i\beta$ to $y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$ (we can call the constants A, B anything we like).

Example 5.2 Find the general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

Solution The auxiliary equation is $m^2 - 2m + 5 = 0$ which has solution

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

Hence the general solution of the differential equation is

$$y = e^{1x}(A\cos 2x + B\sin 2x)$$

(where the 1 in the e^{1x} comes from the real part of $1 \pm 2i$ and the 2 in the cosine and sine terms comes from the imaginary part). Written slightly more simply this gives

$$y = e^{x}(A\cos 2x + B\sin 2x)$$

○

We now turn to the other difficulty:

(ii) Suppose that the auxiliary equation has a pair of equal roots i.e. $m = \lambda$ (twice). The auxiliary equation is equivalent to $(m-\lambda)^2 = 0$ and so the differential equation is equivalent to $(D-\lambda)^2 y = 0$ (i.e. to $(D^2 - 2\lambda D + \lambda^2)y = 0$).

To solve $(D-\lambda)^2y=0$ we write it as $(D-\lambda)(D-\lambda)y=0$ and make the substitution $z=(D-\lambda)y$ (i.e. $z=\frac{dy}{dx}-\lambda y$) so that the equation gives $(D-\lambda)z=0$, i.e.

$$\frac{dz}{dx}-\lambda z=0$$

This is first order linear (in fact it is also separable). We can solve it by calculating the integrating factor:

$$R(x)=e^{\int -\lambda dx}=e^{-\lambda x}$$

and re-writing the equation as:

$$\frac{d}{dx}(R(x).z)=R(x).0=0$$

Hence

$$\frac{d}{dx}(e^{-\lambda x}.z)=0$$

and so:

$$e^{-\lambda x}z=A \qquad \text{(where A is a constant)}$$

Therefore, multiplying by $e^{\lambda x}$, we get

$$z=Ae^{\lambda x}$$

But, remember, $z=\frac{dy}{dx}-\lambda y$. Hence

$$\frac{dy}{dx}-\lambda y=Ae^{\lambda x}$$

This is again a first order linear equation and again we can solve by using the (same) integrating factor

$$R(x)=e^{\int -\lambda dx}=e^{-\lambda x}$$

We re-write the equation as:

$$\frac{d}{dx}(R(x).y)=R(x).Ae^{\lambda x}$$

Hence

$$\frac{d}{dx}(e^{-\lambda x}.y)=e^{-\lambda x}.Ae^{\lambda x}=A$$

and so:

$$e^{-\lambda x}y=\int Adx+B \qquad \text{(where B is a constant)}$$

Integrating and once again multiplying by $e^{\lambda x}$ gives

$$y = (Ax + B)e^{\lambda x}$$

To summarise: the solution of $(D-\lambda)^2 y = 0$ is $y = (Ax+B)e^{\lambda x}$. Naturally, we do not need to wade through all this theory every time.
Example 5.3 Find the general solution of

$$(D^2 + 4D + 4)y = 0$$

Solution The auxiliary equation is $m^2 + 4m + 4 = 0$, i.e. $(m + 2)^2 = 0$. Hence $m = -2$ (twice). The solution of the differential equation is therefore

$$y = (Ax + B)e^{-2x}$$

Higher order equations may be dealt with in a similar fashion. Thus an equation such as

$$(D - 3)(2D + 1)^2(5D - 7)^3 y = 0$$

will have solution

$$y = Ae^{3x} + (Bx + C)e^{-\frac{1}{2}x} + (Ex^2 + Fx + G)e^{\frac{7}{5}x}$$

where the $(5D - 7)^3$ gives rise to a triplicated root of the auxiliary equation and hence to a quadratic expression times $e^{\frac{7}{5}x}$.

We conclude this section by reminding ourselves that what we have been doing in finding the general solution of homogeneous equations is one of the two ingredients in determining the general solution of non-homogeneous equations. Using the examples we have covered:
a) The complementary function for the equation

$$(D^2 - 5D + 6)y = f(x)$$

is given by

$$y_c = Ae^{2x} + Be^{3x}$$

b) The complementary function for the equation

$$(D^2 - 2D + 5)y = f(x)$$

is given by

$$y_c = e^x(A\cos 2x + B\sin 2x)$$

c) The complementary function for the equation

$$(D^2 + 4D + 4)y = f(x)$$

is given by

$$y_c = (Ax + B)e^{-2x}$$

In each case y_c is independent of $f(x)$; i.e. changing $f(x)$ has no effect on the complementary function.

We shall now look at the other ingredient in solving non-homogeneous equations; namely particular integrals.

(The video covers the use of the auxiliary equation and the three cases: a) unequal real roots, b) a pair of complex conjugate roots, and c) equal real roots. V

At this point you should try examples involving the general solution of homogeneous equations and obtaining the complementary function for non-homogeneous equations.) EX

6 Particular Integrals

There are several groups of methods for determining particular integrals. These notes cover two of these groups. You do not need to know both. Your lecturer will indicate which s/he is following. The former group is sometimes called the method of undetermined coefficients or the method of trial functions. Essentially it consists of guesswork, albeit intelligent guesswork! The latter group is sometimes called the D-operator method.

Both approaches have advantages and disadvantages. The undetermined coefficient method requires some background knowledge of the likely form of the particular integral and there are tricky cases which are not easy to explain. However, the algebraic manipulation required is relatively simple. The D-operator method breaks into three procedures or algorithms which will deliver a particular integral without any prior knowledge of the form of the solution. However, these procedures are difficult to justify rigorously. We shall not attempt such justification here (although it can be done). Generally speaking, the D-operator method also requires more algebraic manipulation.

Whichever method is used for determining a particular integral and however dissatisfied you may be with that method, it is usually easy to check that the particular integral is correct : all you have to do is to verify that it satisfies the equation. We will use this to justify any "strange" procedures we adopt although we shall point out the difficulties when we encounter them.

(The videos do not cover the trial function (undetermined coefficient) method, they only cover the D-operator method.)

6.1 The method of undetermined coefficients (or trial functions).

The method applies to n^{th} order linear constant coefficient equations. For convenience we will restrict ourselves to the second order equation

$$(a_2D^2 + a_1D + a_0)y = f(x) \tag{5}$$

We consider a number of cases each involving a different form of $f(x)$.

6.1.1 When $f(x)$ is a polynomial

In this case we can read (5) as saying that if we apply the differential operator

$$a_2 D^2 + a_1 D + a_0$$

to $y(x)$, we should produce a polynomial $f(x)$. The obvious conclusion is that $y(x)$ is itself a polynomial because experience tells us that the only function we can differentiate to produce a polynomial is another polynomial. Normally we can take y to have the same polynomial degree as f. (e.g. if f is quadratic then, normally, y is quadratic.) There is actually an important exception to this equality of degrees which is covered in sub-section 6.1.5; we shall ignore this problem for the moment.

The method is best illustrated by an example

Example 6.1 Find a particular integral for the equation

$$y'' + y = x^2$$

Solution Here $f(x) = x^2$, a quadratic. We try for a particular integral of the form

$$y_p = ax^2 + bx + c \qquad \text{(i.e. another quadratic)}$$

Here a, b, c are the undetermined coefficients and ax^2+bx+c the trial function. We have

$$\begin{array}{lrcl} & y_p' & = & 2ax + b \\ \text{and} & y_p'' & = & 2a \\ \text{therefore} & y_p'' + y_p & = & 2a + ax^2 + bx + c \\ & & = & ax^2 + bx + (c + 2a) \end{array}$$

For this latter expression to reduce to x^2 we require $a = 1, b = 0$ and $c = -2a = -2$. Thus $y_p = x^2 - 2$ is a particular integral of the given equation. ○

Note that we obtained three equations for the three unknowns a, b, c. Normally, for a polynomial of degree d we will get $(d+1)$ equations in $(d+1)$ unknowns. The exceptional case referred to above is related to the problem of consistency of these equations.

(Now try some examples for yourself involving polynomial functions $f(x)$. If you meet the problem of inconsistent equations then leave that example until you've covered the material in section 6.1.5) EX

6.1.2 When $f(x)$ is an exponential function

Equation (5) then says that the differential operator

$$a_2D^2 + a_1D + a_0$$

applied to $y(x)$ produces something involving an exponential $e^{\alpha x}$. The obvious conclusion now is that $y(x)$ also involves $e^{\alpha x}$. If $f(x) = Ae^{\alpha x}$ for some constant A we might reasonably expect $y(x) = ae^{\alpha x}$ to be a particular integral for some appropriate value of a. Again there is an important exception covered in section 6.1.5 which we ignore for the moment.

We illustrate the method by means of an example.

Example 6.2 Find a particular integral for the equation

$$y'' + 3y' + 2y = 3e^{2x}$$

Solution Here $f(x) = 3e^{2x}$. We try for a particular integral of the form $y_p = ae^{2x}$. Here a is the undetermined coefficient and ae^{2x} the trial function. We have

$$y_p' = 2ae^{2x}, \qquad y_p'' = 4ae^{2x}$$

Therefore

$$y_p'' + 3y_p' + 2y_p = 4ae^{2x} + 6ae^{2x} + 2ae^{2x} = 12ae^{2x}$$

For this latter expression to reduce to $3e^{2x}$ we require $12a = 3$, i.e. $a = \frac{1}{4}$. Thus $y_p = e^{2x}/4$ is a particular integral of the given differential equation. ○

Note that we obtained one equation for the one unknown a. Sometimes this equation is inconsistent and it takes the form $0.a = A$ where $A \neq 0$. The exceptional case referred to above is related to this problem.

(Now try some examples for yourself involving exponential functions $f(x)$. If you meet the problem of an inconsistent equation then leave that example until you've covered the material in section 6.1.5) EX

6.1.3 When $f(x)$ is a sine or cosine function

Suppose $f(x) = A\sin\alpha x$ where A, α are constants. Then equation (5) says that the differential operator

$$a_2D^2 + a_1D + a_0$$

applied to $y(x)$ produces a multiple of $\sin\alpha x$. We would anticipate that $y(x)$ might contain terms such as $a\sin\alpha x$ and $b\cos\alpha x$, i.e.

$$y = a\sin\alpha x + b\cos\alpha x$$

This will be the form of our trial function with a, b as the undetermined coefficients. Exactly the same form applies when $f(x)$ is a multiple of $\cos \alpha x$, or a combination of both sine and cosine terms (with the same argument αx). Again there is an important exception covered in section 6.1.5 which we ignore for the moment.

We illustrate the method by means of an example.

