Surface embeddings of Steiner triple systems

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Abstract

A Steiner triple system of order n (STS(n)) is said to be embeddable in an orientable surface if there is an orientable embedding of the complete graph K_n whose faces can be properly 2-coloured (say, black and white) in such a way that all black faces are triangles and these are precisely the blocks of the STS(n). If, in addition, all white faces are triangular, then the collection of all white triangles forms another STS(n); the pair of such STS(n)s is then said to have an (orientable) bi-embedding. We study several questions related to embeddings and bi-embeddings of STSs.

1 Introduction

The study of the relationship between block designs and graph embeddings dates back to Heffter who in 1891 realized the connection between twofold triple systems and surface triangulations. Later work in this field was done by Emch [8], Alpert [1], White [18], Anderson and White [4], Anderson [2], Jungerman, Stahl and White [12], Rahn [15], and most recently White [19]. These authors considered various aspects of the above relationship, including embeddings into closed surfaces, pseudosurfaces and generalized pseudosurfaces, or embeddings of BIBDs with block size greater than 3, or focusing on symmetry properties of the resulting embeddings.

Although we discuss briefly the general idea of hypergraph (hence also block design) embeddings, we are primarily interested in embeddings of Steiner triple systems. The basic observation which comes from [1] is the 1–1 correspondence between twofold triple systems of order n and triangular embeddings of the complete graph K_n into generalised pseudosurfaces. A generalised pseudosurface may be obtained from a finite number of identifications, of finitely many points each (the *singular points*), on a collection of compact surfaces so that the resulting topological space is connected. As shown in [18], the homeomorphism class of the generalised pseudosurface is an invariant of the twofold triple system. But the main reason why surface embeddings of triple systems (and designs in general) have been considered seems to be a kind of "geometrical" 2-dimensional pictorial representation of these objects in a 3-dimensional space. That is why we focus in this paper on embeddings in *orientable surfaces*, and do not allow any kind of degeneracies (such as singular points in pseudosurfaces).

The organization of the paper is as follows. In Section 2 we review basic facts concerning graph and hypergraph embeddings, with emphasis on Steiner triple systems (STSs). Section 3 deals with embeddings and bi-embeddings of STSs, phrased in terms of the corresponding embeddings of complete graphs. Bi-embeddings of STSs are considered in Section 4. In Section 5 we focus our attention on the particular value $n = 19$ and give a complete catalogue of *cyclic* (bi-)embeddings of STS(19)s; we also construct examples of cyclic bi-embedded STS(n)s for other values of $n \equiv 7$ (*mod* 12). Finally, Section 6 contains a list of open problems in this area.

2 Preliminaries

We review some of the basic notions on graph embeddings, which may not be familiar to design theorists. Let $\nu : G \to S$ be an embedding of a graph G in an orientable surface S. Connected components of the set $S \setminus \nu(G)$ are called *regions* or *(open) faces* of ν; the embedding ν is called *2-cell* or *cellular* if all its regions are homeomorphic to an open disc. We do not distinguish between G and its embedded copy $\nu(G)$; no confusion will be likely. Assume that $\nu : G \to S$ is a cellular embedding (which implies that G is connected). Fixing an orientation of S induces, for each vertex v of G, a cyclic permutation P_v of *directed edges* emanating from v (that is, edges endowed with a direction pointing out of v). Since every $e \in E(G)$ gives rise to one pair of oppositely directed edges, every directed edge appears in precisely one cyclic permutation P_v. The product $P = \prod_{v \in V(G)} P_v$, called a *rotation system* in [11], carries the complete information about the cellular embedding $\nu : G \to S$. The face boundaries may be recovered from the rotation system by means of the involutory permutation I that sends every directed edge to its reverse: if P is the rotation system for ν, orbits of the composition PI correspond to face boundaries of the embedding $\nu : G \to S$.

We note that rotation systems can be simplified when working with graphs without loops or multiple edges. In such a case, the cyclic permutation P_v can be replaced by the corresponding cyclic permutation Q_v of *neighbours* of v. The collection $\{Q_v;\ v \in V(G)\}$ is then called a *rotation scheme*. Since we are only interested in embeddings of complete graphs, we shall take advantage of using rotation schemes to describe embeddings.

Now we briefly recall how surface embeddings of hypergraphs can be conveniently defined [12]. Let H be a connected hypergraph with vertex set $V(H)$ and hyperedge set $E(H)$. By [17], there is a 1–1 correspondence between connected hypergraphs and connected bipartite graphs; the bipartite graph $G(H)$ associated with H has vertex set $V(G(H)) = V(H) \cup E(H)$ (which is, at the same time, the bipartition) and edge set $E(G(H)) = \{ve;\ v \in e,\ v \in V(H),\ e \in E(H)\}$. A surface embedding of the hypergraph H is constructed as follows. Take an embedding $\eta : G(H) \to S$ in some surface S. For each vertex $e \in E(H)$ of the bipartite graph $G(H)$, replace e by a small circle (centered at e) on the surface and suppress the part of the drawing η which lies *inside* this small circle. (Such a modification of an embedding is known as *truncation*.) We thus obtain a new embedding η' of a graph G'; note that this new embedding has $|E(H)|$ more faces than

η. As the last step, we contract (to a point on the surface S) each edge incident with some vertex $v \in V(H)$, that is, we contract each edge *not lying* in any of the "small circles". The result is a *hypergraph embedding* $\bar{\eta}: H \to S$. The regions corresponding to the ones bounded by small circles represent the *hyperedges* of H while the vertices in the embedding represent the original vertices of H. For more details we refer the reader to [12].

