Maximizing the number of Pasch configurations in a Steiner triple system

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Abstract

Let $P(v)$ denote the maximum number of Pasch configurations in any Steiner triple system on v points. It is known that $P(v) \leq M(v) = v(v-1)(v-3)/24$, with equality if and only if v is of the form $2^n - 1$. It is also known that $\limsup_{\substack{v\to\infty \\ v\neq 2^n-1}} \dfrac{P(v)}{M(v)} = 1$. We give a new proof of this result and improved lower bounds on $P(v)$ for certain values of v.

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1 Introduction

A Steiner triple system of order v, STS(v), is an ordered pair $(V, \mathcal{B})$ where V is a v-element set (the *points*) and $\mathcal{B}$ is a set of triples from V (the *blocks*), such that each pair from V appears in precisely one block. The necessary and sufficient condition for the existence of an STS(v) is that $v \equiv 1$ or 3 (mod 6) [6], and these values of v are said to be *admissible*. We often omit set brackets and commas from triples so that $\{x, y, z\}$ may be written as xyz when no confusion is likely, and pairs may be treated similarly.

A *Pasch configuration* or *quadrilateral* is a set of four triples on six distinct points having the form $\{abc, ade, bdf, cef\}$. For each admissible $v \neq 7, 13$, it is known that there is an STS(v) containing no Pasch configurations [4]. At the other extreme, no STS(v) can contain more than $M(v) = v(v-1)(v-3)/24$ Pasch configurations, and this upper bound is only achieved when $v = 2^n - 1$ and the corresponding STS(v) is the projective system of order $2^n - 1$ [7]. Apart from the values $v = 2^n - 1$, the maximum number of Pasch configurations $P(v)$ in systems of order v is generally unknown. In fact the only known values when $v \neq 2^n - 1$ appear to be $P(9) = 0, P(13) = 13$ and $P(19) = 84$ (for the latter see [2]). However, it was shown by Gray and Ramsay [5] that $\limsup_{\substack{v\to\infty \\ v\neq 2^n-1}} \frac{P(v)}{M(v)} = 1$, and the proof of this rests on some recursive constructions. Various papers, in particular [5, 7], give some lower bounds for $P(v)$ for specific values of v, to which the recursive constructions may be applied to extend these bounds to infinite classes of admissible values v.

In this current note we improve the lower bounds for $P(v)$ for certain infinite classes of v. Our bounds provide an alternative direct proof of the asymptotic result mentioned above. The constructions we use might best be described as "Add c" constructions since they can, in suitable circumstances, produce an STS($v + c$) from an STS(v) for small values of c. These constructions are not new; the case $c = 2$ is described in [3]. We start by describing the cases $c = 4$ and $c = 6$. A *parallel class* in an STS($6s + 3$) is a set of $2s + 1$ mutually disjoint blocks.

2 Results

Suppose we have an STS($6s + 3$)= $(V, \mathcal{B})$ with two parallel classes $\mathcal{P}_1$ and $\mathcal{P}_2$ intersecting in a single common block $\infty_1\infty_2\infty_3$. Form a regular graph of degree 7, G_7, on the vertex set $V' = V \setminus \{\infty_1, \infty_2, \infty_3\}$. Let ab be an edge of G_7 if $abc \in \mathcal{P}_i$ ($i = 1$ or 2) or if $ab\infty_i \in \mathcal{B}$ (where $i = 1, 2$ or 3). By Vizing's Theorem, the chromatic index χ' of G_7 must be either seven or eight. Suppose that $\chi'(G_7) = 7$. Then each colour class is a one-factor, and so we have seven one factors $\mathcal{F}_j$, $j = 1, 2, \ldots, 7$. Now delete those blocks of $\mathcal{B}$ that lie in $\mathcal{P}_1$ or $\mathcal{P}_2$, or contain any of the points $\infty_1, \infty_2, \infty_3$, to form the set of blocks $\mathcal{B}'$. Then introduce seven new points, say α_j, $j = 1, 2, \ldots, 7$, and form new blocks by appending α_j to each pair $ab \in \mathcal{F}_j$. Finally, add seven new blocks forming an STS(7) on the points α_j. The new blocks together with the blocks of $\mathcal{B}'$ form an STS($6s+7$) on the point set $V' \cup \{\alpha_j : j = 1, 2, \ldots, 7\}$. In a similar way an STS($6s + 9$) might be constructed from an STS($6s + 3$) having three parallel classes intersecting in a common block. The principal difficulty in applying these constructions lies in determining whether or not the graph G_7 (or the corresponding 9-regular graph G_9 in the Add 6 case) has the required chromatic index. We will show that both the Add 4 and the Add 6 constructions may be applied to the projective STS($2^{2k}-1$) when $k \geq 2$. We will denote the projective STS(2^n-1) by S_n.

