Maximum genus embeddings of Steiner triple systems

Mike Grannell, Terry Griggs
Department of Pure Mathematics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM

Jozef Širáň
Department of Mathematics
SvF, Slovak Technical University
Radlinského 11
813 68 Bratislava
SLOVAKIA

Abstract

We prove that for $n > 3$ every STS(n) has both an orientable and a nonorientable embedding in which the triples of the STS(n) appear as triangular faces and there is just one additional large face. We also obtain detailed results about the possible automorphisms of such embeddings.

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Maximum genus embeddings

Corresponding author:
M. J. Grannell, 17 Royal Avenue, Leyland, Lancashire, PR25 1BQ, England.
Tel: 01772-433430 email: m.j.grannell@open.ac.uk

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1 Introduction

Recent papers [3, 6, 7] have dealt with *biembeddings* of *Steiner triple systems*. Such a biembedding comprises a face 2-colourable triangulation of a complete graph K_n in an orientable or in a nonorientable surface S. The triangular faces in each of the two colour classes determine the triples of a Steiner triple system of order n, STS(n). In the orientable case, at least one biembedding exists if and only if $n \equiv 3$ or $7 \pmod{12}$. In the nonorientable case, a necessary condition is that $n \equiv 1$ or $3 \pmod 6$ and $n \geq 9$. Whenever such a biembedding exists, it represents a minimum orientable (or nonorientable) genus face 2-colourable embedding of K_n in a surface and hence may be considered to be a minimum orientable (or nonorientable) genus embedding of each of the two STS(n)s involved. From Euler's formula, it is easy to deduce that in the orientable case the minimum genus is $(n-3)(n-4)/12$ and in the nonorientable case it is $(n-3)(n-4)/6$.

Our focus in this current paper lies at the opposite extreme, namely on cellular embeddings of Steiner triple systems of maximum orientable (or nonorientable) genus. To be precise, we seek a face 2-colourable embedding of a complete graph K_n in an orientable (or nonorientable) surface in which the faces in one of the two colour classes are triangles and so determine an STS(n), while there is just one face in the second colour class and the interior of that face is homeomorphic to an open disc. This latter condition ensures that the embedding is cellular and it precludes artificial inflation of the genus by the addition of unnecessary handles or crosscaps. In the orientable case the corresponding genus is $(n-1)(n-3)/6$ and in the nonorientable case it is $(n-1)(n-3)/3$. To avoid trivialities, we shall assume that $n > 3$ and then the single face in the second colour class, which has $n(n-1)/2 > 3$ edges may be referred to unambiguously as the *large face*. In topological graph theory, graphs which are cellularly embeddable with precisely one face are called "upper embeddable". By analogy with this usage, we use the term *upper embedding* for embeddings of STS(n)s of the type just described, appending the qualifier "orientable" or "nonorientable" as appropriate.

The problem can also be formulated in terms of embeddings of complete graphs in which certain face boundaries (corresponding to the triples of an STS(n)) are prescribed. Orientable embeddings where certain directed closed walks are required to be face boundaries (such that their orientation agrees with a fixed orientation of the surface) are called oriented relative embeddings and have been studied before [1, 2, 9, 10]. The big difference between these oriented relative embeddings and the orientable upper embeddings considered in the current paper is that the prescribed boundaries of our triangular faces have no specified orientation. In the nonorientable case, however, our upper embeddings may be regarded as particular instances of relative embeddings.

An important aspect of topological graph theory is the study of automorphism groups of embedded graphs. An automorphism group of an embedding is fully determined by the image of a single edge having a specified orientation and a specified side of that edge. Consequently, the order of the automorphism group of an embedding cannot exceed four times the number of edges of the embedded graph. An embedding which achieves this bound is called a regular map; such embeddings have the greatest possible symmetry and they are closely related to quotients of triangle groups and Riemann surfaces. For an embedding which is not regular, one may nevertheless ask if the automorphism group contains a subgroup acting regularly on the vertices of the embedded graph. Embeddings of this type are called Cayley maps and have been the focus of much study. These considerations provide a motivation for analysing the possible automorphisms of our upper embeddings of STS(n)s.

We prove that for $n > 3$ every STS(n) has both an orientable and a nonorientable upper embedding. We also obtain detailed results about the possible automorphisms of such embeddings.