Example 6.3 Find a particular integral for the equation

$$y'' - 2y' + 2y = \sin 2x$$

Solution Here $f(x) = \sin 2x$. We try for a particular integral of the form

$$y_p = a \sin 2x + b \cos 2x$$

We then have

$$\begin{array}{lrcl}
 & y_p' & = & 2a\cos 2x - 2b\sin 2x \\
\text{and} & y_p'' & = & -4a\sin 2x - 4b\cos 2x \\
\text{Hence} & y_p'' - 2y_p' + 2y_p & = & -4a\sin 2x - 4b\cos 2x \\
 & & & -\, 4a\cos 2x + 4b\sin 2x \\
 & & & +\, 2a\sin 2x + 2b\cos 2x \\
 & & = & (-4a + 4b + 2a)\sin 2x \\
 & & & +\, (-4b - 4a + 2b)\cos 2x \\
 & & = & (4b - 2a)\sin 2x + (-2b - 4a)\cos 2x
\end{array}$$

For this latter expression to reduce to $\sin 2x$ we require

$$4b - 2a = 1 \qquad \text{and} \qquad -2b - 4a = 0$$

These give $a = -\frac{1}{10}, b = \frac{2}{10}$. Hence

$$y_p = \frac{-\sin 2x + 2\cos 2x}{10}$$

is a particular integral of the given differential equation. ○

In general, this method gives two equations for two unknowns. Problems arise if the equations are inconsistent and the exceptional case mentioned above is related to this difficulty.

(Now try some examples for yourself in which $f(x)$ is either a sine or a cosine function. If you meet the problem of inconsistent equations then leave that example until you've covered the material in section 6.1.5) EX

6.1.4 Combinations of polynomials, exponentials and sine/cosine functions

If $f(x)$ is a product of an exponential with a sine or a cosine then we look for a particular integral which also combines these terms, i.e. if $f(x) = Ae^{\alpha x}\sin\beta x$ or if $f(x) = Ae^{\alpha x}\cos\beta x$ (where A, α, β are constants) we try

$$y_p = e^{\alpha x}(a\sin\beta x + b\cos\beta x)$$

where a, b are constants.

If $f(x)$ is a product of a polynomial with an exponential or a sine or a cosine then we look for a particular integral containing similar combinations. For example, if $f(x) = P(x)e^{\alpha x}$, where $P(x)$ is a quadratic and α is a constant, we try

$$y_p = (ax^2 + bx + c)e^{\alpha x}$$

where a, b, c are constants.

If $f(x) = P(x)\sin\alpha x$ or $P(x)\cos\alpha x$ where $P(x)$ is a quadratic and α is a constant, we try

$$y_p = (ax^2 + bx + c)\sin\alpha x + (gx^2 + hx + i)\cos\alpha x$$

where a, b, c, g, h, i are constants.

Finally we deal with the case when $f(x)$ is a product of a polynomial, an exponential and a sine (or cosine). Suppose, for example

$$f(x) = P(x)e^{\alpha x}\sin\beta x$$

where $P(x)$ is a quadratic and α, β are constants. We try

$$y_p = (ax^2 + bx + c)e^{\alpha x}\sin\beta x + (gx^2 + hx + i)e^{\alpha x}\cos\beta x$$

where a, b, c, g, h, i are constants.

If in any of the above cases $P(x)$ is not a quadratic we make appropriate adjustments - for example if $P(x)$ is a cubic we replace all the quadratic expressions such as $(gx^2 + hx + i)$ by cubics. As in the previous sections inconsistent equations can arise in all the above cases. We shall look at this in section 6.1.5.

Example 6.4 Find a particular integral for the equation

$$y'' - 3y' + 2y = x\sin x$$

Solution We try for a particular integral of the form

$$y_p = (ax + b)\sin x + (gx + h)\cos x$$

We then have

$$\begin{aligned}
y_p' &= (ax+b)\cos x + a\sin x - (gx+h)\sin x + g\cos x \qquad \text{and} \\
y_p'' &= -(ax+b)\sin x + a\cos x + a\cos x - (gx+h)\cos x - g\sin x - g\sin x
\end{aligned}$$

Hence

$$\begin{aligned}
y_p'' - 3y_p' + 2y_p &= -(ax+b)\sin x + 2a\cos x - (gx+h)\cos x - 2g\sin x \\
&\quad - 3(ax+b)\cos x - 3a\sin x + 3(gx+h)\sin x - 3g\cos x \\
&\quad + 2(ax+b)\sin x + 2(gx+h)\cos x
\end{aligned}$$

Grouping together the sine terms and the cosine terms on the right-hand side gives

$$\begin{aligned}
y_p'' - 3y_p' + 2y_p &= [ax + b - 3a - 2g + 3gx + 3h]\sin x \\
&\quad + [gx + h + 2a - 3ax - 3b - 3g]\cos x \\
&= [(a+3g)x + (b - 3a - 2g + 3h)]\sin x \\
&\quad + [(g-3a)x + (h + 2a - 3b - 3g)]\cos x
\end{aligned}$$

For this to equal $x\sin x$ we require

$$\begin{aligned}
a + 3g &= 1 \\
b - 3a - 2g + 3h &= 0 \\
g - 3a &= 0 \\
h + 2a - 3b - 3g &= 0
\end{aligned}$$

The first and third of these give $a = \frac{1}{10}, g = \frac{3}{10}$. The second and fourth then become

$$\begin{aligned}
b + 3h &= \frac{9}{10} \\
h - 3b &= \frac{7}{10} \\
\text{Hence} \qquad b + 3(\frac{7}{10} + 3b) &= \frac{9}{10} \\
\text{i.e.} \qquad 10b &= \frac{9}{10} - \frac{21}{10} = -\frac{12}{10} \\
\text{so} \qquad b &= -\frac{12}{100} \\
\text{But then} \qquad h &= 3b + \frac{7}{10} \\
&= -\frac{36}{100} + \frac{70}{100} = \frac{34}{100}
\end{aligned}$$

Altogether these give

$$y_p = (\frac{x}{10} - \frac{12}{100}) \sin x + (\frac{3x}{10} + \frac{34}{100}) \cos x$$

as a particular integral for the differential equation. ○

If $f(x)$ is a sum (or difference) of functions of the preceding types then the particular integral can be found as the sum (or difference) of the appropriate components. For example if $f(x) = 3x^2 + 2\cos x$ we would try for a particular integral:

$$y_p = (ax^2 + bx + c) + (g \cos x + h \sin x)$$

where a, b, c, g, h are all constants. The first bracketed term deals with $3x^2$ and the second deals with $2\cos x$. This technique enables us to deal with the cases when $f(x)$ is one or other of the hyperbolic functions $A \sinh \alpha x$ or $A \cosh \alpha x$ - we simply expand

$$\sinh \alpha x = \frac{e^{\alpha x} - e^{-\alpha x}}{2} = \frac{1}{2}e^{\alpha x} - \frac{1}{2}e^{-\alpha x}$$

$$\cosh \alpha x = \frac{e^{\alpha x} + e^{-\alpha x}}{2} = \frac{1}{2}e^{\alpha x} + \frac{1}{2}e^{-\alpha x}$$

[In fact we could go back to basics with these functions and treat them like we did sine and cosine in the previous section. This would result in trying

$$y_p = a \sinh \alpha x + b \cosh \alpha x \quad]$$

(Now try some examples for yourself involving functions $f(x)$ which are combinations of polynomials, exponentials and sine/cosine functions. If you meet the problem of inconsistent equations then leave that example until you've covered the material in section 6.1.5) EX

6.1.5 Failing Cases

As we remarked in 6.1.1 - 6.1.4, the methods given sometimes fail. We shall see that it is possible to predict such failures in advance and adjust the form of the trial function in order to obtain a particular integral. We shall start by considering an example, then we shall give the general rule which we shall summarize in a table. **The rule and the table are difficult to follow unless you understand the example.**

Example 6.5 Find a particular integral for the differential equation

$$y'' - 3y' + 2y = e^{2x}$$

Solution Following section 6.1.2 we might try $y_p = ae^{2x}$. This gives

$$y_p'' - 3y_p' + 2y_p = 4ae^{2x} - 6ae^{2x} + 2ae^{2x} = 0$$

Arrgh! There is no choice for a which will make the right-hand side equal e^{2x}. To see why this was predictable consider the complementary function; firstly the auxiliary equation is $m^2 - 3m + 2 = 0$, giving $(m-1)(m-2) = 0$ and so $m = 1$ or 2. Hence the complementary function is

$$y_c = Ae^x + Be^{2x}$$

This contains an e^{2x} term. Our initial choice of y_p is therefore doomed to failure because substituting an e^{2x} term into the differential equation is bound to produce zero.

We must therefore revise our "guess" for y_p. The question we need to address is what function can be differentiated to produce e^{2x} other than e^{2x} itself. A possible candidate is xe^{2x}, so we try

$$y_p = axe^{2x}$$

This gives

$$\begin{array}{lrcl} & y_p' & = & 2axe^{2x} + ae^{2x} \\ \text{and} & y_p'' & = & 4axe^{2x} + 4ae^{2x} \\ \text{Hence} & y_p'' - 3y_p' + 2y_p & = & 4axe^{2x} + 4ae^{2x} \\ & & & -6axe^{2x} - 3ae^{2x} + 2axe^{2x} \\ & & = & ae^{2x} \end{array}$$

Notice how the xe^{2x} terms cancel out - with a lot of insight you might have been able to predict this given that e^{2x} forms part of the complementary function. Whether or not you think it was predictable, we are now left with just ae^{2x} and this reduces to e^{2x} if we take $a = 1$. Hence

$$y_p = xe^{2x}$$

is a particular integral for the differential equation. **Notice how our second choice for y_p (namely axe^{2x}) was simply x times our original choice.**

The general rule is as follows.

Firstly we compute the complementary function $y_c = Ay_1 + By_2$, say, where A, B are constants and y_1, y_2 are particular solutions of the corresponding homogeneous equation. We then try to apply one of the rules given in 6.1.1 - 6.1.4 to determine a particular integral.

If any part of the trial function consists of y_1 or y_2 then these rules will fail. [By "part" of the trial function we mean one of the terms forming the sum e.g. $x\sin x$ is "part" of $(ax+b)\sin x+(gx+h)\cos x$ whilst $\sin x$ is not "part" of $e^x(a\sin x+b\cos x)$.] **We remedy the situation by multiplying the trial function by an appropriate power of x (i.e. by x^r) The value of r is chosen to be the lowest positive integer value which will ensure that the revised trial function does not contain y_1 or y_2.**

For example we might find that

$$y_c = e^x(A\sin 2x + B\cos 2x)$$

so that $y_1 = e^x \sin 2x$ and $y_2 = e^x \cos 2x$. If section 6.1.4 suggests a test function of the form

$$y_p = e^x(a\sin 2x + b\cos 2x)$$

then, clearly, this will fail.. We replace it by

$$y_p = xe^x(a\sin 2x + b\cos 2x)$$

which does not contain y_1 or y_2.