The point set of S_n may be taken as $\mathbb{Z}_2^n \setminus \{0\}$ and abc is a block if and only if $a+b+c=0$. For $n \geq 3$, it will be convenient to denote a point $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}_2^n$ as $a_{i_1 i_2}$ where $a = (i_3, i_4, \ldots, i_n) \in \mathbb{Z}_2^{n-2}$. For example, 0_{11} denotes $(1,1,0,\ldots,0)$. Our first step is to prove the following lemma.

Lemma 2.1 *For $k \geq 2$, the projective system S_{2k} has four parallel classes intersecting in a single common block.*

Proof Note first that S_2, the unique STS(3), comprises a single parallel class. Suppose, inductively, that for some $k \geq 2$, S_{2k-2} has a parallel class $\mathcal{P}$, and choose an ordering for the points in each of the blocks of $\mathcal{P}$. Then the following are four parallel classes of S_{2k} intersecting in the single common block $0_{01}0_{10}0_{11}$ (here $0 \in \mathbb{Z}_2^{2k-2}$).

$$\mathcal{P}_1 = \{a_{00}b_{11}c_{11},\ a_{10}b_{00}c_{10},\ a_{01}b_{01}c_{00},\ a_{11}b_{10}c_{01} : abc \in \mathcal{P}\} \cup \{0_{01}0_{10}0_{11}\}.$$
$$\mathcal{P}_2 = \{a_{00}b_{10}c_{10},\ a_{10}b_{01}c_{11},\ a_{01}b_{00}c_{01},\ a_{11}b_{11}c_{00} : abc \in \mathcal{P}\} \cup \{0_{01}0_{10}0_{11}\}.$$
$$\mathcal{P}_3 = \{a_{00}b_{01}c_{01},\ a_{10}b_{10}c_{00},\ a_{01}b_{11}c_{10},\ a_{11}b_{00}c_{11} : abc \in \mathcal{P}\} \cup \{0_{01}0_{10}0_{11}\}.$$
$$\mathcal{P}_4 = \{a_{00}b_{00}c_{00},\ a_{10}b_{11}c_{01},\ a_{01}b_{10}c_{11},\ a_{11}b_{01}c_{10} : abc \in \mathcal{P}\} \cup \{0_{01}0_{10}0_{11}\}.$$

(We will henceforth refer to the points $0_{01}, 0_{10}, 0_{11}$ as *infinity points*.) □

Lemma 2.2 *The Add 4 construction may be applied to the projective system S_{2k} for $k \geq 2$.*

Proof To apply the Add 4 construction, use the parallel classes $\mathcal{P}_1$ and $\mathcal{P}_2$. The corresponding graph G_7 is disconnected, with each component on the 12 points derived from a block abc of S_{2k-2}, namely the points a_{ij}, b_{ij}, c_{ij} for $i,j = 0,1$. We must show that G_7 has chromatic index $\chi'(G_7) = 7$. Clearly it suffices to do this for each component of G_7. We will denote the seven colours by the integers $1, 2, \ldots, 7$. A 7-edge-colouring of the component corresponding to abc is shown in Table 1 where, for example, $a_{00}a_{10}1$ denotes that the edge $a_{00}a_{10}$ receives colour 1. These triples, together with those of an STS(7) on the colours, are the new blocks used in the construction to form an STS($2^{2k}+3$). □