We here recall that an STS(n) may be formally defined as an ordered pair $(V, \mathcal{B})$, where V is an n-element set (the *points*) and $\mathcal{B}$ is a set of 3-element subsets of V (the *triples*), such that every 2-element subset of V appears in precisely one triple. A necessary and sufficient condition for the existence of an STS(n) is that $n \equiv 1$ or 3 (mod 6); such values of n are called *admissible*. An STS(n) is said to be *cyclic* if it has an automorphism comprising a single cycle of length n. A cyclic STS(n) exists for every admissible n apart from $n = 9$. (See [4] for details.)

We assume that the reader is familiar with basic facts concerning graph embeddings in surfaces, in particular with lifts of embeddings by means of voltage assignments as treated in Chapters 2 - 4 of [8]. When working with embedded designs and graphs, we shall use the same notation for points and vertices of the abstract designs and graphs as well as for the embedded versions; no confusion will be likely. Some of our constructions involve the addition of handles or crosscaps to existing surfaces; we represent these in figures as shown in Figure 1.

A *face 2-colourable* embedding is one which admits a 2-colouring of faces (black and white) such that no two faces of the same colour share an edge.

Figure 1: Representation of handles and crosscaps.

Our embeddings will portray triangular faces as black and the large face as white. Two embeddings are *isomorphic* if there is a bijection between the corresponding vertex sets, preserving all incidences between vertices, edges and faces. An isomorphism may reverse orientation. In principle, an isomorphism may reverse the colour classes of a face 2-colourable embedding but, in practice, such isomorphisms will not arise in this paper since the colour classes we consider here contain faces of different types, namely triangles and the large face. An *automorphism* of an embedding is an isomorphism of that embedding with itself. An embedding of K_n is said to be *cyclic* if it has an automorphism comprising a single cycle of length n.

2 Existence of embeddings

Theorem 2.1 *Every STS(n) has an orientable upper embedding.*

Proof: The triples of the STS(n) will be represented as black triangles of the embedding. The initial step is to take all the black triangles containing a fixed point ∞ of the STS(n). From these one may construct a face 2-coloured planar embedding of a connected simple graph G on n points, having for its faces the $(n-1)/2$ black triangles incident with ∞, and one white face. The graph G and its embedding are illustrated in Figure 2.

We now proceed to add the remaining $(n-1)(n-3)/6$ triples of the STS(n), one at a time, increasing the genus by 1 at each step. Consider at any stage the boundary of the white face. We will assume that every point of the STS(n) appears on this boundary at least once. This assumption is certainly true for the initial embedding illustrated in Figure 2. If the next triple to be added is $\{u, v, w\}$ then we locate one occurrence of each of these points on the boundary of the white face, add a handle to the white face,

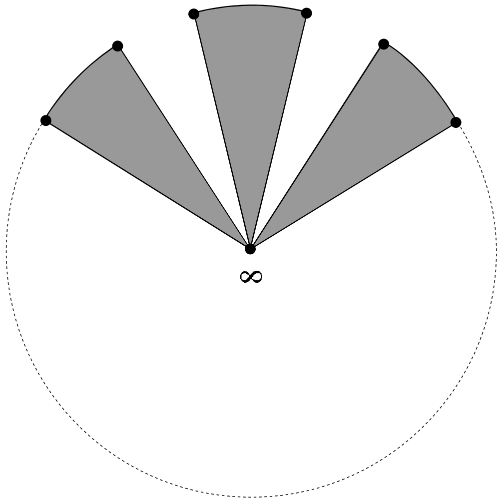

Figure 2: The planar embedding of G.

and paste on the triangle (u, v, w) (or (u, w, v), depending on the order of the selected points around the white face). This is illustrated in Figure 3.