The (orientable) genus of a connected hypergraph H is naturally defined as the smallest genus of an (orientable) surface on which the corresponding bipartite graph $G(H)$ embeds. In this way, one can study questions concerning embeddings (and genera) of block designs, considering the design as a hypergraph whose vertices are points of the design and whose hyperedges are blocks of the design. There has been a lot of activity in this field, see for instance [2, 4, 9, 15, 18, 19].

The situation is particularly interesting in the case of Steiner triple systems. We recall that a *Steiner triple system* on a set V is a collection $\mathcal{B}$ of 3-element subsets (blocks) of V such that each 2-element subset of V is contained in exactly one block of $\mathcal{B}$. Elements of the set V are *points*, and blocks of B are often called *triples* of the system. We will use the acronym STS for a Steiner triple system; if we want to emphasize that $\mathcal{B}$ is an STS on a point set V we use the extended notation $(V, \mathcal{B})$. Two STSs $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ are *isomorphic* if there is a bijection $f: V \to V'$ which maps blocks of $\mathcal{B}$ to blocks of $\mathcal{B}'$, that is, $f(\mathcal{B}) = \mathcal{B}'$. It is a well known fact that if $(V, \mathcal{B})$ is an STS then $|V| \equiv 1$ or 3 $(mod\ 6)$.

There is a natural 1–1 correspondence between Steiner triple systems and edge-decompositions of complete graphs into triangles. If $(V, \mathcal{B})$ is an STS then the block set $\mathcal{B}$ induces a decomposition of the edge set of the complete graph K on the vertex set V into triangles (i.e., complete graphs on 3 vertices), and vice versa. We shall refer to this correspondence throughout.

As regards surface embeddings of a Steiner triple system $(V, \mathcal{B})$, one can either view the system as a hypergraph H with vertex set V and hyperedge set $\mathcal{B}$ and embed it as described before, or one may directly consider embeddings of the complete graph K on the vertex set V with the property that each block of $\mathcal{B}$ appears on the surface as a triangle which bounds a region of the embedding. The fact that both approaches yield the same family of embeddings can be checked easily; we prefer here the second one because of its more explicit links to topological graph theory.

3 Steiner triple systems on surfaces

Let $(V, \mathcal{B})$ be an STS on n points and let $K = K(\mathcal{B})$ be the associated complete graph with $V(K) = V$ whose edge set is decomposed into triangles corresponding to the triples in $\mathcal{B}$. By an *embedding of the STS* $(V, \mathcal{B})$ *in an orientable surface* S we understand any embedding $\phi : K(\mathcal{B}) \to S$ with the property that for each $\{u, v, w\} \in \mathcal{B}$, the 3-cycle (uvw) constitutes a boundary of some face of ϕ. For the sake of convenience, we shall abbreviate the above definition by just saying that in the embedding ϕ, *every triple of* $\mathcal{B}$ *is facial.* Since every edge of K belongs to precisely one facial triple, the faces of ϕ can be properly two-coloured. Standardly, we colour the facial triples of $\mathcal{B}$ *black* and the remaining faces *white*.

Conversely, let $\psi : K_n \to S$ be an embedding whose faces can be properly 2-coloured (black and white) and such that all black faces are bounded by 3-cycles. Then ψ is an embedding of some STS on n points. Indeed, let $\mathcal{B}$ be the collection of the 3-subsets of $V = V(K_n)$ that correspond to the boundary triangles of black faces. Since our face colouring is proper, there is no edge that appears on the boundary of just one face. Thus, each edge of K_n is incident to precisely one black face, which translates to the fact that each pair of elements of V belongs to precisely one 3-subset of $\mathcal{B}$. Hence $(V, \mathcal{B})$ is an STS, as claimed.

A particularly interesting case occurs when the family of all *white* faces constitutes an STS as well. Let $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ be two STSs with $|V| = |V'| = n$. We say that the pair $\mathcal{B}, \mathcal{B}'$ is *bi-embeddable* in some orientable surface S if there is an embedding ϕ of the STS $(V, \mathcal{B})$ whose white faces are 3-cycles which constitute the blocks of an STS isomorphic to $(V', \mathcal{B}')$. Briefly, in a *bi-embedding* ϕ of the pair $\mathcal{B}, \mathcal{B}'$, facial triples of $\mathcal{B}$ are black while those corresponding to $\mathcal{B}'$ are white. Necessarily, the bi-embedding ϕ is then an orientable triangular embedding of the complete graph on n vertices, and so $n \equiv 0, 3, 4$ or 7 $(mod\ 12)$ (see e.g. [16]) and the surface has minimum genus. Combining this with the existence condition for STSs, we see that a pair of STSs on n points can have an orientable bi-embedding *only if* $n \equiv 3$ or 7 $(mod\ 12)$. Conversely, each orientable triangular embedding $\psi : K_n \to S$ whose faces can be properly 2-coloured induces a bi-embedding of a pair of STSs.

In order to illustrate the above concepts, Fig. 1 depicts a bi-embedding of the pair $\mathcal{B}, \mathcal{B}'$ where $\mathcal{B} = \mathcal{B}'$ is the (unique) STS on 7 points. Specifically, $\mathcal{B} = \{013, 124, 235, 346, 450, 561, 602\}$, and the isomorphic STS $\mathcal{B}' =$

$\{023, 134, 245, 356, 460, 501, 612\}$.