The number of Pasch configurations in a projective system of order v is $\frac{v^3}{24}(1-o(1))$ as $v \to \infty$, while the number containing a particular block is linear in v. So any construction that removes only a linear number of

$a_{00}a_{10}1$	$a_{00}a_{01}2$	$a_{00}a_{11}3$	$a_{10}a_{01}3$	$a_{10}a_{11}2$	$a_{01}a_{11}1$
$b_{00}b_{10}1$	$b_{00}b_{01}2$	$b_{00}b_{11}3$	$b_{10}b_{01}3$	$b_{10}b_{11}2$	$b_{01}b_{11}1$
$c_{00}c_{10}1$	$c_{00}c_{01}2$	$c_{00}c_{11}3$	$c_{10}c_{01}3$	$c_{10}c_{11}2$	$c_{01}c_{11}1$
$a_{00}b_{11}6$	$a_{00}c_{11}7$	$b_{11}c_{11}4$	$a_{00}b_{10}4$	$a_{00}c_{10}5$	$b_{10}c_{10}6$
$a_{10}b_{00}5$	$a_{10}c_{10}4$	$b_{00}c_{10}7$	$a_{10}b_{01}7$	$a_{10}c_{11}6$	$b_{01}c_{11}5$
$a_{01}b_{01}6$	$a_{01}c_{00}7$	$b_{01}c_{00}4$	$a_{01}b_{00}4$	$a_{01}c_{01}5$	$b_{00}c_{01}6$
$a_{11}b_{10}5$	$a_{11}c_{01}4$	$b_{10}c_{01}7$	$a_{11}b_{11}7$	$a_{11}c_{00}6$	$b_{11}c_{00}5$

Table 1: A 7-edge colouring of G_7.

blocks will leave a system containing $\frac{v^3}{24}(1-o(1))$ Pasch configurations. This observation and the result of the previous lemma are sufficient to provide an alternative proof of the asymptotic result of Gray and Ramsay. However, we aim for a rather more precise count of the Pasch configurations. So let T_k denote the system of order $2^{2k}+3=v+4$ created by the Add 4 construction as explained above and taking the STS(7) on the colours to have the blocks $123, 145, 167, 246, 257, 347, 356$. Our first step is to count the Pasch configurations that are destroyed in forming T_k.

Lemma 2.3 *For $k \geq 2$, the number of Pasch configurations of S_{2k} that contain one of the infinity points $0_{01}, 0_{10}, 0_{11}$ or a block from either of the two parallel classes $\mathcal{P}_1, \mathcal{P}_2$ is $(v-3)(17v-150)/12$, where $v = 2^{2k}-1$.*

Proof We first calculate the number of Pasch configurations in S_{2k} containing at least one infinity point. Since any pair of intersecting blocks determines exactly two Pasch configurations in S_{2k}, the number of Pasch configurations that contain any particular point is $(v-1)(v-3)/4$. We also need to know the number of configurations that contain exactly two and exactly three infinity points respectively. Configurations with exactly two infinity points have one of the forms:

$$\begin{array}{llllll}
(i) & a_{00}a_{10}0_{10} & (ii) & a_{00}a_{10}0_{10} & (iii) & a_{00}a_{11}0_{11} \\
 & a_{01}a_{11}0_{10} & & a_{01}a_{11}0_{10} & & a_{01}a_{10}0_{11} \\
 & a_{00}a_{01}0_{01} & & a_{00}a_{11}0_{11} & & a_{00}a_{01}0_{01} \\
 & a_{10}a_{11}0_{01} & & a_{01}a_{10}0_{11} & & a_{10}a_{11}0_{01}
\end{array}$$

where $a \in \mathbb{Z}_2^{2k-2} \setminus \{0\}$. There are therefore $(v-3)/4$ configurations of each type. Configurations with exactly three infinity points are of the forms:

$$\begin{array}{llllllll}
(i) & a_{00}a_{10}0_{10} & (ii) & a_{00}a_{10}0_{10} & (iii) & a_{01}a_{11}0_{10} & (iv) & a_{01}a_{11}0_{10} \\
 & a_{00}a_{11}0_{11} & & a_{01}a_{10}0_{11} & & a_{00}a_{11}0_{11} & & a_{01}a_{10}0_{11} \\
 & a_{10}a_{11}0_{01} & & a_{00}a_{01}0_{01} & & a_{00}a_{01}0_{01} & & a_{10}a_{11}0_{01} \\
 & 0_{10}0_{11}0_{01} & & 0_{10}0_{11}0_{01} & & 0_{10}0_{11}0_{01} & & 0_{10}0_{11}0_{01}
\end{array}$$

and there are therefore $(v-3)$ configurations with three infinity points. Putting this together, the number of configurations containing at least one infinity point is:

$$\frac{3(v-1)(v-3)}{4} - \frac{3(v-3)}{4} - 2(v-3) = \frac{(3v-14)(v-3)}{4}. \qquad (1)$$

Next we calculate the number of Pasch configurations that contain at least one block from either $\mathcal{P}_1$ or $\mathcal{P}_2$, but no infinity points. In these two parallel classes there are $2(v-3)/3$ blocks, excluding the common infinity block. Each of these blocks lies in $(v-3)$ Pasch configurations, which would give a total of $2(v-3)^2/3$ Pasch configurations, were it not for the facts that some of these are counted twice and some contain an infinity point.

First we determine how many of the $2(v-3)^2/3$ correspond to a Pasch configuration containing an infinity point. So consider how a block from $\mathcal{P}_\alpha$ ($\alpha = 1, 2, 3$ or 4) may lie in a Pasch configuration with a block containing an infinity point (excluding the common infinity block). Suppose that $a_{ij}b_{h\ell}c_{mn}$ is a block from $\mathcal{P}_\alpha$ and that $a_{ij}a_{pq}\infty$ is a block containing an infinity point $\infty = 0_{01}, 0_{10}$ or 0_{11}. There are two Pasch configurations containing these two blocks and each has precisely one other block containing the point ∞, while the other block is necessarily in $\mathcal{P}_\beta$ for some $\beta \neq \alpha$. We may therefore count these Pasch configurations by choosing intersecting blocks from $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$. The block from $\mathcal{P}_\alpha$ may be chosen in $(v-3)/3$ ways, there are 3 choices for β, and the intersecting block from $\mathcal{P}_\beta$ may be chosen in 3 ways; the remaining two blocks, which must contain an infinity point, are then determined. So there are $3(v-3)$ distinct Pasch configurations containing a block from $\mathcal{P}_\alpha$ and a block containing an infinity point (excluding the common infinity block). By applying this result for $\alpha = 1, 2$, it is apparent that $6(v-3)$ must be subtracted from $2(v-3)^2/3$ to take account of Pasch configurations containing an infinity point. Note this also takes into account the double counting of those configurations with an infinity point that have been counted twice because they have blocks from each of $\mathcal{P}_1$ and $\mathcal{P}_2$.

It remains to determine the number of Pasch configurations that contain blocks from both $\mathcal{P}_1$ and $\mathcal{P}_2$, but no infinity point, since these have been counted twice. Such a Pasch configuration must also contain blocks from $\mathcal{P}_3$ and $\mathcal{P}_4$. So each choice of a block from $\mathcal{P}_1$ and intersecting block from $\mathcal{P}_2$ gives rise to only one Pasch configuration of this type. Consequently, the number of such configurations is $3 \times (v-3)/3 = (v-3)$, Hence a further $(v-3)$ must be subtracted from $2(v-3)^2/3$ to take account of double counting of Pasch configurations having blocks from all four parallel classes.