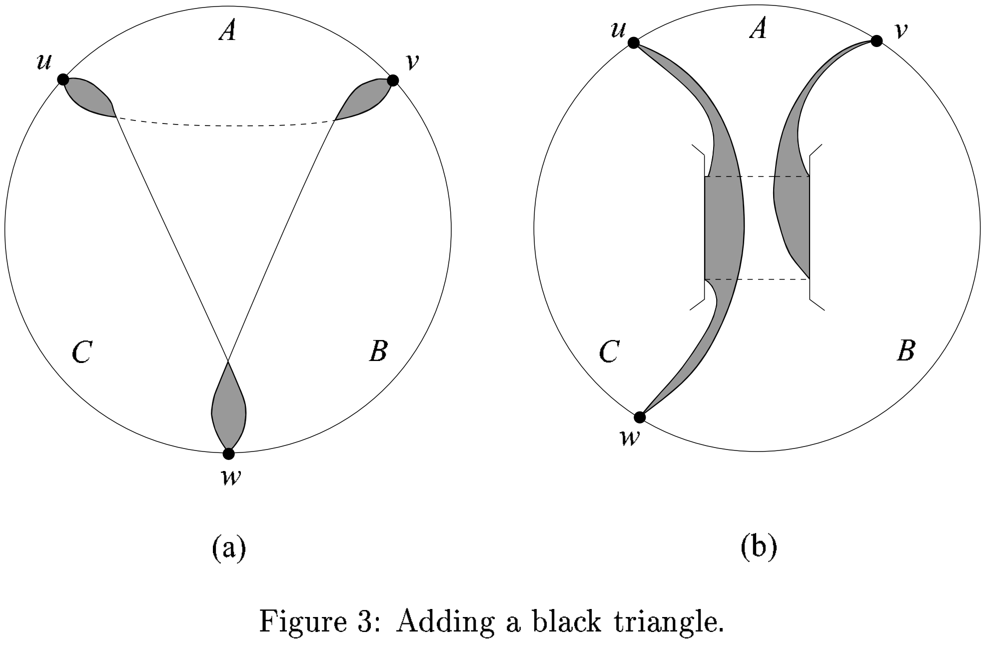

Figure 3: Adding a black triangle.

Figure 3(a) gives a schematic representation of the operation while Figure 3(b) shows the location of the triangle relative to the handle. If the points

u, v, w originally divided the boundary of the white face into three sections A, B and C, then it is easy to see that after the addition of the black triangle (u, v, w) as shown in Figure 3, there still remains just one white face with boundary $A(vw)C(uv)B(wu)$. This face has three more edges than at the previous stage and every point of the STS(n) still appears on the boundary. It is also clear that if the interior of the white face was homeomorphic to an open disc prior to the addition of the black triangle, then it remains so after this addition. □

We remark that it is not necessary to start with the planar embedding specified in the proof. All that is required is a planar embedding of a graph G containing only black triangles from the STS(n) and a single white face incident with all the points of the STS(n).

Theorem 2.2 *Every STS(n) (with $n > 3$) has a nonorientable upper embedding.*

Proof: The proof is identical with that of Theorem 2.1 up to the addition of the final black triangle. This is added to the white face using two crosscaps rather than one handle. Figure 4 illustrates this final step. For clarity, the edges uv, vw and wu are labelled a, b and c respectively.

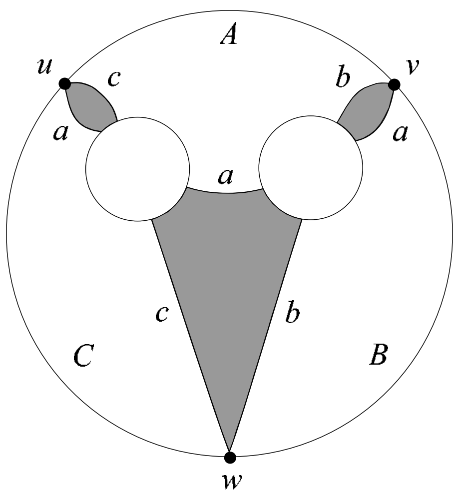

Figure 4: Adding the final black triangle.

Using the same notation as in the proof of Theorem 2.1, the boundary of the white face after the addition of the black triangle (u, v, w) is $A(vw)B(vu)$

$C(wu)$. The resulting surface has $((n-1)(n-3)/6)-1$ handles and 2 crosscaps giving nonorientable genus $(n-1)(n-3)/3$. □