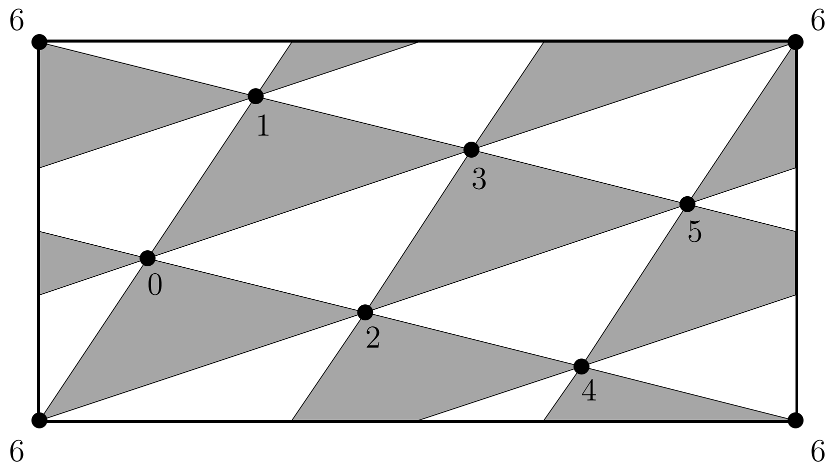

Fig. 1. A bi-embedding of STS(7) in a torus.

Because of the connection between orientable bi-embeddings of STSs and face 2-colourable orientable triangular embeddings of complete graphs, we recall a few facts about the latter. Constructions of minimum genus embeddings (which include triangulations) of complete graphs in orientable and nonorientable surfaces have a rich history. They form the essential part of the solution of the famous Heawood problem of determining the chromatic number of a surface, or, equivalently, determining the genus of a complete graph. Most of the solution (which also gave birth to modern topological graph theory as treated in [11]) is due to Ringel and Youngs; we recommend Ringel's book [16] for details. However, the majority of the known minimum genus orientable embeddings of complete graphs are *not* face 2-colourable.

In the case when $n \equiv 3$ $(mod\ 12)$ the embeddings of K_n found in [16] are indeed face 2-colourable. The proof technique of [16] uses the theory of current graphs. However no information is yielded concerning the STS(n)s which have been embedded, which is the main focus of this paper. In the next section we present a proof of this result using exclusively design-theoretic methods. To our mind this is not only simpler and more transparent, it also positively identifies the STS(n)s so embedded. They are those obtained from the well-known Bose construction based on a Latin square constructed as the square-root Cayley table of an odd-order cyclic group.

In the case when $n \equiv 7$ $(mod\ 12)$ there is the toroidal embedding of K_7 given above and the face 2-colourable triangular embedding of K_{19} given in [18] (see also [13]). Youngs [20] produces triangular embeddings of K_n by means of current assignments on ladder graphs. Amongst the variety of ladder graphs used in [20] it is possible to find, for each $n \equiv 7$ $(mod\ 12)$, one which is bipartite [c.f. especially pages 39-44 of [20]]. Anderson [3] points out the significance of a bipartition; for our purposes this ensures that the corresponding triangular embedding is face 2-colourable. Indeed, the embedding is cyclic (see below) and the bipartition ensures that the face 2-colourability condition for cyclic embeddings (also described below) is satisfied. Thus it is known that there are biembeddings for all $n \equiv 7$ $(mod\ 12)$. But here also, no information is produced about the STSs which have been embedded. In Section 4 we give a design-theoretic proof that such embeddings exist for half of the residue class $n \equiv 7$ $(mod\ 12)$. In another paper [10] we also construct such embeddings by topological methods.

The method of constructing face 2-colourable triangulations directly from Steiner triple system seems to have one more advantage. In contrast with the known topological methods, it seems to produce a large number of new embeddings, as will be documented in Section 5.

4 Bi-embeddings of STSs

We start with rephrasing a well known result in topological graph theory (see [16]).

Theorem 1 *Let $n \equiv 3$ $(mod\ 12)$. Then there exists a pair of bi-embedded STSs of order n in an orientable surface.*

Proof. Take the group $\mathcal{Z}_{4s+1}$ and define on it the operation $\circ$ by $i \circ j = (i+j)/2 = (2s+1)(i+j)$. Use the classical Bose construction [5] to build a STS $(V, \mathcal{B})$ on the point set $V = \mathcal{Z}_{4s+1} \times \mathcal{Z}_3$. The block set $\mathcal{B}$ consists of $4s+1$ triples of the form $(i,0),(i,1),(i,2)$, $i \in \mathcal{Z}_{4s+1}$, together with $3 \times (4s+1)2s$ triples of the form $(i,k),(j,k),(i \circ j, k+1)$ where $i,j \in \mathcal{Z}_{4s+1}$, $i \neq j$ and $k \in \mathcal{Z}_3$. Let $n = 12s+3$. We define two STSs $(\mathcal{Z}_n, \mathcal{B}_0)$ and $(\mathcal{Z}_n, \mathcal{B}_1)$, both isomorphic to $(V, \mathcal{B})$, using the bijections $f_m : V \to \mathcal{Z}_n$, $m = 0, 1$, given by $f_m(i,k) = 3i + (-1)^m kt$ where $t = 6s+1$; naturally, $\mathcal{B}_m = f_m(\mathcal{B})$. (It is understood that on the right side of the equation for $f_m(i,k)$ we have

$i \in \{0, 1, \ldots, 4s\}$, $k \in \{0, 1, 2\}$, and the addition is *mod* n.) It can easily be checked that $\mathcal{B}_0 \cap \mathcal{B}_1 = \emptyset$, i.e., the two STSs are disjoint.