Example 6.6 Find a particular integral for the differential equation

$$y'' - 2y' + y = 2e^x$$

Solution Here the auxiliary equation is $m^2 - 2m + 1 = 0$ giving $m = 1$ (twice). Therefore the complementary function is

$$y_c = (Ax + B)e^x$$

The form of particular integral normally employed for an equation of the given type would be (see section 6.1.2)

$$y_p = ae^x$$

Plainly this will fail here because y_c contains e^x. If we multiply by x and try

$$y_p = axe^x$$

this will also fail because y_c contains xe^x. We therefore multiply by a further factor x and try

$$y_p = ax^2e^x$$

This gives

$$\begin{array}{lrcl} & y_p' & = & ax^2e^x + 2axe^x \\ \text{and} & y_p'' & = & ax^2e^x + 4axe^x + 2ae^x \\ \text{Therefore} & y_p'' - 2y_p' + y_p & = & ax^2e^x + 4axe^x + 2ae^x \\ & & & -2ax^2e^x - 4axe^x + ax^2e^x \\ & & = & 2ae^x \end{array}$$

and this gives $2e^x$ if $a = 1$. Hence a (correct) particular integral is

$$y_p = x^2e^x$$

Notice how in the working above on the right hand side the x^2e^x and xe^x terms cancelled out. With the complementary function to hand you might just be able to see why this was bound to happen here - but if you can't, don't worry, such insight isn't essential. Look back now to the general rule. It should be clear why in this case we multiplied the original test function by x^2 (and not x or, for that matter x^3 which would have been "overkill").

Example 6.7 Find a particular integral for the differential equation

$$y'' - 2y' = x$$

Solution Here the auxiliary equation is $m^2 - 2m = 0$ which gives $m = 0$ or 2 and so the complementary function is

$$y_c = Ae^{0x} + Be^{2x} = A + Be^{2x}$$

The usual form of particular integral for the given equation (see section 6.1.1) is

$$y_p = ax + b$$

However, both this and y_c contain constant terms (A and b). We therefore multiply by x to obtain

$$y_p = ax^2 + bx$$

which now has no terms in common with y_c. It gives

$$\begin{array}{lrcl} & y_p' & = & 2ax + b \\ & y_p'' & = & 2a \\ \text{Hence} & y_p'' - 2y_p' & = & 2a - 4ax - 2b \end{array}$$

This yields x provided $a = -\frac{1}{4}$ and $b = -\frac{1}{4}$. Hence

$$y_p = -\frac{x^2}{4} - \frac{x}{4}$$

is a (correct) particular integral. $\bigcirc$

In order to determine whether or not we are in one of the failing cases of sections 6.1.1 - 6.1.4, it is necessary to look at the complementary function. For this reason, when solving a differential equation by the method of complementary functions and particular integrals, it is best to compute the complementary function first.

(Now try some examples for yourself involving functions $f(x)$ for which the methods of sections 6.1.1 - 6.1.4 fail to work.) EX

We conclude this section by giving a tabulation of forms for the particular integral.

Particular Integrals Summary

For a linear constant - coefficient differential equation, the solution has the form $y = y_c + y_p$ (complementary function + particular integral). y_p can often be determined by use of a "trial" function. This technique is sometimes called the method of undetermined coefficients.

$f(x)$	For y_p try $g(x)$. (SEE FOOTNOTE)
x^k e.g. x^2	$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ $g(x) = a_0 + a_1x + a_2x^2$
$e^{\alpha x}$ e.g. e^{3x}	$g(x) = ae^{\alpha x}$ $g(x) = ae^{3x}$
$x^ke^{\alpha x}$ e.g. xe^{3x}	$g(x) = (a_0 + a_1x + a_2x^2 + \cdots + a_kx^k)e^{\alpha x}$ $g(x) = (a_0 + a_1x)e^{3x}$
$A\sin\beta x$ or $A\cos\beta x$ e.g. $5\sin 2x$	$g(x) = a\sin\beta x + b\cos\beta x$ $g(x) = a\sin 2x + b\cos 2x$
$Ax^k\sin\beta x$ or $Ax^k\cos\beta x$ e.g. $5x\sin 2x$	$g(x) = (a_0 + a_1x + a_2x^2 + \cdots + a_kx^k)\sin\beta x$ $+ (b_0 + b_1x + b_2x^2 + \cdots + b_kx^k)\cos\beta x$ $g(x) = (a_0 + a_1x)\sin 2x + (b_0 + b_1x)\cos 2x$
$Ae^{\alpha x}\sin\beta x$ or $Ae^{\alpha x}\cos\beta x$ e.g. $5e^{3x}\cos 2x$	$g(x) = e^{\alpha x}(a\sin\beta x + b\cos\beta x)$ $g(x) = e^{3x}(a\sin 2x + b\cos 2x)$
$Ax^ke^{\alpha x}\sin\beta x$ or $Ax^ke^{\alpha x}\cos\beta x$ e.g. $5xe^{3x}\cos 2x$	$g(x) = e^{\alpha x}([a_0 + a_1x + a_2x^2 + \cdots + a_kx^k]\sin\beta x$ $+ [b_0 + b_1x + b_2x^2 + \cdots + b_kx^k]\cos\beta x)$ $g(x) = e^{3x}([a_0 + a_1x]\sin 2x + [b_0 + b_1x]\cos 2x)$

Footnote. In all cases if y_c already contains the suggested $g(x)$ or even one of the terms forming $g(x)$, then multiply $g(x)$ by the lowest power of x, x^r say, such that y_c does not contain $x^rg(x)$ or any term forming it. For example, if $f(x) = x^3e^{2x}$ and y_c contains xe^{2x} then we multiply the suggested $g(x)$ in the table by x^2 in order to avoid an xe^{2x} in y_p; thus our trial function for y_p would be

$$y_p = x^2(a_0 + a_1x + a_2x^2 + a_3x^3)e^{2x}$$

6.2 The D-Operator Method

The method applies to n^{th} order linear constant-coefficient equations. We shall find it convenient to write the equation

$$(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = f(x)$$

in the form

$$P(D)y = f(x) \tag{6}$$

Here $P(D)$ is the differential operator

$$a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$$

which is a polynomial in D.

Equation (6) can be read as saying that a certain operator ($P(D)$) applied to y produces $f(x)$. It is plausible that some other (inverse?) operator applied to $f(x)$ will produce y (i.e. a solution of the differential equation). We shall write this other operator as $[P(D)]^{-1}$ or as $\frac{1}{P(D)}$, although the process involved is **not** simply dividing 1 by $P(D)$, because $P(D)$ is not a number. If we denote the solution produced as y_p, we have

$$y_p = \frac{1}{P(D)}\{f(x)\} \tag{7}$$

We shall normally use { } to enclose the object which an operator acts upon. If the object is a single letter we may omit the brackets as in D^2y rather than $D^2\{y\}$.

It remains to determine how to interpret $\frac{1}{P(D)}$ in equation (7) in various particular circumstances. There are three techniques for handling this problem, each appropriate to a different type of function $f(x)$.

(The video recaps the D-operator notation and discusses the inverse operator.) V

6.2.1 Use of the Binomial Expansion

This is appropriate whenever $f(x)$ is a polynomial in x.

Reminder:	$(1+u)^\alpha = 1 + \alpha u + \dfrac{\alpha(\alpha-1)}{2!}u^2 + \cdots$
-in particular:	$\dfrac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$
Both the above formulas are valid when the number u satisfies $\|u\| < 1$.	

The central idea is to expand $\frac{1}{P(D)} = [P(D)]^{-1}$ by the binomial theorem even though D is not a number.

Example 6.8 Find a particular integral for the differential equation

$$(D^2 + D + 1)y = x^2$$

Solution

$$\begin{aligned}
y_p &= \frac{1}{D^2+D+1}\{x^2\} \\
&= \frac{1}{1+[D+D^2]}\{x^2\} \qquad \text{(now think of } [D+D^2] \text{ as } u) \\
&= (1 - [D+D^2] + [D+D^2]^2 - [D+D^2]^3 + \cdots)\{x^2\} \\
&= x^2 - [D+D^2]\{x^2\} + [D+D^2]^2\{x^2\} - [D+D^2]^3\{x^2\} + \cdots \\
&= x^2 - [2x+2] + [D^2 + 2D^3 + D^4]\{x^2\} - \cdots \\
&= x^2 - 2x - 2 + [2+0+0] - \cdots \\
&= x^2 - 2x
\end{aligned}$$

Note how we can ignore - in this problem - powers of D higher than D^2 since $D^3\{x^2\} = 0, D^4\{x^2\} = 0$, etc., etc. In general, if $f(x)$ is a polynomial of degree n we can ignore powers of D greater than the n^{th} power.

As a check we can compute

$$(D^2 + D + 1)\{x^2 - 2x\} = 2 + (2x-2) + (x^2 - 2x) = x^2$$

as required (such checks are useful but not essential). ○

Example 6.9 Find a particular integral for the differential equation

$$(D^2 + 5D + 6)y = x + 1$$

Solution

$$\begin{aligned}
y_p &= \frac{1}{D^2+5D+6}\{x+1\} \\
&= \frac{1}{6}.\frac{1}{1+\left[\frac{5D+D^2}{6}\right]}\{x+1\} \\
&= \frac{1}{6}\left(1-\left[\frac{5D+D^2}{6}\right]+\left[\frac{5D+D^2}{6}\right]^2+\cdots\right)\{x+1\} \\
&= \frac{1}{6}\left(1-\frac{5}{6}D+\cdots\right)\{x+1\} \qquad \text{ignoring } D^2, D^3, \ldots \\
&= \frac{1}{6}\left([x+1]-\frac{5}{6}.1\right)
\end{aligned}$$

and this gives

$$y_p = \frac{x}{6}+\frac{1}{36}$$

As an alternative, we could factorise (D^2+5D+6) as $(D+3)(D+2)$. Then

$$\begin{aligned}
y_p &= \frac{1}{(D+3)(D+2)}\{x+1\} \\
&= \frac{1}{6}.\frac{1}{(1+\frac{D}{3})(1+\frac{D}{2})}\{x+1\} \\
&= \frac{1}{6}(1-\frac{D}{3}+\frac{D^2}{9}+\cdots)(1-\frac{D}{2}+\frac{D^2}{4}+\cdots)\{x+1\} \\
&= \frac{1}{6}(1-\frac{D}{3}+\frac{D^2}{9}+\cdots)\{(x+1)-\frac{1}{2}.1\} \\
&= \frac{1}{6}(1-\frac{D}{3}+\frac{D^2}{9}+\cdots)\{x+\frac{1}{2}\} \\
&= \frac{1}{6}(x+\frac{1}{2}-\frac{1}{3}.1)
\end{aligned}$$

and this gives, as before

$$y_p = \frac{x}{6}+\frac{1}{36}$$

A further alternative is provided by the partial fractions

$$
\begin{aligned}
\frac{1}{(D+3)(D+2)} &= \frac{1}{D+2} - \frac{1}{D+3}, \qquad \text{so} \\
y_p &= \left(\frac{1}{2} . \frac{1}{1+\frac{D}{2}} - \frac{1}{3} . \frac{1}{1+\frac{D}{3}} \right) \{x+1\} \\
&= \frac{1}{2}(1 - \frac{D}{2} + \frac{D^2}{4} + \cdots)\{x+1\} \\
&\quad - \frac{1}{3}(1 - \frac{D}{3} + \frac{D^2}{9} + \cdots)\{x+1\} \\
&= \frac{1}{2}(x+1-\frac{1}{2}.1) - \frac{1}{3}(x+1-\frac{1}{3}.1)
\end{aligned}
$$

and once again this gives

$$y_p = \frac{x}{6} + \frac{1}{36}$$

Rather than looking for factors and partial fractions, it is generally best to stick to the original method of solution.