Both Theorems 2.1 and 2.2 have alternative proofs based on embeddings of the *vertex-block incidence graph* of an $\mathrm{STS}(n) = (V, \mathcal{B})$. This graph is the bipartite graph whose vertex set is $V \cup \mathcal{B}$ and whose edges are determined by joining $p \in V$ to $T \in \mathcal{B}$ if and only if $p \in T$. Given an embedding of the vertex-block incidence graph with precisely one face, an upper embedding of the associated $\mathrm{STS}(n) = (V, \mathcal{B})$ may be obtained by surrounding each vertex $T \in \mathcal{B}$ by a small disc on the surface and each of the three edges incident with T by a thin strip, so that the union of the disc and the three strips forms a triangle whose vertices are the three points $p_1, p_2, p_3 \in T$. To prove Theorems 2.1 and 2.2 it therefore suffices to show the existence of a single-face embedding of the vertex-block incidence graph of every $\mathrm{STS}(n)$. In the nonorientable case this is trivial because every graph has such an embedding [5, 11]. In the orientable case we may appeal to a theorem of Xuong [13] which asserts the existence of a single-face embedding of a graph G provided that the Betti number $\beta(G) = 1 - |V(G)| + |E(G)|$ is even and that G has a spanning tree whose complement in G has no component with an odd number of edges. If G is the vertex-block incidence graph of an $\mathrm{STS}(n) = (V, \mathcal{B})$ then $|V(G)| = n + n(n-1)/6$ and $|E(G)| = 3 \cdot n(n-1)/6$, so $\beta(G) = (n-1)(n-3)/3$ is even. A spanning tree of G having the required property may be constructed by choosing a vertex $x \in V$, taking all edges of G incident with x, all edges of G incident with each $T \in \mathcal{B}$ for which $x \in T$, and finally any choice from the remaining edges of G which results in a spanning tree. In the complement, each $T \in \mathcal{B}$ has degree 0 or 2, and so each component of the complement has an even number of edges. We thank Volodymyr Korzhyk who originally suggested this alternative approach to proving Theorems 2.1 and 2.2.

Theorems 2.1 and 2.2 enable us to remark that for each admissible n, the number of non-isomorphic orientable (or nonorientable) upper embeddings of $\mathrm{STS}(n)$s is at least as great as the number of non-isomorphic $\mathrm{STS}(n)$s. This number is asymptotic to $n^{n^2/6}$ [12].

3 Automorphisms

In this section we obtain detailed results about the possible automorphisms of orientable and nonorientable upper embeddings of STS(n)s. We repeat the assumption that $n > 3$.

Theorem 3.1 *If ϕ is an automorphism of an orientable (or nonorientable) upper embedding of an STS(n) then ϕ, represented as a permutation of the points, has one of two forms:*
(a) ϕ comprises a product of disjoint cycles of equal length, or
(b) ϕ comprises a single fixed point together with a product of disjoint cycles of equal length.
Furthermore, ϕ preserves the direction around the large face and the common cycle length is odd.

Proof: Suppose that ϕ has two fixed points, a and b. Since ϕ must preserve the large face and the edge ab appears somewhere on the boundary of this face, it must fix the points adjacent to the edge ab on this boundary. By repetition of this argument, ϕ fixes every point of the STS(n). Thus ϕ is the identity mapping and so is both of type (a) and type (b). It follows that if ϕ is not the identity mapping then it can have at most one fixed point.

Next suppose that ϕ contains two disjoint cycles of lengths p and q, where $1 < p < q$. Then ϕ^p is an automorphism with p fixed points and a cycle of length at least 2. By the previous paragraph, this is not possible. Hence ϕ must take one of the forms (a) or (b) defined in the statement of the theorem.

Now assume that ϕ has the form (a) and that it reverses the direction around the large face. Clearly ϕ is not the identity. Consider any edge ab which is mapped by ϕ to an edge $a'b'$ appearing on the boundary of the large face as shown in Figure 5.

If c is adjacent to b on this boundary then it must be mapped to c' adjacent to b' as shown. Proceeding in this fashion we deduce that $\phi(a') = a$ and, further, that $\phi^2(x) = x$ for every point x of the STS(n). Since ϕ is not the identity and has the form (a), we see that ϕ must be the product of disjoint transpositions, contradicting the fact that n is odd.