Thus the number of Pasch configurations in S_{2k} having at least one block from either $\mathcal{P}_1$ or $\mathcal{P}_2$, but no infinity points is

$$\frac{2(v-3)^2}{3} - 6(v-3) - (v-3) = \frac{(v-3)(2v-27)}{3}. \qquad (2)$$

Finally, by combining equations (1) and (2), it follows that for $k \geq 2$, the number of Pasch configurations of S_{2k} that contain one of the infinity points, or a block from $\mathcal{P}_1$ or $\mathcal{P}_2$ is

$$\frac{(3v-14)(v-3)}{4} + \frac{(v-3)(2v-27)}{3} = \frac{(v-3)(17v-150)}{12}. \qquad \square$$

The second step in obtaining the Pasch count for T_k is to determine the number of Pasch configurations added by the inclusion of the new blocks.

Lemma 2.4 *For $k \geq 2$ the new blocks lie in a total of $\frac{(v-3)(3v-28)}{4}+7$ Pasch configurations in T_k.*

Proof A Pasch configuration that contains a new block (i.e. with a colour from $\{1, 2, \ldots, 7\}$) must contain either exactly two, or exactly four, new blocks.

In the case of exactly two new blocks, these must have the form $a_{ij}a_{h\ell}N$, $b_{pq}b_{rs}N$, where $a \neq b$, $N \in \{1, 2, 3\}$ and $i+h = p+r,\ j+\ell = q+s \pmod 2$. Either a and b lie in different blocks of $\mathcal{P}$, or they lie in the same block of $\mathcal{P}$. In the former case, ab is a pair not covered by $\mathcal{P}$ and so it lies in some block $abx \notin \mathcal{P}$. The remaining two blocks of this Pasch configuration must then consist of abx appropriately subscripted. For example, when $N = 1$ there are eight Pasch configurations of this form in T_k:

(i)	$a_{00}a_{10}1$ $b_{00}b_{10}1$ $a_{00}b_{00}x_{00}$ $a_{10}b_{10}x_{00}$	(ii)	$a_{00}a_{10}1$ $b_{00}b_{10}1$ $a_{00}b_{10}x_{10}$ $a_{10}b_{00}x_{10}$	(iii)	$a_{00}a_{10}1$ $b_{01}b_{11}1$ $a_{00}b_{01}x_{01}$ $a_{10}b_{11}x_{01}$	(iv)	$a_{00}a_{10}1$ $b_{01}b_{11}1$ $a_{00}b_{11}x_{11}$ $a_{10}b_{01}x_{11}$
(v)	$a_{01}a_{11}1$ $b_{00}b_{10}1$ $a_{01}b_{00}x_{01}$ $a_{11}b_{10}x_{01}$	(vi)	$a_{01}a_{11}1$ $b_{00}b_{10}1$ $a_{01}b_{10}x_{11}$ $a_{11}b_{00}x_{11}$	(vii)	$a_{01}a_{11}1$ $b_{01}b_{11}1$ $a_{01}b_{01}x_{00}$ $a_{11}b_{11}x_{00}$	$(viii)$	$a_{01}a_{11}1$ $b_{01}b_{11}1$ $a_{01}b_{11}x_{10}$ $a_{11}b_{01}x_{10}$

The number of choices for a is $(v-3)/4$, and then b cannot be a or either of the two points occurring with a in $\mathcal{P}$, so there are $(v-3)/4 - 3 = (v-15)/4$ choices for b. Hence the pair ab may be chosen in $\frac{1}{2}(\frac{v-3}{4})(\frac{v-15}{4}) = \frac{(v-3)(v-15)}{32}$ ways and the colour selected may be 1, 2 or 3. So the number of Pasch configurations created in this way is

$$3 \times 8 \times \frac{(v-3)(v-15)}{32} = \frac{3(v-3)(v-15)}{4}. \qquad (3)$$

The second possibility for a Pasch configuration having exactly two new blocks is that a and b lie in the same block $abc \in \mathcal{P}$. The remaining blocks of this Pasch configuration must then consist of abc appropriately subscripted. Since these must intersect, one must lie in $\mathcal{P}_3$, and the other in $\mathcal{P}_4$. Each pair of intersecting blocks from $\mathcal{P}_3$ and $\mathcal{P}_4$ (excluding the infinity block) gives a Pasch configuration with two new blocks containing one of the colours 1, 2 or 3. For example, $a_{00}b_{01}c_{01} \in \mathcal{P}_3$ and $a_{00}b_{00}c_{00} \in \mathcal{P}_4$ lie in a Pasch configuration with the blocks $b_{00}b_{01}2$ and $c_{00}c_{01}2$. There are $(v-3)/3$ non-infinity blocks in $\mathcal{P}_3$ and each intersects three blocks in $\mathcal{P}_4$. So the number of Pasch configurations of this type in T_k is $(v-3)$.