We claim that the pair $\mathcal{B}_0, \mathcal{B}_1$ is bi-embeddable in an orientable surface. To show this, let us think of the triples in $\mathcal{B}_0$ ($\mathcal{B}_1$) as pairwise disjoint black (white) topological triangles, i.e., objects homeomorphic to a closed disc. For *each* pair of distinct points $u, v \in \mathcal{Z}_n$ we now take the corresponding black and white triangle, both containing u and v as vertices, and glue these triangles together along the side uv. Let S be the resulting topological space; then S is certainly a generalized pseudosurface. Our aim is to prove that, in fact, S is an *orientable surface*. This will be done by exhibiting a rotation scheme $Q = \{Q_i;\ i \in \mathcal{Z}_n\}$ for an orientable triangular embedding of the complete graph K_n in which the facial triangles will be in a 1–1 correspondence with the triples in $\mathcal{B}_0 \cup \mathcal{B}_1$.

The scheme can be obtained by identifying all triples in $\mathcal{B}_0 \cup \mathcal{B}_1$ (=triangles in the embedding) that contain a given fixed element of $\mathcal{Z}_n$. Taking into account the obvious action of the group $\mathcal{Z}_{4s+1}$ on the triples of $\mathcal{B}$ (and hence also on $\mathcal{B}_0$ and $\mathcal{B}_1$), it suffices to do that for the three points $0, 1, 2$. The computation is elementary but cumbersome, and a patient reader can convince himself that the cyclic permutations Q_i for $i = 0, 1, 2$ can be described as follows. Let us define three auxiliary 6-term sequences A_q, B_q and C_q as follows: $A_q = q, 2q+1, q+1, -q-3, -2q-7, -q-4$, $B_q = q, 2q+1, q-t, -q-t, -2q+1, -q$, and $C_q = q, 3-q, 2-2q, t+3-q, t+3+q, 2q+2$. Then,

$$Q_0 \;=\; (t, A_{-6s+1}, A_{-6(s-1)+1}, \ldots, A_{-5}, -t, -A_{-6s+1}, \ldots, -A_{-5}) \;,$$

$$Q_1 \;=\; (B_3, B_6, \ldots, B_{6s}, -t, 0) \;, \text{ and}$$

$$Q_2 \;=\; (C_3, C_6, \ldots, C_{6s}, t+2, t+3) \;.$$

The remaining cyclic permutations are defined by the recursion

$$Q_i(j) = Q_{i-3}(j-3) + 3 \quad \text{for each } i, j \in \mathcal{Z}_n,\ i \neq j.$$

An easy but tedious checking shows that the family $Q = \{Q_i;\ i \in \mathcal{Z}_n\}$ is indeed a rotation scheme for a triangular embedding of K_n whose triangular faces are precisely the triples in $\mathcal{B}_0 \cup \mathcal{B}_1$. □

We note here that a similar approach to the above, constructing triangular embeddings of K_n using the Bose construction, can also be found in [7]. However the proof given there, which applies for all $n \equiv 3$ (*mod* 6), $n \geq$

9, always produces an embedding in a *non-orientable* surface which is not suitable for our purposes.

Theorem 2 *If* $m \equiv 3$ (*mod* 12) *then, from an orientable bi-embedding of a pair of STS(m)s, we may construct an orientable bi-embedding of a pair of STS(*$3m-2$*)s.*

Proof. Let $m = 12s+3$. Consider an STS(m) $(V, \mathcal{B})$ where $V = \mathcal{Z}_{m-1} \cup \infty$. Suppose further that $(V, \mathcal{B})$ is embedded in an orientable surface of minimum genus (i.e., the corresponding face 2-colourable embedding of K_m is triangular) and that $Q = \{Q_i;\ i \in V\}$ is the rotation scheme. (The existence of such an embedding for all s is proved in Theorem 1.) Let $\{<\infty, b_i, a_i>:\ i = 0, 1, 2, \ldots, 6s\}$ be the set of (oriented) blocks containing the point ∞. Without loss of generality, suppose that

$$Q_\infty = (a_0, b_0, a_1, b_1, a_2, b_2, \ldots, a_{6s}, b_{6s}) \ .$$

For $i = 0, 1, 2, \ldots, 6s$ let

$$Q_{a_i} = (b_i, \infty, c_{i,1}, d_{i,1}, c_{i,2}, d_{i,2}, \ldots, c_{i,6s}, d_{i,6s}) \ ,$$

$$Q_{b_i} = (\infty, a_i, e_{i,1}, f_{i,1}, e_{i,2}, f_{i,2}, \ldots, e_{i,6s}, f_{i,6s}) \ .$$

Now let $n = 36s + 7 = 3(12s + 2) + 1$, and consider further an STS(n) $(\bar{V}, \bar{\mathcal{B}})$ where $\bar{V} = \mathcal{Z}_{m-1} \cup \mathcal{Z}'_{m-1} \cup \mathcal{Z}''_{m-1} \cup \{\infty\}$. We describe a rotation scheme $\bar{Q} = \{\bar{Q}_i;\ i \in \bar{V}\}$ in terms of the rotation scheme Q, which will determine an embedding of $(\bar{V}, \bar{\mathcal{B}})$ within a face 2-colourable orientable triangular embedding of K_n. Let

$$\begin{aligned}\bar{Q}_\infty = (&a_0, b'_0, a'_1, b''_1, a''_2, b_2, a_3, b'_3, a'_4, b''_4, a''_5, b_5, \ldots, a_{6s}, b'_{6s},\\ &a'_0, b''_0, a''_1, b_1, a_2, b'_2, a'_3, b''_3, a''_4, b_4, a_5, b'_5, \ldots, a'_{6s}, b''_{6s},\\ &a''_0, b_0, a_1, b'_1, a'_2, b''_2, a''_3, b_3, a_4, b'_4, a'_5, b''_5, \ldots, a''_{6s}, b_{6s}) \ ,\end{aligned}$$