Next, assume that ϕ has the form (b) and that it reverses the direction around the large face. Again, ϕ is clearly not the identity. Suppose that ϕ fixes the point ∞ (and no other point). Arguing as before we see that ϕ fixes ∞ and contains $(n-1)/2$ disjoint transpositions. Suppose that three of these are $(a_1\ b_1), (a_2\ b_2)$ and $(a_3\ b_3)$. Consider the edge a_1b_1. Since this edge is

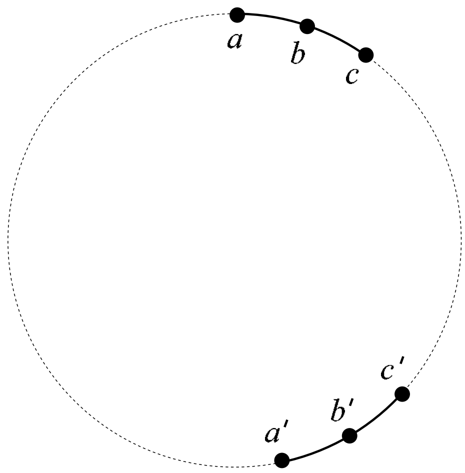

Figure 5: The large face.

stabilized by ϕ, it must appear midway between two successive occurrences of ∞ on the boundary of the large face. But the edge a_2b_2 must also appear midway between the same two successive occurrences of ∞, and the same is true of the edge a_3b_3. Since there are only two midway positions, we have a contradiction. We conclude that ϕ preserves the direction around the large face.

Finally, consider the cycle length. If ϕ has the form (a), then the cycle length is necessarily odd. If ϕ has the form (b) and the cycle length is k, suppose that k is even. Then $\psi = \phi^{k/2}$ is an automorphism which comprises a fixed point and $(n-1)/2$ transpositions. If $(a_1\ b_1)$ is one of these transpositions, then ψ will reverse the direction of the edge a_1b_1 and so fails to preserve the direction around the large face, a contradiction. Thus k must be odd. □

Theorem 3.2 *If ϕ is an automorphism of an orientable upper embedding of an STS(n), and if ϕ comprises a product of disjoint cycles of equal length k, then either $k = 1$ (in which case ϕ is the identity permutation) or $k = 3$. In the case $k = 3$ we must have $n \equiv 3$ (mod 6).*

Proof: Suppose $k > 1$ so that ϕ is not the identity permutation. Then $k|n$ and ϕ preserves orientation since it preserves direction around the large face. Let M denote the embedding in question. The group $\langle\phi\rangle$ generated by ϕ is the cyclic group of order k, Z_k, which acts on vertices of M semi-regularly as a group of orientation-preserving automorphisms. As ϕ preserves colours, the group $\langle\phi\rangle$ also acts semi-regularly on edges of M, and hence on arcs (edges with direction) of M. The action of $\langle\phi\rangle$ on M therefore defines a quotient embedding $M' = M/\langle\phi\rangle$ in some orientable surface. The associated natural projection $\pi : M \to M'$ is known to be a k-fold regular covering, with possible branch points in some face centres. Therefore the underlying base graph of the quotient embedding M' may contain parallel edges and loops, but no semi-edges. Each vertex and each edge of the base graph have exactly k pre-images under π^{-1} in M; we also note that π preserves vertex valencies.

Let F be the large face of M; we recall that the boundary walk of F contains each edge of M exactly once. It follows from the covering properties of the projection that the boundary walk of the face $\pi(F)$ in the quotient map M' must also contain every edge of the base graph exactly once. Consequently, the quotient embedding M' can also be properly 2-coloured (the face $\pi(F)$ white and the remaining faces black), and then π projects the 2-colouring of M onto the 2-colouring of M'. Since the face lengths in M must be multiples of face lengths in M', we see that black faces of M' have length 1 or 3. Let t and l be the number of black triangles and black loops (faces of length 1), respectively. By the regularity of π, each component of the pre-image of a black loop is a black triangle of M (with a branch point of order 3 inside). It follows that if $l > 0$ then $3|k$ and each black loop of M' has $k/3$ pre-images (black triangles) in M. On the other hand, each black triangle of M' has k pre-image black triangles in M. Consideration of the black faces tells us that the parameters k, l, n must satisfy $kt + kl/3 = n(n-1)/6$. As regards the white faces F and $\pi(F)$ that contain each edge of M and M' (respectively) exactly once, from the k-fold covering property of π we see that the boundary of $\pi(F)$ has length $n(n-1)/2k$.

It follows from [5, proof of Theorem 2.2.2] that the embedding M (as a regular covering space of M') may be re-constructed from the quotient M' by a voltage lift, using a voltage assignment on arcs of the base graph of M' in the group Z_k. From the preceding considerations it then follows that the voltages assigned to (directed) edges around each of the t black triangles in M' must sum to 0 and those on any loops must be $k/3$ or $2k/3$.