Occasionally a "$\frac{1}{D}$" appears naturally in the solution. Since "D" means "differentiate" and "$\frac{1}{D}$" represents the inverse operator, we interpret "$\frac{1}{D}$" as meaning "integrate". The following example provides an illustration.

Example 6.10 Find a particular integral for the differential equation

$$(D^2 + D)y = x^2 + x + 1$$

Solution We write

$$y_p = \frac{1}{D^2 + D}\{x^2 + x + 1\}$$

but we cannot expand $\frac{1}{D^2+D}$ directly by the binomial theorem because there is no constant term in the denominator; instead we firstly extract the factor D and then expand $\frac{1}{1+D}$:

$$
\begin{aligned}
y_p &= \frac{1}{D(D+1)}\{x^2+x+1\} \\
&= \frac{1}{D}.\frac{1}{1+D}\{x^2+x+1\} \\
&= \frac{1}{D}.(1-D+D^2+\cdots)\{x^2+x+1\} \\
&= \frac{1}{D}\{(x^2+x+1)-(2x+1)+2\} \\
&= \frac{1}{D}\{x^2-x+2\} \\
&= \int\{x^2-x+2\}dx
\end{aligned}
$$

This gives

$$y_p = \frac{x^3}{3} - \frac{x^2}{2} + 2x$$

It is true that we could have added an arbitrary constant of integration to the above expression. We have chosen to add zero. Remember we are only asked for **a** particular integral and so our answer is just as good as $\frac{x^3}{3} - \frac{x^2}{2} + 2x + 3$, say. How can two particular integrals differ like this? The answer lies in the complementary function. Here the auxiliary equation is $m^2 + m = 0$, so $m = 0$ or -1 and hence

$$y_c = Ae^{0x} + Be^{-x} = A + Be^{-x}$$

The general solution to the differential equation is $y = y_c + y_p$, i.e.

$$y = A + Be^{-x} + \frac{x^3}{3} - \frac{x^2}{2} + 2x$$

Since any particular solution to the differential equation is necessarily of this form, it follows that by choosing $A = 3$ and $B = 0$ we should have a particular integral $\frac{x^3}{3} - \frac{x^2}{2} + 2x + 3$. In general, the difference between any two particular integrals may be obtained from the complementary function by suitable choice of the constants contained within it. It is important to note that $\boxed{\text{two different particular integrals may both be correct!}}$

(The video covers the general principles and all the examples in the above section, including a discussion of the constant in the last example. V

Now try some examples for yourself involving polynomial functions $f(x)$.)

EX

6.2.2 The "Exponential Theorem"

This is appropriate (apart from one particular circumstance) for $f(x) = e^{\alpha x}$, $\sin\beta x$, $\cos\beta x$, $e^{\alpha x}\sin\beta x$, $e^{\alpha x}\cos\beta x$, $\sinh\beta x$, $\cosh\beta x$, $e^{\alpha x}\sinh\beta x$, $e^{\alpha x}\cosh\beta x$, and one or two similar forms.

The "theorem" arises from the following observations:

$$D\{e^{\alpha x}\} = \alpha e^{\alpha x},\ D^2\{e^{\alpha x}\} = \alpha^2 e^{\alpha x},\ D^3\{e^{\alpha x}\} = \alpha^3 e^{\alpha x}, \ldots$$

In general $D^n\{e^{\alpha x}\} = \alpha^n e^{\alpha x}$. This suggests that to evaluate $\frac{1}{P(D)}\{e^{\alpha x}\}$ we might simply change all the D's to α's. Thus

$$\boxed{\frac{1}{P(D)}\{e^{\alpha x}\} = \frac{1}{P(\alpha)}e^{\alpha x}}$$

Clearly the method require $P(\alpha) \neq 0$. (The "one particular circumstance" referred to above is the case $P(\alpha) = 0$.) The method is easy to apply.

Example 6.11 Find a particular integral for the differential equation

$$(D^2 + D + 1)y = e^{2x}$$

Solution

$$y_p = \frac{1}{D^2 + D + 1}\{e^{2x}\} = \frac{1}{2^2 + 2 + 1}.e^{2x} = \frac{e^{2x}}{7}$$

As a check we can compute

$$(D^2 + D + 1)\{\frac{e^{2x}}{7}\} = \frac{1}{7}(D^2 + D + 1)\{e^{2x}\} = \frac{1}{7}(2^2 + 2 + 1)e^{2x} = e^{2x}$$

as required. Note how one can "see" the method working: the "7" itself arose as $2^2 + 2 + 1$.

For dealing with sin and cos we use the formulae:

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ \cos\theta &= \mathcal{R}e^{i\theta} \\ \sin\theta &= \mathcal{I}e^{i\theta} \end{aligned}$$

where $\mathcal{R}, \mathcal{I}$ denote real and imaginary parts. For dealing with sinh and cosh we reduce to exponential form by using:

$$\begin{aligned} \sinh\theta &= \frac{e^{\theta} - e^{-\theta}}{2} \\ \cosh\theta &= \frac{e^{\theta} + e^{-\theta}}{2} \end{aligned}$$

Example 6.12 Find a particular integral for the differential equation

$$(D^2 + D + 1)y = \sin 2x$$

Solution

$$\begin{aligned}
y_p &= \frac{1}{D^2 + D + 1}\{\sin 2x\} \\
&= \frac{1}{D^2 + D + 1}\{\mathcal{I}e^{2ix}\} \\
&= \mathcal{I}\frac{1}{D^2 + D + 1}\{e^{2ix}\} \\
&= \mathcal{I}\frac{1}{(2i)^2 + 2i + 1}e^{2ix} \\
&= \mathcal{I}\frac{1}{-3 + 2i}e^{2ix} \\
&= \mathcal{I}\frac{-3 - 2i}{(-3)^2 + 2^2}e^{2ix} \\
&= \mathcal{I}\frac{(-3 - 2i)(\cos 2x + i\sin 2x)}{9 + 4} \\
&= \frac{(-3\sin 2x - 2\cos 2x)}{13}
\end{aligned}$$

[A particular integral for $(D^2+D+1)y = \cos 2x$ would be obtained by taking the real part at the final stage.] ○

Example 6.13 Find a particular integral for the differential equation

$$(D^2 + D + 1)y = \sinh 3x$$

Solution

$$\begin{aligned}
y_p &= \frac{1}{D^2 + D + 1}\{\sinh 3x\} \\
&= \frac{1}{D^2 + D + 1}\{\frac{e^{3x} - e^{-3x}}{2}\} \\
&= \frac{1}{D^2 + D + 1}\{\frac{e^{3x}}{2}\} - \frac{1}{D^2 + D + 1}\{\frac{e^{-3x}}{2}\} \\
&= \frac{1}{2}.\frac{1}{D^2 + D + 1}\{e^{3x}\} - \frac{1}{2}.\frac{1}{D^2 + D + 1}\{e^{-3x}\} \\
&= \frac{1}{2}.\frac{1}{9 + 3 + 1}.e^{3x} - \frac{1}{2}.\frac{1}{9 - 3 + 1}.e^{-3x}
\end{aligned}$$

Thus we obtain

$$y_p = \frac{e^{3x}}{26} - \frac{e^{-3x}}{14}$$ ○

Example 6.14 Find a particular integral for the differential equation

$$(D^2 + D + 1)y = e^{-2x}\sin 3x$$

Solution

$$\begin{aligned}
y_p &= \frac{1}{D^2+D+1}\{e^{-2x}\sin 3x\} \\
&= \mathcal{I}\frac{1}{D^2+D+1}\{e^{-2x}e^{3ix}\} \\
&= \mathcal{I}\frac{1}{D^2+D+1}\{e^{(-2+3i)x}\} \\
&= \mathcal{I}\frac{1}{(-2+3i)^2+(-2+3i)+1}.e^{(-2+3i)x} \\
&= \mathcal{I}\frac{1}{-6-9i}e^{-2x}e^{3ix} \\
&= \mathcal{I}\frac{-6+9i}{36+81}e^{-2x}e^{3ix} \\
&= \frac{e^{-2x}}{117}\mathcal{I}(-6+9i)(\cos 3x + i\sin 3x) \\
&= \frac{e^{-2x}}{117}(-6\sin 3x + 9\cos 3x) \\
&= \frac{e^{-2x}(-2\sin 3x + 3\cos 3x)}{39}
\end{aligned}$$

[A particular integral for $(D^2 + D + 1)y = e^{-2x}\cos 3x$ would be obtained by taking the real part at the final stage]. ○

The method described above and applied to

$$P(D)y = e^{\alpha x}$$

will fail if $P(\alpha) = 0$. This can indeed happen as is shown by $(D - 1)y = e^x$, which might suggest

$$y_p = \frac{1}{D-1}\{e^x\} = \frac{1}{1-1}.e^x \qquad \text{whoops!}$$

The following method is applicable in such failing cases and it can also deal with various other functions $f(x)$.

(The video covers the general principles and all the examples in the above section, including the failing case immediately above. V

Now try some examples for yourself involving functions $f(x)$ which are exponentials, sines, cosines, or combinations of these functions. If you encounter an example where $P(\alpha) = 0$ then leave it until after you've covered the next section.) EX

6.2.3 The "Shift Theorem"

This is appropriate for dealing with the failing case of the exponential theorem and for $f(x) = g(x)e^{\alpha x}$, $g(x)\sin\beta x$, $g(x)\cos\beta x$, $g(x)\sinh\beta x$, $g(x)\cosh\beta x$, (and similar forms), where $g(x)$ is a polynomial in x.

The "theorem" arises from the following observations:

$$\begin{aligned} D\{g(x)e^{\alpha x}\} &= g(x)\alpha e^{\alpha x} + g'(x)e^{\alpha x} \\ &= e^{\alpha x}(\alpha g(x) + D\{g(x)\}) \\ &= e^{\alpha x}(D+\alpha)\{g(x)\} \end{aligned}$$

Taking the first line above and differentiating again gives

$$\begin{aligned} D^2\{g(x)e^{\alpha x}\} &= (g(x)\alpha^2 e^{\alpha x} + g'(x)\alpha e^{\alpha x}) + (g'(x)\alpha e^{\alpha x} + g''(x)e^{\alpha x}) \\ &= e^{\alpha x}(\alpha^2 + 2\alpha D + D^2)\{g(x)\} \\ &= e^{\alpha x}(D+\alpha)^2\{g(x)\} \end{aligned}$$

Similarly,

$$D^3\{g(x)e^{\alpha x}\} = e^{\alpha x}(D+\alpha)^3\{g(x)\}$$

In general:

$$D^n\{g(x)e^{\alpha x}\} = e^{\alpha x}(D+\alpha)^n\{g(x)\}$$

This suggests that to evaluate $\frac{1}{P(D)}\{g(x)e^{\alpha x}\}$ we might bring $e^{\alpha x}$ to the front (outside the scope of the operator), change all the D's to $(D+\alpha)$'s and operate on what is left (namely $g(x)$). Thus

$$\boxed{\frac{1}{P(D)}\{g(x)e^{\alpha x}\} = e^{\alpha x}\frac{1}{P(D+\alpha)}\{g(x)\}}$$

Note that the right hand side has still to be evaluated by one of the earlier methods (usually the binomial method).