$$\bar{Q}_{a_i} = (b'_i, \infty, c_{i,1}, d'_{i,1}, c''_{i,1}, d''_{i,1}, c'_{i,1}, d_{i,1}, c_{i,2}, d'_{i,2}, c''_{i,2}, d''_{i,2}, c'_{i,2}, d_{i,2}, \ldots, c_{i,6s}, d'_{i,6s}, c''_{i,6s}, d''_{i,6s}, c'_{i,6s}, d_{i,6s}, b_i, b''_i, a'_i, a''_i)\ ,$$

$$\bar{Q}_{b_i} = (\infty, a''_i, a'_i, b'_i, b''_i, a_i, e_{i,1}, f'_{i,1}, e''_{i,1}, f''_{i,1}, e'_{i,1}, f_{i,1}, e_{i,2}, f'_{i,2}, e''_{i,2}, f''_{i,2}, e'_{i,2}, f_{i,2}, \ldots, e_{i,6s}, f'_{i,6s}, e''_{i,6s}, f''_{i,6s}, e'_{i,6s}, f_{i,6s})\ ,$$

where $i = 0, 1, 2, \ldots, 6s$. The permutations $\bar{Q}_{a'_i}$ (resp. $\bar{Q}_{a''_i}$) are constructed from $\bar{Q}_{a_i}$ by replacing undashed elements by their corresponding dashed (double-dashed) elements, dashed elements by their corresponding double-dashed (undashed) elements and double-dashed elements by their corresponding undashed (dashed) elements. Similarly for $\bar{Q}_{b'_i}$ and $\bar{Q}_{b''_i}$.

Again what remains is to check that $\bar{Q}$ is a rotation scheme with the required properties. This may be deduced from the corresponding properties of Q by considering firstly the collection of $n(n-1)/2$ pairs obtained from alternate pairs of adjacent entries in the rotations forming $\bar{Q}$, i.e. the pairs formed from the entries in positions 1 and 2, 3 and 4,..., $12s+1$ and $12s+2$ in each $\bar{Q}_z$ ($z \in \bar{V}$). It is feasible, if somewhat tedious, to verify that every pair of distinct elements from $\bar{V}$ appears precisely once as such an entry-pair and that if $\bar{Q}_z$ "contains" such a pair (x, y) in that order then $\bar{Q}_x$ "contains" (y, z) in that order. The checking is then repeated for the remaining adjacent pairs, i.e. those from positions 2 and 3, 4 and 5,...,$12s+2$ and 1. We suggest that the reader first works through the easy cases where $s = 0$ and $s = 1$, using the embedding of the STS(15) constructed in Theorem 1. □

Theorem 3 *If $m \equiv 7 \pmod{12}$ then, from an orientable bi-embedding of a pair of STS(m)s, we may construct an orientable bi-embedding of a pair of STS(3m − 2)s.*

Proof. The construction is similar to that of the previous Theorem but now with $m = 12s + 7$ and the suffices i, j on a, b, c, d, e, f running up to $6s + 2$. We proceed as before except that we select precisely one of the pairs (a_i, b_i), say (a_0, b_0), for special treatment. Take $\bar{Q}_{a_0}$, $\bar{Q}_{b_0}$ as before and then apply the permutation $(a'_0\ \ a''_0)(b'_0\ \ b''_0)$. Having done this, $\bar{Q}_{a'_0}, \bar{Q}_{a''_0}, \bar{Q}_{b'_0}, \bar{Q}_{b''_0}$ are formed as before, but from the modified $\bar{Q}_{a_0}$, $\bar{Q}_{b_0}$. Next take

$$\bar{Q}_\infty = (a_0, b''_0, a''_1, b_1, a_2, b'_2, a'_3, b''_3, a''_4, b_4, a_5, b'_5, \ldots, a_{6s+2}, b'_{6s+2},$$

$$a'_0, b_0, a_1, b'_1, a'_2, b''_2, a''_3, b_3, a_4, b'_4, a'_5, b''_5, \ldots, a'_{6s+2}, b''_{6s+2},$$
$$a''_0, b'_0, a'_1, b''_1, a''_2, b_2, a_3, b'_3, a'_4, b''_4, a''_5, b_5, \ldots, a''_{6s+2}, b_{6s+2}) \ ,$$

The remaining rotation schemes follow the same pattern as before. The checking procedure is as previously described; we suggest that the reader first works through the case $s = 0$ using the bi-embedding of the STS(7) given earlier. □

We note here that applying Theorem 2 to the bi-embedding produced by Theorem 1 gives a bi-embedding for $n \equiv 7$ (*mod* 36). Applying Theorem 3 to this gives a bi-embedding for $n \equiv 19$ (*mod* 108). Proceeding in this fashion we obtain bi-embeddings for $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{2}$ of the residue class $n \equiv 7$ (*mod* 12). We also observe that the "twist" given to the pair (a_0, b_0) in the proof of Theorem 3 may be applied to any individual pair. In fact, in the constructions of both Theorems 2 and 3 we may "twist" any number, say k, of the pairs (a_i, b_i) provided in the former case that $k \equiv 0$ or 1 (*mod* 3) and in the latter case that $k \equiv 1$ or 2 (*mod* 3). We can thereby produce a large number of different bi-embeddings which may or may not be isomorphic.