Next consider the 42 blocks in Table 1, together with the 7 blocks of the STS(7) on the colours. An exhaustive search shows that there are 46 Pasch configurations that have their four blocks taken from these 49 blocks. Seven of these Pasch configurations come from the STS(7). Thus each block $abc \in \mathcal{P}$ in S_{2k-2} gives rise to 39 distinct Pasch configurations, and the total number arising in this way is therefore $39(v-3)/12 = 13(v-3)/4$.

Finally, the STS(7) on the seven colours has a further 7 Pasch configurations. So the total number of Pasch configurations in T_k containing a new point is

$$\frac{3(v-3)(v-15)}{4} + (v-3) + \frac{13(v-3)}{4} + 7 = \frac{(v-3)(3v-28)}{4} + 7.$$

□

Theorem 2.1 *For $k \geq 2$, the number of Pasch configurations in T_k is* $\dfrac{(v-3)(v^2-17v+132)}{24} + 7$, *where $v = 2^{2k} - 1$.*

Proof The projective system S_{2k} contains $v(v-1)(v-3)/24$ Pasch configurations. In the construction of T_k, by Lemma 2.3 $(v-3)(17v-150)/12$ have been removed, and by Lemma 2.4 $(v-3)(3v-28)/4 + 7$ have been added. So the total in T_k is

$$\frac{v(v-1)(v-3)}{24} - \frac{(v-3)(17v-150)}{12} + \frac{(v-3)(3v-28)}{4} + 7$$
$$= \frac{(v-3)(v^2-17v+132)}{24} + 7.$$

□

Next we turn our attention to the Add 6 construction.

Lemma 2.5 *The Add 6 construction may be applied to the projective system S_{2k} for $k \geq 2$.*

Proof To apply the Add 6 construction, use the parallel classes $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$. The corresponding graph G_9 is disconnected, with each component on the 12 points derived from a block abc of S_{2k-2}, namely the points a_{ij}, b_{ij}, c_{ij} for $i, j = 0, 1$. We must show that G_9 has chromatic index $\chi'(G_9) = 9$. Clearly it suffices to do this for each component of G_9. We will denote the nine colours by the integers $1, 2, \ldots, 9$. A 9-edge-colouring of the component corresponding to abc is shown in Table 2 where, for example, $a_{00}a_{10}1$ denotes that the edge $a_{00}a_{10}$ receives colour 1. These

$a_{00}a_{10}1$	$a_{00}a_{01}2$	$a_{00}a_{11}3$	$a_{10}a_{01}3$	$a_{10}a_{11}2$	$a_{01}a_{11}1$
$b_{00}b_{10}1$	$b_{00}b_{01}2$	$b_{00}b_{11}3$	$b_{10}b_{01}3$	$b_{10}b_{11}2$	$b_{01}b_{11}1$
$c_{00}c_{10}1$	$c_{00}c_{01}2$	$c_{00}c_{11}3$	$c_{10}c_{01}3$	$c_{10}c_{11}2$	$c_{01}c_{11}1$
$a_{00}b_{11}5$	$a_{00}c_{11}4$	$b_{11}c_{11}6$	$a_{00}b_{10}6$	$a_{00}c_{10}7$	$b_{10}c_{10}9$
$a_{10}b_{00}4$	$a_{10}c_{10}5$	$b_{00}c_{10}6$	$a_{10}b_{01}6$	$a_{10}c_{11}7$	$b_{01}c_{11}9$
$a_{01}b_{01}5$	$a_{01}c_{00}4$	$b_{01}c_{00}7$	$a_{01}b_{00}7$	$a_{01}c_{01}6$	$b_{00}c_{01}8$
$a_{11}b_{10}4$	$a_{11}c_{01}5$	$b_{10}c_{01}7$	$a_{11}b_{11}7$	$a_{11}c_{00}6$	$b_{11}c_{00}8$
$a_{00}b_{01}8$	$a_{00}c_{01}9$	$b_{01}c_{01}4$	$a_{10}b_{10}8$	$a_{10}c_{00}9$	$b_{10}c_{00}5$
$a_{01}b_{11}9$	$a_{01}c_{10}8$	$b_{11}c_{10}4$	$a_{11}b_{00}9$	$a_{11}c_{11}8$	$b_{00}c_{11}5$