Example 6.15 Find a particular integral for the differential equation

$$(D^2 + D + 1)y = xe^{2x}$$

Solution

$$\begin{aligned}
y_p &= \frac{1}{D^2 + D + 1}\{xe^{2x}\} \\
&= e^{2x}\frac{1}{(D+2)^2 + (D+2) + 1}\{x\} \qquad \text{(shift theorem)} \\
&= e^{2x}\frac{1}{D^2 + 5D + 7}\{x\} \\
&= e^{2x}.\frac{1}{7}.\frac{1}{1 + [\frac{5D+D^2}{7}]}\{x\} \\
&= \frac{e^{2x}}{7}\left(1 - \left[\frac{5D + D^2}{7}\right]\cdots\right)\{x\} \qquad \text{(binomial expansion)}
\end{aligned}$$

This gives

$$y_p = \frac{e^{2x}}{7}(x - \frac{5}{7})$$

○

Example 6.16 (When the exponential theorem fails) Find a particular integral for the differential equation

$$(D^2 - 3D + 2)y = e^{2x}$$

Solution (Note: $2^2 - 3 \times 2 + 2 = 0$ so the exponential theorem fails.) We introduce a "1" in order to use the shift theorem: we write e^{2x} as $1.e^{2x}$. Thus

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 3D + 2}\{e^{2x}\} \\
&= \frac{1}{D^2 - 3D + 2}\{1.e^{2x}\} \\
&= e^{2x}\frac{1}{(D+2)^2 - 3(D+2) + 2}\{1\} \\
&= e^{2x}\frac{1}{D^2 + D}\{1\}
\end{aligned}$$

(The absence of a constant in the denominator above is typical of the application of the shift theorem to the failing case of the exponential theorem. Can you see why?)

So we may write

$$\begin{aligned} y_p &= e^{2x}\frac{1}{D}.\frac{1}{1+D}\{1\} \\ &= e^{2x}\frac{1}{D}(1-D+\cdots)\{1\} \\ &= e^{2x}\frac{1}{D}\{1\} = e^{2x}.x \end{aligned}$$

Hence $y_p = xe^{2x}$. Note that by introducing a constant of integration at the last stage, we can obtain different valid particular integrals of the form $y_p = (x+c)e^{2x}$ for any constant c. ○

Example 6.17 Find a particular integral for the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (x+1)\sin 2x$$

Solution

$$\begin{aligned} y_p &= \frac{1}{D^2+D+1}\{(x+1)\sin 2x\} \\ &= \frac{1}{D^2+D+1}\{\mathcal{I}(x+1)e^{2ix}\} \\ &= \mathcal{I}\frac{1}{D^2+D+1}\{(x+1)e^{2ix}\} \\ &= \mathcal{I}e^{2ix}\frac{1}{(D+2i)^2+(D+2i)+1}\{x+1\} \\ &= \mathcal{I}e^{2ix}\frac{1}{D^2+(4i+1)D+(2i-3)}\{x+1\} \\ &= \mathcal{I}\frac{e^{2ix}}{2i-3}.\frac{1}{1+[\frac{D^2+(4i+1)D}{(2i-3)}]}\{x+1\} \\ &= \mathcal{I}\frac{e^{2ix}}{2i-3}\left(1-\left[\frac{D^2+(4i+1)D}{(2i-3)}\right]+\cdots\right)\{x+1\} \\ &= \mathcal{I}\frac{e^{2ix}}{2i-3}\left(1-\left[\frac{4i+1}{2i-3}\right]D+\cdots\right)\{x+1\} \\ & \quad \text{(ignoring } D^2, D^3, \text{ etc.)} \end{aligned}$$

This gives

$$\begin{aligned}
y_p &= \mathcal{I}\frac{e^{2ix}}{2i-3}\left(x+1-\frac{4i+1}{2i-3}\right)\\
&= \mathcal{I}\frac{(\cos 2x + i\sin 2x)(-2i-3)}{4+9}\left(x+1-\frac{(4i+1)(-2i-3)}{4+9}\right)\\
&= \mathcal{I}\frac{([-3\cos 2x + 2\sin 2x] + i[-3\sin 2x - 2\cos 2x])}{13}.\left(x+1-\frac{(5-14i)}{13}\right)\\
&= \mathcal{I}\frac{([-3\cos 2x + 2\sin 2x] + i[-3\sin 2x - 2\cos 2x])([13x+8]+14i)}{13^2}\\
&= \frac{([-3\cos 2x + 2\sin 2x].14 + [-3\sin 2x - 2\cos 2x][13x+8])}{169}
\end{aligned}$$

Finally this gives

$$y_p = \frac{([-58-26x]\cos 2x + [4-39x]\sin 2x)}{169} \qquad \bigcirc$$

(The video covers the general principles and the first two examples in the above section, including a discussion of the failing case of the exponential theorem. V

Now try some examples for yourself involving functions $f(x)$ which are products of polynomials with exponential or sine/cosine functions. Try also some examples where the exponential theorem fails.) EX

7 The general solution, boundary and initial conditions

In this section we try to draw together some loose ends. In section 5 we saw how to determine complementary functions. Section 6 gave two distinct approaches to finding particular integrals (you only need to know one of these: either the method of trial functions or the D-operator method).

For any particular problem we determine the complementary function y_c and a particular integral y_p. The general solution is then given by

$$y = y_c + y_p$$

This solution will contain arbitrary constants. If additional information is provided then it may be possible to determine some or all of these constants. For a second order equation with two constants in the general solution, two independent pieces of information (of a suitable form) will suffice to determine the constants. The two most common cases are

- We require the solution $y(x)$ to satisfy conditions of the form

  $$y = y_1 \text{ at } x = x_1 \textbf{ and } y = y_2 \text{ at } x = x_2$$

  where x_1, x_2, y_1, y_2 are all known numerical values. Conditions like these are called **boundary conditions** (x_1 and x_2 are regarded as the boundaries).

- We require the solution $y(x)$ to satisfy conditions of the form

  $$y = y_0 \textbf{ and } \frac{dy}{dx} = y_0' \text{ at } x = x_0$$

  where x_0, y_0, y_0' are all known numerical values. Conditions like these are called **initial conditions** (x_0 is regarded as the initial value of x).

The examples below illustrate obtaining the general solution and determining the solution which satisfies given boundary or initial conditions. In each case we give two methods for determining a particular integral but you only need to follow one of these.

Example 7.1 Find the general solution of the differential equation

$$(D^2 - 4D + 3)y = x^2 + 2x - 3$$

Hence find the solution which satisfies the boundary conditions

$$y(0) = \frac{23}{27} \qquad y(1) = 0$$

Solution The auxiliary equation is $m^2 - 4m + 3 = 0$, i.e. $(m-3)(m-1) = 0$. Therefore $m = 1$ or 3. Hence the complementary function y_c is

$$y_c = Ae^x + Be^{3x}$$

For a particular integral use a) or b) below:

a) Try $y_p = ax^2 + bx + c$ as a trial solution. Then

$$\begin{aligned}(D^2 - 4D + 3)y_p &= 2a - 4(2ax + b) + 3(ax^2 + bx + c)\\ &= 3ax^2 + (3b - 8a)x + (2a - 4b + 3c)\end{aligned}$$

This gives $x^2 + 2x - 3$ provided

$$3a = 1, \qquad 3b - 8a = 2, \qquad 2a - 4b + 3c = -3$$

These give

$$a = \frac{1}{3} \qquad b = \frac{14}{9} \qquad c = \frac{23}{27}$$

Hence

$$y_p = \frac{x^2}{3} + \frac{14x}{9} + \frac{23}{27}$$

b) A particular integral y_p is given by

$$\begin{aligned}
y_p &= \frac{1}{D^2-4D+3}\{x^2+2x-3\} \\
&= \frac{1}{3}.\frac{1}{1+[\frac{D^2-4D}{3}]}\{x^2+2x-3\} \\
&= \frac{1}{3}\left(1-\left[\frac{D^2-4D}{3}\right]+\left[\frac{D^2-4D}{3}\right]^2+\cdots\right)\{x^2+2x-3\} \\
&= \frac{1}{3}\left(1-\frac{D^2}{3}+\frac{4D}{3}+\frac{16D^2}{9}+\cdots\right)\{x^2+2x-3\} \\
&\quad \text{(ignoring } D^3, D^4, \text{ etc.)} \\
&= \frac{1}{3}\left(1+\frac{4D}{3}+\frac{13D^2}{9}+\cdots\right)\{x^2+2x-3\} \\
&= \frac{1}{3}\left([x^2+2x-3]+\frac{4}{3}[2x+2]+\frac{13}{9}.2\right) \\
&= \frac{x^2}{3}+\frac{14x}{9}+\frac{23}{27}
\end{aligned}$$

Whichever method is used for y_p we obtain the general solution

$$\begin{aligned}
y &= y_c + y_p \\
\text{i.e.} \quad y &= Ae^x + Be^{3x} + \frac{x^2}{3} + \frac{14x}{9} + \frac{23}{27}
\end{aligned}$$

We are told $y(0) = \frac{23}{27}$, i.e. when $x = 0$ the value of y is $\frac{23}{27}$. Putting these values into the general solution we get

$$\frac{23}{27} = A + B + \frac{23}{27}$$

and so $A + B = 0$, or $B = -A$.
We are also told $y(1) = 0$ so putting $x = 1,\ y = 0$ into the general solution we get

$$\begin{aligned}
0 &= Ae + Be^3 + \frac{1}{3} + \frac{14}{9} + \frac{23}{27} \\
\text{i.e.} \quad 0 &= Ae + Be^3 + \frac{74}{27}
\end{aligned}$$

Since $B = -A$ we obtain

$$A(e^3 - e) = \frac{74}{27}$$

giving

$$A = \frac{74}{27(e^3 - e)} \qquad B = \frac{-74}{27(e^3 - e)}$$

Returning these values to the general solution we find

$$y = \frac{74}{27(e^3 - e)}[e^x - e^{3x}] + \frac{x^2}{3} + \frac{14x}{9} + \frac{23}{27}$$

Example 7.2 Find the general solution of the differential equation

$$(D^2 - 4D + 4)y = e^{2x}\cos x$$

Hence find the solution which satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$.