If $n \not\equiv 3 \pmod 6$ there can be no loops and the voltage sum around the boundary of the face F must then be zero modulo k since it is the sum of the voltages on all the triangles. But then the face F lifts to k faces having boundary length $n(n-1)/2k$ rather than one face having boundary length

$n(n-1)/2$. If $n \equiv 3 \pmod 6$ there may be some loops and the voltage sum around the boundary of the face F must be one of $0, k/3$ or $2k/3 \pmod k$. But F can only lift to a face having boundary length $n(n-1)/2$ in the latter two cases, and then only if $k = 3$. □

4 Automorphisms with no fixed point

We start with an existence result for orientable upper embeddings which links with the case $k = 3$ of Theorem 3.2.

Theorem 4.1 *If $n \equiv 3$ (mod 6) then there exists an orientable upper embedding of an $STS(n)$ having an automorphism that is a product of disjoint cycles of length* 3.

Proof: Suppose that $n = 6s + 3$ and assume initially that $3 \not| (2s+1)$. Working modulo $2s+1$, consider all ordered triples in Z_{2s+1}^3 of the form $(x, y, (x+y)/2)$ with $x < y$. (In "$x < y$" we intend that x and y are represented as integers in $[0, 2s]$.) These triples will form the black triangular faces of a face 2-coloured graph embedding. The initial step in constructing this is to take those triangles of the form $(x, -x, 0)$ for $x = 1, 2, \ldots, s$, and construct from these a face 2-coloured planar embedding of a simple connected graph H on $2s+1$ vertices having for its faces the s black triangles described and one white face as illustrated in Figure 6(a).

Next add a black loop to each vertex including 0, to form the graph H' and the planar embedding of H' illustrated in Figure 6(b). This has $2s+1$ vertices, s black triangles and $2s+1$ black loops. We now add the remaining black triangular faces $(x, y, (x+y)/2)$, one at a time, increasing the genus by 1 at each step in the manner described in the proof of Theorem 2.1. The result is a cellular face 2-coloured embedding of a multigraph H'' having $2s+1$ vertices, $\binom{2s+1}{2}$ black triangles, $2s+1$ black loops and one white face which has all $3\binom{2s+1}{2} + (2s+1) = (2s+1)(3s+1)$ edges.

Now assign voltages in the group Z_3 to the edges of H'' as follows. On each loop place the voltage 1 with a consistent orientation (e.g. all clockwise). On the edges of a black triangle $(x, y, (x+y)/2)$ place the voltages 0 and 1 as shown in Figure 7.