Table 2: A 9-edge colouring of G_9.

triples, together with those of an STS(9) on the colours, are the new blocks used in the construction to form an STS($2^{2k} + 5$). □

Now let U_k denote the system of order $2^{2k}+5 = v+6$ created by the Add 6 construction as explained above and taking the STS(9) on the colours to have the blocks $123, 148, 157, 169, 249, 256, 278, 345, 368, 379, 467, 589$. The STS(9), which is unique up to isomorphism, contains no Pasch configurations. Our first step is to count the Pasch configurations that are destroyed in forming U_k.

Lemma 2.6 *For $k \geq 2$, the number of Pasch configurations of S_{2k} that contain one of the infinity points $0_{01}, 0_{10}, 0_{11}$ or a block from any of the three parallel classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ is $7(v-3)(v-10)/4$, where $v = 2^{2k} - 1$.*

Proof As in the proof of Lemma 2.3, the number of configurations containing at least one infinity point is $(3v-14)(v-3)/4$. We must also calculate the number of Pasch configurations that contain at least one block from $\mathcal{P}_1, \mathcal{P}_2$ or $\mathcal{P}_3$, but no infinity points. In these three parallel classes there are $(v-3)$ blocks, excluding the common infinity block. Each of these blocks lies in $(v-3)$ Pasch configurations, which would give a total of $(v-3)^2$ Pasch configurations, were it not for the facts that some of these are counted two or three times and some contain an infinity point.

Again, as in the proof of Lemma 2.3, the number of Pasch configurations containing a block from $\mathcal{P}_\alpha$ and a block containing an infinity point (excluding the common infinity block) is $3(v-3)$. By applying this result for $\alpha = 1, 2, 3$, it is apparent that $9(v-3)$ must be subtracted from $(v-3)^2$ to take account of Pasch configurations containing an infinity point. Note this also takes into account the double counting of those configurations with an infinity point that have been counted twice because they have blocks from $\mathcal{P}_1$ and $\mathcal{P}_2$, or from $\mathcal{P}_1$ and $\mathcal{P}_3$, or from $\mathcal{P}_2$ and $\mathcal{P}_3$.

It remains to determine the number of Pasch configurations that contain blocks from $\mathcal{P}_i$ and $\mathcal{P}_j$ ($(i, j) = (1, 2), (1, 3), (2, 3)$), but no infinity point, since these have been subject to multiple counting. Such a Pasch configuration must contain blocks from all four parallel classes and, as shown in the proof of Lemma 2.3, the number of such configurations is $(v-3)$. Each is counted 3 times, once for the block in $\mathcal{P}_1$, once for the block in $\mathcal{P}_2$, and once for the block in $\mathcal{P}_3$. So a further $2(v-3)$ must be subtracted from $(v-3)^2$ to take account of multiple counting of Pasch configurations having blocks from all four parallel classes.

Thus the number of Pasch configurations in S_{2k} having at least one block from $\mathcal{P}_1$ or $\mathcal{P}_2$ or $\mathcal{P}_3$, but no infinity points, is $(v-3)^2 - 9(v-3) - 2(v-3) = (v-3)(v-14)$.