Solution The auxiliary equation is $m^2 - 4m + 4 = 0$, i.e. $(m - 2)^2 = 0$. Therefore $m = 2$ (twice). Hence the complementary function y_c is

$$y_c = (Ax + B)e^{2x}$$

For a particular integral use a) or b) below:

a) Try $y_p = e^{2x}(a\cos x + b\sin x)$ as a trial solution. Then

$$\begin{aligned} y_p' &= e^{2x}(-a\sin x + b\cos x) + 2e^{2x}(a\cos x + b\sin x) \\ &= e^{2x}([b + 2a]\cos x + [2b - a]\sin x) \\ \text{and} \qquad y_p'' &= e^{2x}([(2b - a) + 2(b + 2a)]\cos x \\ &\qquad + [2(2b - a) - (b + 2a)]\sin x) \\ &= e^{2x}([4b + 3a]\cos x + [3b - 4a]\sin x) \end{aligned}$$

Hence

$$\begin{aligned} (D^2 - 4D + 4)y_p &= e^{2x}([4b + 3a - 4(b + 2a) + 4a]\cos x \\ &\qquad + [3b - 4a - 4(2b - a) + 4b]\sin x) \\ &= e^{2x}(-a\cos x - b\sin x) \end{aligned}$$

This gives $e^{2x}\cos x$ provided $a = -1,\ b = 0$. Hence $y_p = -e^{2x}\cos x$.

b) A particular integral y_p is given by

$$\begin{aligned}
y_p &= \frac{1}{D^2-4D+4}\{e^{2x}\cos x\} \\
&= \mathcal{R}\frac{1}{D^2-4D+4}\{e^{2x}e^{ix}\} \\
&= \mathcal{R}\frac{1}{D^2-4D+4}\{e^{(2+i)x}\} \\
&= \mathcal{R}\frac{1}{(2+i)^2-4(2+i)+4}.e^{(2+i)x} \\
&= \mathcal{R}\frac{1}{-1+0i}.e^{2x}e^{ix} \\
&= -e^{2x}\mathcal{R}\{e^{ix}\} \\
&= -e^{2x}\cos x
\end{aligned}$$

Whichever method is used for y_p we obtain the general solution

$$\begin{aligned}
y &= y_c + y_p \\
&= (Ax+B)e^{2x} - e^{2x}\cos x
\end{aligned}$$

We are told $y(0) = 1$, i.e. when $x = 0$ the value of y is 1. Putting these values into the general solution we get

$$1 = B - 1$$

Therefore $B = 2$.
We are also told $y'(0) = 0$. To use this fact we have to obtain an expression for $y'(x)$, which we do by differentiating the general solution. Here this gives

$$\frac{dy}{dx} = 2(Ax+B)e^{2x} + Ae^{2x} + e^{2x}\sin x - 2e^{2x}\cos x$$

Substituting $y'(0) = 0$ gives

$$0 = 2B + A - 2$$

Replacing B by 2 gives $A = -2$. Returning the values of A and B to the general solution gives

$$y = (2 - 2x)e^{2x} - e^{2x}\cos x = e^{2x}(2 - 2x - \cos x) \qquad \bigcirc$$

(The video covers the example $(D^2 + D + 1)y = e^x$, given the initial conditions $y(0) = 3,\ y'(0) = 0$. V

You should now try some examples involving initial conditions and boundary conditions.) EX

8 Applications

(This section is not covered in the videos.)

In this section we shall look in some detail at the application of the foregoing theory to mechanical vibrations. In particular we consider the mathematical modelling of a mass moving on the end of a spring. The equations and their solutions are also relevant to analogous mechanical systems both small and large (such as clock pendulums and bridges). We shall mention these briefly at appropriate points. We conclude the section by considering an electrical circuit also subject to the same type of equation; such circuits have an important role in radio transmission and reception.

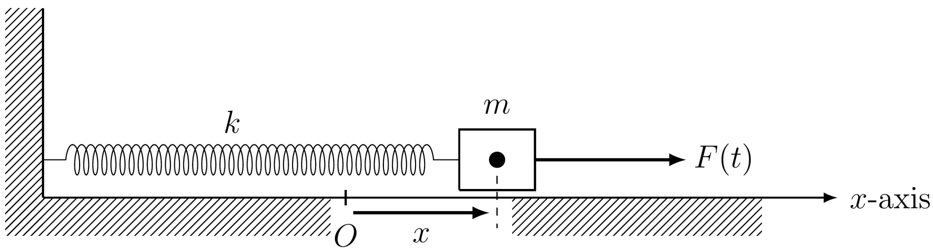

Figure 1: Mass on a spring.

Consider a mass m attached to the end of a spring as shown in the diagram. (Figure 1.) The origin O corresponds to the position of the centre of the mass when the spring is in its unstretched position. When the spring is stretched (or compressed) we let x denote the displacement of the centre of mass. The quantity x will vary with the time t, i.e. $x = x(t)$.

The mass m may be subject to the action of the following forces:

1. A restoring force: $-kx$
   Here k is a positive constant related to the stiffness of the spring; the force is proportional to the extension but in the opposite direction to it (hence the minus sign),

2. A resistance: $-c\frac{dx}{dt}$
   Here c is a positive constant related to the viscosity of the medium through which the mass travels; the force is proportional to the velocity of the mass but in the opposite direction,

3. An external applied force: $F(t)$
   This force may vary with time.

The equation of motion of the mass is obtained from Newton's Law:

$$\text{mass} \ \times \ \text{acceleration} = \text{force}$$

This gives

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t)$$

The term on the right-hand side is simply the sum of all the forces acting on the mass. The equation can be written as

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

This equation is called the **oscillator equation**. If $c = 0$, the oscillator is said to be undamped; in physical terms this means that energy will not be dissipated by resistance.

The **homogeneous** oscillator equation is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

8.1 The homogeneous oscillator equation

We shall start our analysis by solving the homogeneous equation, i.e. we shall find the complementary function. The auxiliary equation is

$$m\lambda^2 + c\lambda + k = 0$$

The solutions of this are

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

The nature of the complementary function therefore depends on whether $c^2 < 4km$, $c^2 > 4km$, or $c^2 = 4km$. We examine each of these three cases.

a) $c^2 < 4km$ **(Weak damping)**
We can write

$$\begin{aligned}\lambda &= \frac{-c \pm i\sqrt{4km - c^2}}{2m} \\ &= -\frac{c}{2m} \pm i\frac{\sqrt{4km - c^2}}{2m}\end{aligned}$$

Hence

$$x(t) = e^{-\frac{ct}{2m}}\left[A\cos\left(\frac{\sqrt{4km - c^2}}{2m}t\right) + B\sin\left(\frac{\sqrt{4km - c^2}}{2m}t\right)\right]$$

It is worth remarking here that an expression such as

$$A\cos\theta + B\sin\theta$$

can always be re-written as

$$R\sin(\theta + \alpha)$$

for suitable R and α; R is called the amplitude and α the phase angle. In fact if we expand

$$R\sin(\theta + \alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

we see that this yields $A\cos\theta + B\sin\theta$ provided $R,\ \alpha$ are chosen to satisfy

$$R\sin\alpha = A, \qquad R\cos\alpha = B$$

Squaring and adding (noting $\sin^2\alpha + \cos^2\alpha = 1$) gives $R^2 = A^2 + B^2$, so we can take $R = \sqrt{A^2 + B^2}$. The value of α can then be selected as necessary to satisfy

$$\sin\alpha = \frac{A}{\sqrt{A^2 + B^2}}, \qquad \cos\alpha = \frac{B}{\sqrt{A^2 + B^2}}$$

(It is tempting to divide these two equations and take

$$\alpha = \arctan(\frac{A}{B})$$

but α might have to differ from this angle by π radians.)

Bearing the above in mind we can write x as

$$x(t) = e^{-\frac{ct}{2m}}R\sin(pt + \alpha)$$

where $R,\ \alpha$ are constants and $p = (\sqrt{4km - c^2})/2m$. If $c \neq 0$ this represents a damped sinusoidal wave as shown below (figure 2); the factor $e^{-\frac{ct}{2m}}$ superimposes an exponential decay on top of a sine wave of amplitude R.

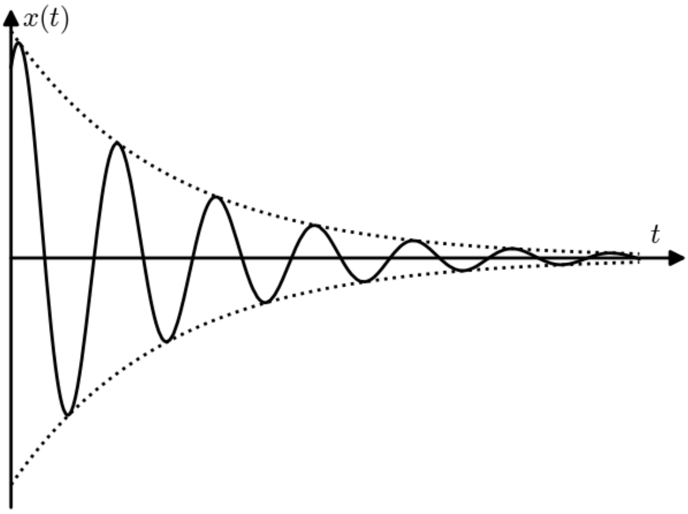

Figure 2: Damped sinusoidal wave.

If $c = 0$ the exponential decay factor is absent and the graph of the solution is simply a sinusoidal wave $x(t) = R\sin(pt + \alpha)$ as shown below (figure 3).

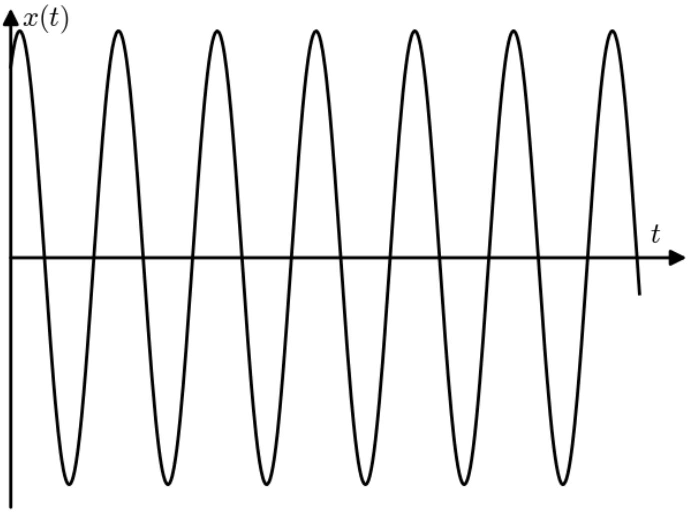

Figure 3: Sinusoidal wave.

b) $c^2 > 4km$ **(Strong damping)**
Here we write

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \lambda_1,\ \lambda_2,\ \text{say (both real).}$$

If $\lambda_1 = \frac{-c-\sqrt{c^2-4km}}{2m}$ then, clearly, $\lambda_1 < 0$. But $\lambda_2 = \frac{-c+\sqrt{c^2-4km}}{2m}$ is also negative because $c^2 > c^2 - 4km$. We then have

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

with both $\lambda_1,\ \lambda_2 < 0$. This represents exponential decay (figure 4).