5 Cyclic bi-embeddings of STSs

Consider a cyclic STS(n) defined on $\mathcal{Z}_n$ and let $Q = \{Q_i;\ i \in \mathcal{Z}_n\}$ be a rotation scheme of an embedding of the system in an orientable surface of minimum genus, given by the formula $\lceil (n-3)(n-4)/12 \rceil$. The embedding is said to be *cyclic* if $Q_i(j) = Q_0(j-i) + i$ for each $i, j \in \mathcal{Z}_n$, $i \neq j$. An example of a cyclic (bi-) embedding is that given in Fig. 1 for STS(7). To exhibit a cyclic embedding it is necessary only to specify Q_0, the rotation scheme at the point 0. We do this below for bi-embeddings with values of $n \equiv 7$ (*mod* 12). (Because of the short orbit, no such embeddings can exist for $n \equiv 3$ (*mod* 12).) In the case when $n = 19$, the listing is complete and all calculations were done by hand. For other values of n, the results were produced by computer calculations and are examples only. The empirical evidence indicates that there are indeed very many such embeddings. For example when $n = 31$ our present estimate is that this is likely to be in excess of $1,000$ non-isomorphic embeddings. We hope to make this the subject of a further paper.

To assist the reader in checking the results, note that the condition on Q_0 which determines that Q_0 is a triangulation is that $Q_0(Q_0(i) - i) = -i$ for all $i \in \mathcal{Z}_n \setminus \{0\}$. This may be seen by considering the faces on the two sides of any edge ab. The condition on Q_0 which ensures that such a cyclic triangulation is face 2-colourable is that, for each $i \in \mathcal{Z}_n \setminus \{0\}$, the equation $Q_0^{k_i}(i) = -i$ should imply that k_i is odd. To see this, suppose that the condition is satisfied; we can then give a procedure for colouring the faces.

Take any (directed) edge $0x$ incident with 0. The orientation of the embedding induces two "sides" to this edge. Colour this edge B/W; that is, colour one side black and the other side white. Now colour the remaining directed edges $0y$ incident with 0 alternating B/W and W/B. For a non-zero vertex a colour the edges incident with a in a similar fashion by taking the colouring of the (directed) edge ab to be that of $0(b-a)$. Now consider any triangular face $\{p, q, r\}$ and suppose, for sake of argument, that the directed edge pq is coloured B/W. Then $0(q-p)$ is colored B/W and, since $Q_0^k(q-p) = p-q$ implies k odd, we have the edge $0(p-q)$ coloured W/B, and hence qp is also coloured W/B. (See Fig. 2a.)

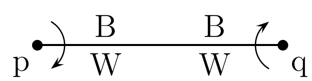

Fig. 2a.

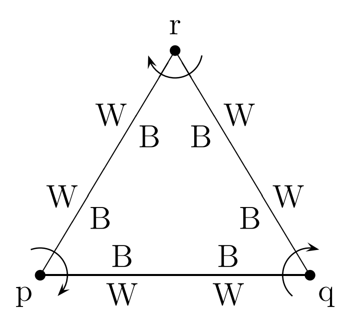

Fig 2b.

Note also that the directed edge pr has the colouring of $0(r-p)$ which is alternate to that of $0(q-p)$, and so pr has the colour W/B. Similarly qr has the colour B/W and the reverse edges, as before, have the reverse colourations. This gives rise to Fig. 2b. Thus the procedure determines a unique colour for any triangle $\{p, q, r\}$ (in this case black) and ensures that all the neighbouring triangles have the opposite colour.

There exist precisely four pairwise non-isomorphic cyclic STS(19)s, [14]. In the interest of completeness we list the base triples for each system below.

$$
\begin{array}{ll}
A1: & 0\ 1\ 4,\quad 0\ 2\ 9,\quad 0\ 5\ 11 \\
A2: & 0\ 1\ 4,\quad 0\ 2\ 12,\quad 0\ 5\ 13 \\
A3: & 0\ 1\ 8,\quad 0\ 2\ 5,\quad 0\ 4\ 10 \\
A4: & 0\ 1\ 8,\quad 0\ 2\ 5,\quad 0\ 4\ 13
\end{array}
$$

We find that there are precisely eight non-isomorphic cyclic bi-embeddings of STS(19)s. These are listed below together with an identification of the two systems. For each pair of the embedded systems, the realization of the first system is as above and so can be immediately checked by the reader. The second system is an isomorphic copy of one of the above systems and this too is fairly easily verified.

1. $Q_0 = (4, 1, 12, 10, 6, 14, 16, 15, 9, 2, 5, 11, 18, 3, 17, 7, 8, 13)$; $A1$ and $A3$.
2. $Q_0 = (4, 1, 8, 13, 9, 2, 16, 15, 6, 14, 17, 7, 18, 3, 5, 11, 12, 10)$; $A1$ and $A3$.
3. $Q_0 = (4, 1, 12, 2, 16, 15, 9, 7, 8, 14, 17, 10, 6, 11, 18, 3, 5, 13)$; $A2$ and $A3$.
4. $Q_0 = (4, 1, 12, 2, 5, 13, 9, 7, 8, 14, 16, 15, 6, 11, 18, 3, 17, 10)$; $A2$ and $A3$.
5. $Q_0 = (4, 1, 8, 14, 16, 15, 6, 11, 12, 2, 5, 13, 9, 7, 18, 3, 17, 10)$; $A2$ and $A3$.
6. $Q_0 = (4, 1, 8, 14, 17, 10, 6, 11, 12, 2, 16, 15, 9, 7, 18, 3, 5, 13)$; $A2$ and $A3$.
7. $Q_0 = (4, 1, 12, 2, 16, 15, 6, 11, 18, 3, 5, 13, 9, 7, 8, 14, 17, 10)$; $A2$ and $A4$.
8. $Q_0 = (4, 1, 8, 14, 16, 15, 9, 7, 18, 3, 17, 10, 6, 11, 12, 2, 5, 13)$; $A2$ and $A4$.