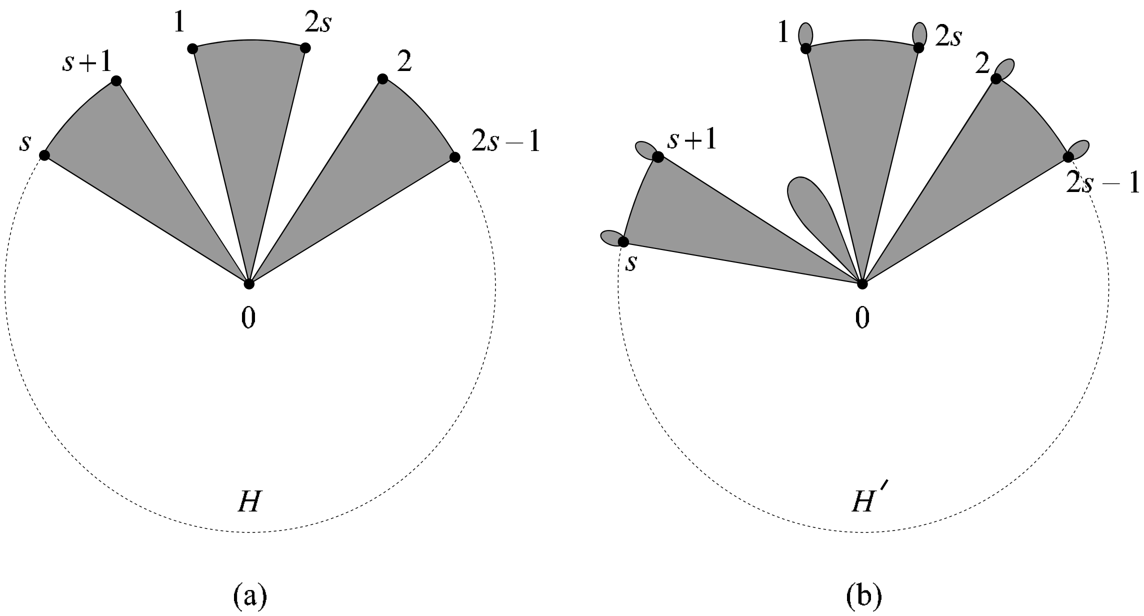

Figure 6: Planar embeddings of H and H'.

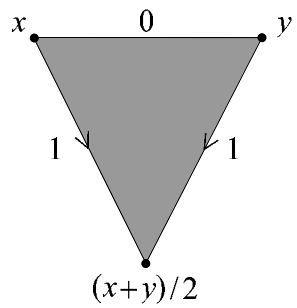

Figure 7: Voltage assignment on H''.

This assignment is unambiguous because if $(x + (x + y)/2)/2 = y$ in Z_{2s+1} then $3(x - y) \equiv 0 \pmod{2s+1}$ and this cannot happen if $x \neq y$ and $3 \not| \ (2s + 1)$. Having completed the voltage assignment on H'', the directed voltage sum over all the black triangles is zero, while that over all the loops is $2s + 1$ or $4s + 2$, depending on the choice of direction, and this is non-zero modulo 3. It follows that the total voltage sum around the white face in the embedding of H'' is non-zero modulo 3.

Consequently, the lifted graph and embedding has $3(2s+1) = 6s+3$ vertices and one white face having all $(6s+3)(6s+2)/2$ edges. The lift of the black faces comprises:

(a) all triangles of the form (x_0, x_1, x_2) from the loops of H'', and

(b) all triangles of the forms $(x_0, y_0, ((x+y)/2)_1)$, $(x_1, y_1, ((x+y)/2)_2)$ and $(x_2, y_2, ((x+y)/2)_0)$ from the triangles of H''.

Altogether these black triangles form the triples of an STS(n), in fact that produced by the well-known Bose construction (see, for example [4]) based on the group Z_{2s+1}. It follows that the lifted embedding is an orientable upper embedding of an STS(n). Since the embedding is obtained from a graph lifting using Z_3, it has an automorphism of order 3 given by $\prod_{x \in Z_{2s+1}} (x_0\ x_1\ x_2)$. This completes the proof in the case $3 \not| \ (2s+1)$.

In the case $3|(2s+1)$ we modify the voltage assignment on H'' as follows. Put $w = (2s+1)/3$. For each $x \in \{0, 1, \ldots, w-1\}$, there will be three black triangles in H'' of the forms $(x, x+w, x+2w)$, $(x, x+2w, x+w)$ and $(x+w, x+2w, x)$. These receive voltage assignments as shown in Figure 8.

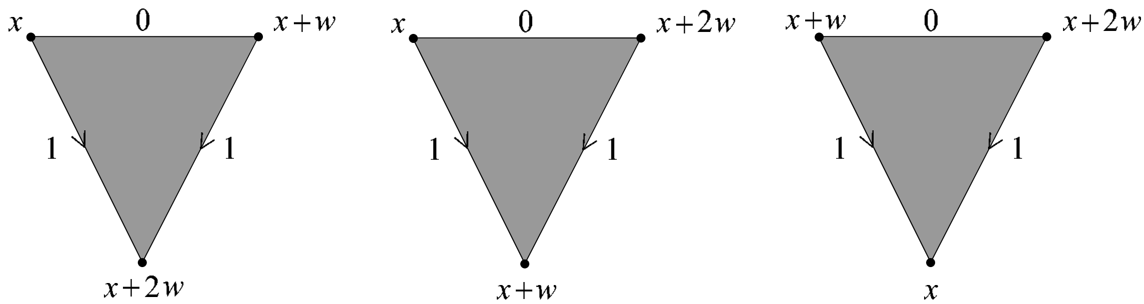

Figure 8: Voltage assignments in H''.