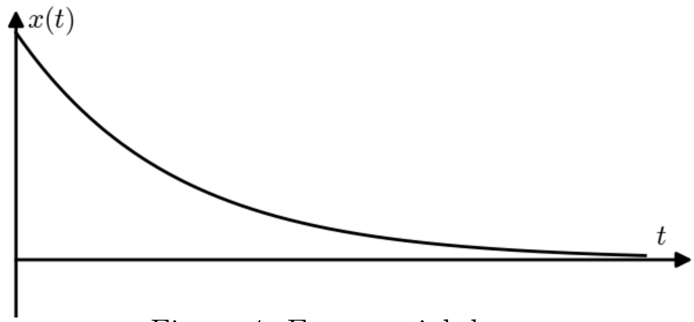

Figure 4: Exponential decay.

This is the sort of result one would expect in a very viscous medium - imagine pulling the end of the spring in treacle!

c) $c^2 = 4km$ **(Critical damping)**
In this case

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} \qquad \text{(twice).}$$

Then we have

$$x(t) = e^{-\frac{ct}{2m}}(At + B)$$

We have the possibility of initial growth (due to the At term) overlaid with exponential decay (figure 5).

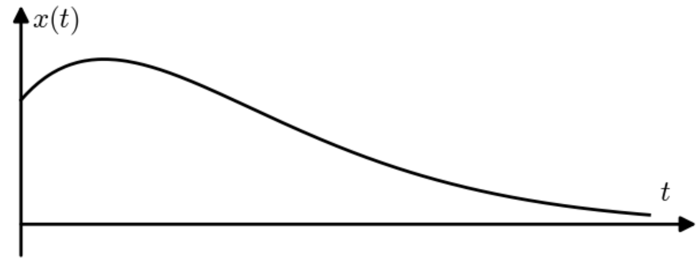

Figure 5: Critical damping.

Note that in all of the three cases above, provided $c > 0$, the solution $x(t)$ tends to zero as t tends to infinity. For this reason the complementary function is often called the **transient** term. It is only when $c = 0$ (the undamped case) that the solution persists with time. This latter case is known as **simple harmonic motion** (SHM). Simple harmonic motion provides a good approximation for the motion of a clock pendulum with $x(t)$ representing the angular displacement at time t. Sometimes the case when $c > 0$ is called **damped harmonic motion**.

8.2 The inhomogeneous equation

The inhomogeneous term $F(t)$ is often called the **forcing term**. We shall only consider the case when the forcing term is periodic and has the form

$$F(t) = a \sin \omega t$$

where a, ω are constants. This is a sine wave of amplitude a and period $\frac{2\pi}{\omega}$.

We start by obtaining a particular integral for our equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = a \sin \omega t$$

Using either the method of trial functions (try $x_p = \alpha \sin \omega t + \beta \cos \omega t$) or the D-operator method, we obtain

$$x_p = \frac{a(k - m\omega^2)}{(k - m\omega^2)^2 + c^2\omega^2} \sin \omega t - \frac{ac\omega}{(k - m\omega^2)^2 + c^2\omega^2} \cos \omega t$$

(You might try this as an exercise!)

We can combine the sine and cosine terms (as we did earlier in the weak damping case of the complementary function) to convert this to the form

$$x_p = \frac{a}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \sin(\omega t + \phi)$$

where ϕ is the phase angle. Note that if $c = 0$ (undamped case) **and** $k = m\omega^2$ then there is something awfully wrong with this expression because the denominator vanishes- we'll look at this case later. Apart from this problem case, we have

$$x_p = A(\omega) \sin(\omega t + \phi)$$

where

$$A(\omega) = \frac{a}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}$$

Thus the particular integral does not tend to zero as t tends to infinity. This part of the solution is called the **steady state term**. It is a sinusoidal wave with amplitude $A(\omega)$; we have written A as $A(\omega)$ to emphasise that the amplitude depends on ω. Differentiation of this expression with respect to ω will show that when $c^2 < 2km$, then $A(\omega)$ has a (local) maximum at

$$\omega = \omega_r = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}$$

This value of ω is called the **resonant frequency**. [If you want to check the maximum, it is easier to minimise $(k - m\omega^2)^2 + c^2\omega^2$.] The corresponding maximum amplitude is

$$A_{max} = \frac{a}{c\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}}$$

We show below (figure 6) the graph of $A(\omega)$ against ω for various values of c. These graphs have a (local) maximum at ω_r when $c^2 < 2km$.

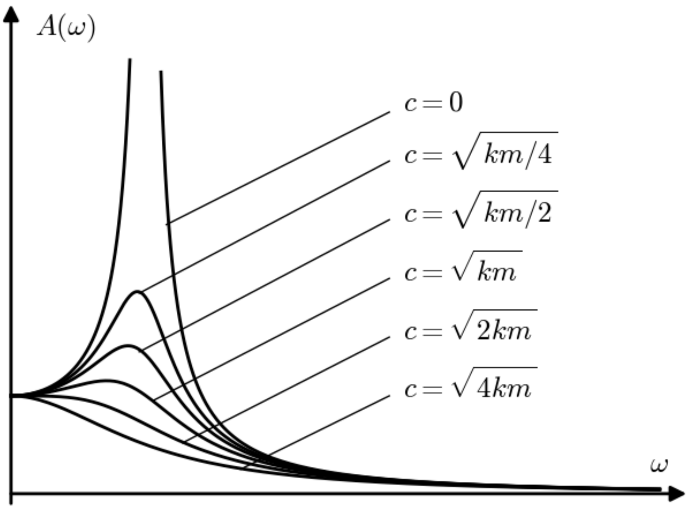

Figure 6: Amplitude against frequency.

(The case of $c = 0$ is covered below).

Selecting the resonant frequency for the forcing term results in the largest possible amplitude in the steady state solution. Physically this is seen in a variety of situations. A good example is a swing: moving your body in time with the natural movement of the swing results in a large amplitude swinging session - it is unproductive to move your body out of time with the natural movement of the swing.

Resonance is a physical phenomenon whose importance is difficult to underestimate. It lies at the root of many engineering disasters. Troops marching over bridges have been known to make them collapse as a result of resonance, wind gusts have had the same effect on suspension bridges. Aircraft engines have fallen off and in some cases the wings as well as a consequence of resonant vibrations. On the lighter side, opera singers have been able to break wine glasses as a party trick simply by emitting a resonant note at high volume close to a glass. To see why the problem can sometimes be of an extreme nature we shall conclude this subsection by briefly examining the case $c = 0$ (undamped systems).

We remarked above that our formula for x_p was clearly erroneous when $c = 0$ **and** $\omega = \sqrt{\frac{k}{m}}$. The difficulty arises because when $c = 0$ the complementary function has the form

$$x_c = R\sin(pt + \alpha)$$

where

$$p = \frac{\sqrt{4km - c^2}}{2m} = \sqrt{\frac{k}{m}}$$

Thus, when $c = 0$, the complementary function is not transient, and expanding $\sin(pt + \alpha)$ we can write it in the form

$$x_c = A\cos(\sqrt{\frac{k}{m}}t) + B\sin(\sqrt{\frac{k}{m}}t)$$

If the forcing term $F(t) = a\sin\omega t$ is now chosen with $\omega = \sqrt{\frac{k}{m}}$ then the particular integral takes a different form from the one quoted earlier. Using either the trial function method (try $x_p = t(\alpha\sin\omega t + \beta\cos\omega t)$) or the D-operator method we obtain

$$x_p = -\frac{at}{2\omega}\cos\omega t = -\frac{at}{2\sqrt{\frac{k}{m}}}\cos(\sqrt{\frac{k}{m}}t)$$

(Again, you might check this as an exercise).

Putting together the complementary function and particular integral we obtain the general solution

$$x = A\cos(\sqrt{\frac{k}{m}}t) + B\sin(\sqrt{\frac{k}{m}}t) - \frac{at}{2\sqrt{\frac{k}{m}}}\cos(\sqrt{\frac{k}{m}}t)$$

The factor t in the final term means that the amplitude of the oscillations will increase linearly with time. Of course, some limit is normally reached as a result of physical constraints (i.e. something breaks!). If you apply a force to an undamped oscillating system exactly in time with the natural oscillations (the frequency of the complementary function) then the resulting oscillations will grow larger and larger - you are set for a disaster! In practice c is never zero but it can sometimes be small enough for the amplitude of the forced oscillations to build up to that critical point where the structure gives way.

8.3 Electrical analogy

We examine briefly the so-called LRC circuit containing a power source, an inductance (L), resistance (R) and capacitance (C).

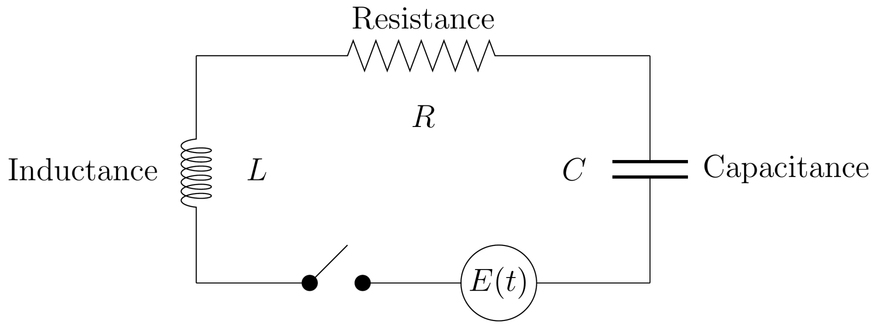

Figure 7: LRC circuit.

Here L, R, C are positive constants and $E(t)$ is the applied electromotive force (voltage).

We let i denote the current flowing through the circuit at time t after the switch is closed. The potential (voltage) drop across an inductance L is $L\frac{di}{dt}$. That across the resistance R is Ri. That across the capacitance C is $\frac{Q}{C}$ where Q is the charge on the capacitance. The total applied potential is $E(t)$ and so

$$L\frac{di}{dt} + Ri + \frac{Q}{C} = E(t)$$

However, the current i is, by definition, the rate of change of charge. That is,

$$i = \frac{dQ}{dt}$$

It follows that

$$\frac{di}{dt} = \frac{d^2Q}{dt^2}$$

Thus the above equation can be written as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Note that this has the same form as the equation governing our earlier example of the spring. Here L replaces the mass m, R replaces the damping constant c and $\frac{1}{C}$ replaces the spring constant k. The solutions to this equation therefore exhibit all the features that we have seen previously. In particular LRC circuits exhibit resonance. The amplitude of the charge Q

can be made large by "tuning" the values of L, R and C to the frequency ω of an applied voltage $E(t) = a \sin \omega t$. In radio circuits this phenomenon is employed for tuning purposes.

9 Miscellaneous

9.1 Simultaneous equations

(This section is not covered in the videos.)

The techniques developed may be applied to solve certain simultaneous differential equations. In particular, we can solve a pair of first order linear equations of the form

$$\begin{aligned} \frac{dx}{dt} &= Ax + By + C(t) \\ \frac{dy}{dt} &= Ex + Fy + G(t) \end{aligned}$$

where $A,\ B,\ E,\ F$ are constants and $C(t),\ G(t)$ are (simple) functions of t. We use the first of the equations to obtain an expression for y in terms of x and t; this is then substituted into the second and a second order differential equation results.