Embedding # 1 is isomorphic to the embedding C given in [13] and embedding # 6 is isomorphic to that given by figure 27 of [20]. Note we have also proved that every cyclic STS(19) can be embedded in an orientable surface of genus $\lceil(19-3)(19-4)/12\rceil = 20$, which is the orientable genus of the complete graph K_{19}.

The following seven rotation schemes describe cyclic bi-embeddings of the projective STS(31). In each example the orientations of the cyclic orbits and therefore triples of the projective system are the same. However, the other bi-embedded systems in each case are pairwise non-isomorphic thus showing that the embeddings are themselves non-isomorphic.

1. $Q_0 = (12, 1, 8, 3, 16, 6, 17, 4, 26, 23, 24, 2, 21, 15, 18, 14, 20, 19,$
$$29, 22, 27, 13, 28, 5, 9, 7, 30, 11, 25, 10)$$
2. $Q_0 = (12, 1, 8, 3, 16, 6, 20, 19, 21, 15, 18, 14, 25, 10, 29, 22, 27, 13,$
$$28, 5, 9, 7, 30, 11, 17, 4, 26, 23, 24, 2)$$
3. $Q_0 = (12, 1, 8, 3, 16, 6, 20, 19, 29, 22, 27, 13, 28, 5, 9, 7, 30, 11,$
$$17, 4, 26, 23, 24, 2, 21, 15, 18, 14, 25, 10)$$
4. $Q_0 = (12, 1, 8, 3, 18, 14, 20, 19, 21, 15, 28, 5, 9, 7, 30, 11, 25, 10,$
$$29, 22, 27, 13, 16, 6, 17, 4, 26, 23, 24, 2)$$
5. $Q_0 = (12, 1, 8, 3, 18, 14, 20, 19, 29, 22, 26, 23, 24, 2, 21, 15, 28, 5,$
$$27, 13, 16, 6, 17, 4, 9, 7, 30, 11, 25, 10)$$
6. $Q_0 = (12, 1, 8, 3, 18, 14, 20, 19, 29, 22, 27, 13, 16, 6, 17, 4, 26, 23,$
$$24, 2, 21, 15, 28, 5, 9, 7, 30, 11, 25, 10)$$
7. $Q_0 = (12, 1, 8, 3, 18, 14, 25, 10, 29, 22, 27, 13, 16, 6, 20, 19, 21, 15,$
$$28, 5, 9, 7, 30, 11, 17, 4, 26, 23, 24, 2)$$

Observe that there are 80 pairwise non-isomorphic cyclic STS(31)s [6], each of which is composed of 5 orbits, giving rise to $2^4 = 16$ different orbit orientations on the surface. Clearly the total number of cyclic bi-embeddings of the STS(31)s is very large.

Finally in this section we list a rotation scheme of a cyclic bi-embedding for a pair of STS(n)s where $n \equiv 7\ (mod 12)$, $43 \leq n \leq 91$.

1. $n = 43$.

$$Q_0 = (9, 1, 12, 2, 21, 6, 14, 3, 28, 22, 24, 7, 20, 4, 38, 13, 36, 17, 33, 31, 32, 29,$$
$$35, 34, 39, 16, 26, 19, 41, 10, 27, 23, 30, 25, 40, 11, 42, 8, 37, 15, 18, 5).$$

2. $n = 55$.

$$Q_0 = (11, 1, 14, 2, 19, 4, 22, 5, 25, 7, 16, 3, 47, 23, 29, 9, 48, 18, 51, 15,$$

$$27, 6, 32, 24, 34, 28, 43, 41, 42, 39, 46, 20, 50, 17, 53, 12, 40, 36,$$
$$38, 33, 37, 30, 35, 26, 49, 21, 45, 44, 52, 13, 54, 10, 31, 8).$$

3. $n = 67$.

$$Q_0 = (13, 1, 16, 2, 21, 4, 24, 5, 35, 8, 18, 3, 26, 6, 60, 22, 33, 10, 59, 27,$$
$$58, 28, 53, 51, 52, 49, 57, 23, 64, 15, 66, 12, 50, 46, 48, 43,$$
$$47, 41, 44, 34, 56, 25, 39, 30, 62, 19, 65, 14, 42, 31, 40, 32,$$
$$37, 9, 36, 11, 45, 38, 55, 54, 61, 20, 63, 17, 29, 7).$$

4. $n = 79$.

$$Q_0 = (15, 1, 18, 2, 23, 4, 26, 5, 40, 13, 43, 32, 42, 12, 31, 7, 71, 25, 28, 6, 20, 3,$$
$$54, 46, 55, 48, 60, 56, 58, 53, 57, 51, 76, 17, 78, 14, 73, 22, 75, 19, 67, 30,$$
$$66, 27, 68, 36, 49, 37, 69, 34, 63, 61, 62, 59, 65, 64, 72, 24, 70, 29, 45, 35,$$
$$74, 21, 77, 16, 50, 41, 52, 39, 44, 10, 47, 11, 38, 9, 33, 8).$$

5. $n = 91$.

$$Q_0 = (17, 1, 20, 2, 25, 4, 28, 5, 42, 11, 38, 9, 22, 3, 35, 8, 48, 12, 33, 7,$$
$$81, 40, 83, 27, 80, 31, 46, 32, 88, 19, 90, 16, 77, 45, 76, 37, 86, 23,$$
$$89, 18, 52, 15, 60, 49, 54, 39, 57, 13, 82, 29, 55, 43, 51, 41, 47, 34,$$
$$73, 71, 72, 69, 78, 44, 85, 24, 87, 21, 79, 36, 65, 58, 70, 66, 68, 63,$$
$$67, 61, 75, 74, 84, 26, 62, 53, 64, 56, 59, 14, 30, 6, 50, 10).$$

6 Concluding remarks

It is one purpose of this paper to establish the study of embeddings of combinatorial designs on surfaces as legitimate mathematical activity. The work is closely related to classical Topological Graph Theory; the Heawood map colour theorem. However the emphasis is different. Our main focus is on the embedding of the design rather than the graph. For this reason we call this study "Topological Design Theory". We point the way forward by outlining some of the fundamental problems. Although they will almost certainly be difficult, we do not believe that all will be intractable, and so progress should be possible.