It follows that for $k \geq 2$, the number of Pasch configurations of S_{2k} that contain one of the infinity points, or a block from $\mathcal{P}_1$ or $\mathcal{P}_2$ or $\mathcal{P}_3$ is

$$\frac{(3v-14)(v-3)}{4} + (v-3)(v-14) = \frac{7(v-3)(v-10)}{4}. \qquad \square$$

The second step in obtaining the Pasch count for U_k is to determine the number of Pasch configurations added by the inclusion of the new blocks.

Lemma 2.7 *For $k \geq 2$ the new blocks lie in a total of $\frac{(v-3)(9v-85)}{12}$ Pasch configurations in U_k.*

Proof As in the proof of Lemma 2.4, consider first a pair ab that is not covered by the parallel class $\mathcal{P}$ in S_{2k-2}. Suppose that this pair lies in a block abx in that system. The number of Pasch configurations that contain two blocks of the form $a_{ij}a_{h\ell}N$, $b_{pq}b_{rs}N$ and two further blocks obtained by subscripting abx appropriately is, as shown previously in Lemma 2.4, $3(v-3)(v-15)/4$. However, unlike Lemma 2.4, there are no Pasch configurations containing exactly two blocks of the form $a_{ij}a_{h\ell}N$, $b_{pq}b_{rs}N$ when abc is a block of $\mathcal{P}$.

Next consider the 54 blocks in Table 2, together with the 12 blocks of the STS(9) on the colours. An exhaustive search shows that there are 60 Pasch configurations that have their four blocks taken from these 66 blocks. None of these Pasch configurations comes from the STS(9) itself. Thus each block $abc \in \mathcal{P}$ in S_{2k-2} gives rise to 60 distinct Pasch configurations, and the total number arising in this way is therefore $60(v-3)/12 = 5(v-3)$. It follows that the total number of Pasch configurations in U_k containing a new point is

$$\frac{3(v-3)(v-15)}{4} + 5(v-3) = \frac{(v-3)(3v-25)}{4}. \qquad \square$$

Theorem 2.2 *For $k \geq 2$, the number of Pasch configurations in U_k is $\dfrac{(v-3)(v^2-25v+270)}{24}$, where $v = 2^{2k}-1$.*

Proof The projective system S_{2k} contains $v(v-1)(v-3)/24$ Pasch configurations. In the construction of U_k, by Lemma 2.6 $7(v-3)(v-10)/4$ have been removed, and by Lemma 2.7 $(v-3)(3v-25)/4$ have been added. So the total in U_k is

$$\frac{v(v-1)(v-3)}{24} - \frac{7(v-3)(v-10)}{4} + \frac{(v-3)(3v-25)}{4}$$
$$= \frac{(v-3)(v^2-25v+270)}{24}. \qquad \square$$

3 Concluding remarks

It is possible to generalize the results of Lemmas 2.1, 2.2 and 2.5. From the proof of Lemma 2.1 it may be seen that, given an STS(u) with a parallel class, the STS(v) with $v = 4u+3$ constructed by two successive applications of the standard doubling construction [1, Construction 2.15] has four parallel classes intersecting in a common block. The Add 4 and Add 6 constructions may then be applied to this STS(v), with proofs following closely those of Lemmas 2.2 and 2.5.

The choices of STS(7) and STS(9) used in our constructions have been optimized for the colourings of G_7 and G_9 shown in Tables 1 and 2 to maximize the number of Pasch configurations obtained.

The bounds obtained in Theorems 2.1 and 2.2 may be expressed as follows.

(i) If $w = 2^{2k}+3$ then $P(w) \geq \dfrac{(w-7)(w^2-25w+216)}{24} + 7$.

(ii) If $w = 2^{2k}+5$ then $P(w) \geq \dfrac{(w-9)(w^2-37w+456)}{24}$.

We believe that these are currently the best lower bounds for $P(w)$ when w has one of the two forms shown. In particular, $P(67) \geq 7582$ and $P(69) \geq 6660$. It appears that the best that can be achieved for $P(67)$ using previous results is a lower bound of 3992 obtained by doubling the STS(33) with 345 Pasch configurations as described by Gray and Ramsay in [5].

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