Example 9.1 Solve the differential equations

$$\begin{aligned} \frac{dx}{dt} &= x + y + t \\ \frac{dy}{dt} &= 3x - y \end{aligned}$$

Solution From the first equation

$$y = \frac{dx}{dt} - x - t$$

and so

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} - \frac{dx}{dt} - 1$$

Putting these two expression into the second equation gives

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 1 = 3x - (\frac{dx}{dt} - x - t)$$

i.e.

$$\frac{d^2x}{dt^2} - 4x = 1 + t$$

The complementary function is

$$x_c = Ae^{-2t} + Be^{2t}$$

A particular integral is

$$x_p = -\frac{1}{4}(1+t)$$

Hence $x = Ae^{-2t} + Be^{2t} - \frac{1}{4}(1+t)$ is the general solution for x. Having found x we can now find y from our first step

$$\begin{aligned} y &= \frac{dx}{dt} - x - t \\ &= -2Ae^{-2t} + 2Be^{2t} - \frac{1}{4} - [Ae^{-2t} + Be^{2t} - \frac{1}{4}(1+t)] - t \end{aligned}$$

and this gives

$$y = -3Ae^{-2t} + Be^{2t} - \frac{3t}{4}$$ ○

Sometimes more complicated equations can also be solved by similar techniques involving higher order equations. For example, the equations

$$\begin{aligned} \frac{d^2x}{dt^2} + 4\frac{dy}{dt} - 2 &= 0 \\ \frac{d^2y}{dt^2} + 4\frac{dx}{dt} - 8 &= 0 \end{aligned}$$

can be solved by obtaining $\frac{dy}{dt}$ from the first equation in terms of $\frac{d^2x}{dt^2}$, differentiating it and inserting the result into the second equation. This gives a third order equation for x. Once this has been solved we can substitute it into the first equation in order to get an expression for $\frac{dy}{dt}$. Then y can be found by integration. The resulting solutions will contain a total of four independent arbitrary constants. A set of initial conditions such as the values of $x,\ y,\ \frac{dx}{dt}$, and $\frac{dy}{dt}$ at some particular value $t = t_0$ will enable the constants to be evaluated.

(Now try solving some simultaneous differential equations.) EX

9.2 Reduction to first order equations

A second order linear constant - coefficient differential equation can always be written in the form

$$(D-\alpha)(D-\beta)y = f(x)$$

for suitable constants α and β. [In fact $\alpha,\ \beta$ are the roots of the auxiliary equation and may be equal or complex]. If we substitute

$$v = (D-\beta)y \tag{8}$$

we can write the equation as

$$(D-\alpha)v = f(x) \tag{9}$$

But this is a first order linear equation for v which can be solved by the integrating factor method. Having obtained v we can then solve the equation (8) for y by recognising that it too is a first order linear equation.

Carrying out this process we see that the integrating factor for equation (9) is $e^{-\alpha x}$ and equation (9) can be expressed as

$$\frac{d}{dx}(e^{-\alpha x}v) = e^{-\alpha x}f(x)$$

giving

$$e^{-\alpha x}v = \int e^{-\alpha x}f(x)dx + A$$

where A is a constant. Therefore

$$v = e^{\alpha x}\left(\int e^{-\alpha x}f(x)dx + A\right)$$

Equation (8) can then be written as

$$(D-\beta)y = e^{\alpha x}\left(\int e^{-\alpha x}f(x)dx + A\right)$$

This has integrating factor $e^{-\beta x}$ and can therefore be expressed as

$$\frac{d}{dx}(e^{-\beta x}y) = e^{(\alpha-\beta)x}\left(\int e^{-\alpha x}f(x)dx + A\right)$$

giving

$$e^{-\beta x}y = \int e^{(\alpha-\beta)x}\left(\int e^{-\alpha x}f(x)dx + A\right)dx + B$$

where B is a constant. Therefore

$$y = e^{\beta x}\left[\int e^{(\alpha-\beta)x}\left(\int e^{-\alpha x} f(x)dx + A\right)dx + B\right]$$

To use this method in practice requires skills of integration. It does show that the second order linear constant coefficient equation will have a general solution for any reasonable function $f(x)$. The same techniques will reduce a general n^{th} order linear constant coefficient differential equation to n simultaneous first order linear equations.

(Now try solving some second order equations by reduction to simultaneous first order ones; stick to ones with "easy" functions $f(x)$!) EX

10 Summary

When you have completed this package you should be able to do the things listed below.

1. recognise a **second order linear constant-coefficient differential equation,**
2. understand what is meant by a **homogeneous** equation,
3. obtain the **complementary function** using the **auxiliary equation,** in each of the three cases: (a) unequal real roots, (b) complex roots, (c) equal real roots,
4. obtain a **particular integral** by one of the two methods described (**trial functions** or **D-operators**),
5. obtain the **general solution** by adding the complementary function and a particular integral,
6. obtain solutions satisfying given **boundary** or **initial** conditions by determining appropriate values for the constants in the general solution,
7. understand the connection with mechanical and electrical vibrational problems, the physical significance of the individual terms, **damping** and **resonance,**
8. solve simple **simultaneous** linear differential equations,
9. understand how to reduce a second order equation to two simultaneous first order equations.

11 Bibliography

For textbooks covering the basic prerequisites for this package (differentiation, integration and first order differential equations) see, for example, one of the following (although there are dozens of other suitable textbooks many of which are in the University library).

Stroud, K. A. Engineering Mathematics (third edition), Macmillan, 1992.

Jeffrey, A. Mathematics for Engineers and Scientists, Van Nostrand, 1989.

Thomas, G. B. and Finney, R. L. Calculus and Analytic Geometry, Addison-Wesley, 1988.

Larson, R. E., Hostetler, R. P. and Edwards, B. H. Calculus, D. C. Heath and Company, 1990.

Gilbert, J. Guide to Mathematical Methods, Macmillan, 1991.

Anton, H. Calculus with Analytic Geometry, Wiley, 1992.

There are many textbooks which cover second order differential equations. All of the books listed above give an elementary treatment. The books by Stroud and by Jeffrey discuss the D-operator method for particular integrals. Jeffrey also covers the method of undetermined coefficients (trial functions) and the remaining books listed above also adopt this approach. Those listed below are devoted exclusively to the subject of differential equations and they give a great deal more detail. However, as is common with U. S. texts, none of them deal with the D-operator method.

Sanchez, D. A., Allen, R. C. and Kyner, W. T. Differential Equations (second edition), Addison-Wesley, 1988.

Zill, D. G. A first course in Differential Equations with Applications (fourth edition), Prindle, Weber, Schmidt - Kent Publishing Company, 1989.

Boyce, W. E. and DiPrima, R. C. Elementary Differential Equations and Boundary Value Problems (fifth edition), Wiley, 1992.

Of the three books listed, you would probably find the volume by Boyce et al. to be too advanced for general use. All three volumes cover a much wider range of topics than this package - for example first order differential equations, Laplace Transforms, partial differential equations and Fourier series. In addition to these three books there are very many other textbooks covering second order equations and, again, many of these are in the University library. Your tutor should be able to advise you which textbooks are suitable for your own needs, but you need never be short of an alternative approach or more questions to try!

12 Appendix - Video Summaries

There are three videos associated with the topic of second order differential equations. The presenter is Mike Grannell from the Department of Mathematics at the University of Central Lancashire. We recommend that you read the preamble to these notes which makes some suggestions about how you should approach viewing the videos.

Video title: Second Order Linear Constant-Coefficient Differential Equations (part 1). (45 minutes)

Prerequisite: you will need to know how to solve easy first order linear differential equations by the use of appropriate integrating factors.

Summary

1. General form, notation and terminology: "linear", "constant - coefficient", "homogeneous", "D-operator". The form of the general solution (containing two independent arbitrary constants).

2. Obtaining the general solution. Definitions of the complementary function and a particular integral (y_c and y_p).

3. The complementary function and the auxiliary equation. Obtaining the auxiliary equation and verifying that the roots give exponential solutions of the homogeneous differential equation.

4. Unequal real roots of the auxiliary equation. The example $(D^2 - 5D + 6)y = 0$.

5. Complex roots of the auxiliary equation (for equations with real coefficients); general theory. The example $(D^2 - 2D + 5)y = 0$.

6. Equal real roots of the auxiliary equation. What goes wrong: the example $(D^2 + 4D + 4)y = 0$. General theory for $(D - \lambda)^2 y = 0$. Completion of the example $(D^2 + 4D + 4)y = 0$.

Video title: Second Order Linear Constant-Coefficient Differential Equations (part 2). (45 minutes)

Summary

1. Recap general form, introduce notation $P(D)y = f(x)$

2. A particular integral and the inverse operator:

$$y_p = \frac{1}{P(D)}\{f(x)\}$$

3. Dependency on the nature of $f(x)$:
a) Binomial expansion,
b) Exponential theorem,
c) Shift theorem.

4. **Binomial expansion:** Reminder of the binomial expansion of $\frac{1}{1+u}$. Appropriateness of the binomial expansion method when $f(x)$ is a polynomial in x. The examples:

$$\begin{aligned}(D^2 + D + 1)y &= x^2 \\ (D^2 + 5D + 6)y &= x + 1 \\ (D^2 + D)y &= x^2 + x + 1\end{aligned}$$

Discussion of the constant in the latter case.

5. **Exponential theorem:** Appropriateness when $f(x)$ is an exponential, a sine, a cosine or a combination of these forms. Plausibility argument for

$$\frac{1}{P(D)}\{e^{\alpha x}\} = \frac{1}{P(\alpha)}\{e^{\alpha x}\}$$

except when $P(\alpha) = 0$. The examples:

$$\begin{aligned}(D^2 + D + 1)y &= e^{2x} \\ (D^2 + D + 1)y &= \sin 2x \\ (D^2 + D + 1)y &= \sinh 3x \\ (D^2 + D + 1)y &= e^{-2x} \sin 3x\end{aligned}$$

(The latter example is not completed.) An example when the method fails because $P(\alpha) = 0$:

$$(D^2 - 3D + 2)y = e^{2x}$$

Video title: Second Order Linear Constant-Coefficient Differential Equations (part 3). (35 minutes)

Summary

1. Recap the form of the general solution and obtaining particular integrals by the binomial and the exponential methods.

2. **Shift theorem:** Appropriateness when $f(x)$ is a product, particularly a product of a polynomial with an exponential. Plausibility argument for
$$\frac{1}{P(D)}\{g(x)e^{\alpha x}\} = e^{\alpha x}\frac{1}{P(D+\alpha)}\{g(x)\}$$
The examples:
$$\begin{aligned}(D^2+D+1)y &= xe^{2x}\\ (D^2-3D+2)y &= e^{2x}\end{aligned}$$
the latter example being a failing case of the exponential theorem.

3. Summary of the methods for obtaining the general solution which have been dealt with above in these videos. Mention of alternative methods.

4. Obtaining a solution satisfying given initial conditions. The example
$$(D^2+D+1)y = e^x, \quad \text{given } y(0)=3 \text{ and } y'(0)=0.$$