Problem 1. Produce a design-theoretic proof that for all $n \equiv 3$ or 7 $(mod\ 12)$, there exists an STS(n) which can be embedded in an orientable surface of genus $(n-3)(n-4)/12$. The theorems proved in this paper leave only half of the cases $n \equiv 7$ $(mod\ 12)$ to be considered.

Much more dificult and probably at this stage beyond the scope of current methods is

Problem 2. Can *every* STS(n) for $n \equiv 3$ or 7 $(mod\ 12)$ be so embedded?

Although the empirical evidence is still flimsy we believe that the answer to this problem is in the affirmative. Our reason for saying this is that we have embeddings of both the unique anti-Pasch STS(15) (Theorem 1) and the projective STS(31) containing the maximum number of quadrilaterals; two systems which structurally are as diverse as possible. A more realistic goal might be to restrict attention to the case $n = 15$. There are 80 pairwise non-isomorphic STS(15)s [14] and at the moment we know only of the embedding of one of them.

Also of interest is the problem of bi-embeddings of pairs of STSs.

Problem 3. For $n \equiv 3$ or 7 $(mod\ 12)$, and a given pair $\mathcal{B}, \mathcal{B}'$ of STS(n)s defined on the same base set, does there exist an orientable face 2-colourable triangular embedding of K_n with the property that the two STSs so formed are isomorphic to $\mathcal{B}$ and $\mathcal{B}'$?

In comparison to Problem 2 we believe that the answer here is in the negative and that the construction of a counter-example may be possible.

Finally, there is the case when $n \equiv 1$ or 9 $(mod\ 12)$. Here any embedding of an STS(n) in any orientable surface is *not* a triangulation and hence no bi-embeddings are possible. Nevertheless we still ask

Problem 4. For a given STS(n), $n \equiv 1$ or 9 (mod 12), what is the minimum genus γ of the orientable surface into which it can be embedded? In particular when does $\gamma = \lceil (n-3)(n-4)/12 \rceil$? The cases $n = 9$ and $n = 13$ might repay further study.

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References

[1] S. R. Alpert, *Twofold triple systems and graph imbeddings*, J. Combinat. Theory (A) **18** (1975), 101-107.

[2] I. Anderson, *Infinite families of bi-embedding numbers*, J. Graph Theory **3** (1979), 263-268.

[3] I. Anderson, *Triples, current graphs and biembeddings*, Aequationes Math. **24** (1982), 230-242.

[4] I. Anderson and A. T. White, *Current graphs and bi-embeddings*, J. Graph Theory **2** (1978), 231-239.

[5] R. C. Bose, *On the construction of balanced incomplete block designs*, Ann. Eugenics **9** (1939), 353-399.

[6] M. J. Colbourn and R. A. Mathon, *On cyclic Steiner 2-designs*, in: "Topics on Steiner Systems" (C. C. Lindner and A. Rosa, Eds.), North Holland, 1980, 215-153.

[7] P. Ducrocq and F. Sterboul, *On G-triple systems*, Publications du Laboratoire de Calcul de l'Université des Sciences at Techniques de Lille, No. 103 (1978).

[8] A. Emch, *Triple and multiple systems, their geometric configurations and groups*, Trans. Amer. Math. Soc. **31** (1929), 25-42.

[9] B. L. Garman, R. D. Ringeisen, A. T. White, *On the genus of strong tensor products of graphs*, Canad. J. Math. **XXVIII** No. 3 (1976), 523-532.

[10] M. J. Grannell, T. S. Griggs, J. Širáň, *Face 2-colourable triangular embeddings of complete graphs*, submitted.

[11] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley, N.Y., 1987.

[12] M. Jungerman, S. Stahl, A. T. White, *Imbeddings of hypergraphs*, Congressus Num. **29** (1980), 545-557.

[13] S. Lawrencenko, S. Negami, A. T. White, *Three nonisomorphic triangulations of an orientable surface with the same complete graph*, Discrete Math. 135 (1994), 367-369.

[14] R. A. Mathon, K. T. Phelps, A. Rosa, *Small Steiner triple systems and their properties*, Ars Combin. **15** (1983), 3-110.

[15] J. M. Rahn, *The genus of a block design*, Congressus Num. **50** (1985), 255-268.

[16] G. Ringel, *Map Color Theorem*, Springer, 1974.

[17] T. S. R. Walsh, *Hypermaps versus bipartite maps*, J. Combinat. Theory (B) **18** (1975), 155-163.

[18] A. T. White, *Block designs and graph imbeddings*, J. Combinat. Theory (B) **25** (1978), 166-183.

[19] A. T. White, *Efficient imbeddings of finite projective planes*, Proc. London Math. Soc. **70** (1995), Part 1, 33-55.

[20] J. W. T. Youngs, *The mystery of the Heawood conjecture*, in: "Graph Theory and its Applications" (B. Harris, Ed.), Acad. Press, 1970, 17-50.