We also reverse the voltage assignment on any one loop. As a consequence of these modifications, the resulting voltage assignment is unambiguously specified and the voltage sum around the white face in the embedding of H'' is again non-zero modulo 3. The remainder of the argument is as previously given. □

The following Theorem contrasts results for the nonorientable case with corresponding results for the orientable case given in Theorem 3.2.

Theorem 4.2 *If $n \equiv 1$ or 3 (mod 6) and $n > 3$ then every cyclic STS(n) has a nonorientable upper embedding with a cyclic automorphism.*

Proof: Suppose initially that $n \equiv 1 \pmod 6$ and that S is a cyclic STS(n). Let $\{\{0, \alpha_i, \beta_i\} : i = 1, 2, \ldots, (n-1)/6\}$ be a set of orbit starters for S. We may assume that $0 < \alpha_i < \beta_i < n$ for each i. Put $a_i = \alpha_i, b_i = \beta_i - \alpha_i$ and $c_i = \beta_i$ so that $a_i + b_i \equiv c_i \pmod n$ and $\{a_i, b_i, c_i, n - a_i, n - b_i, n - c_i : i = 1, 2, \ldots, (n-1)/6\} = \{1, 2, \ldots, n-1\}$. In effect, $\{(a_i, b_i, c_i) : i = 1, 2, \ldots, (n-1)/6\}$ is a set of difference triples for S.

Given one of these triples (a_i, b_i, c_i), we define two directed graphs G_i^1 and G_i^2 each having a single vertex labelled z_i and whose edges are labelled with a_i, b_i and c_i. The graph G_i^1 has an orientable embedding of genus 1 and the graph G_i^2 has a nonorientable embedding of genus 2. These graphs and their embeddings are illustrated in Figure 9.

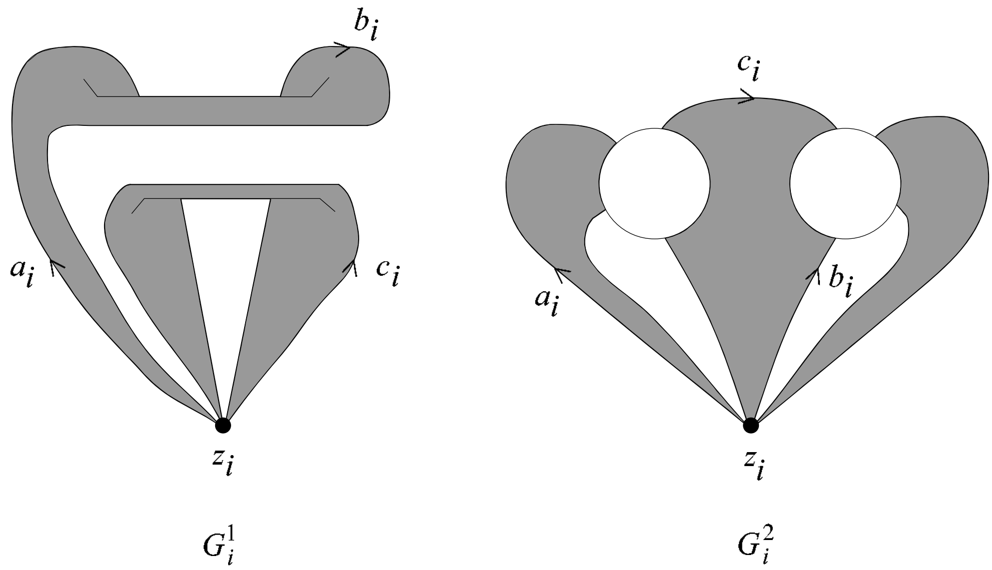

Figure 9: Embeddings of G_i^1 and G_i^2.

Each of G_i^1 and G_i^2 has three edges and their embeddings each have two faces, here coloured black and white. The directed sums of the edge labels of the black faces of G_i^1 and G_i^2 are both $\pm(a_i + b_i - c_i) \equiv 0 \pmod n$. For the white face of G_i^1 the same is true, but for the white face of G_i^2 the directed sum of the edge labels is $\pm(a_i + b_i + c_i)$.

We now form a graph G with an associated cellular face 2-coloured embedding by attaching for each $i = 1, 2, \ldots, (n-1)/6$ a copy of $G_i^{k_i}$, where $k_i = 1$ or 2, to a common root vertex x. That is, we identify the vertices z_i with x for each i; we carry out this operation so that the embedding which results has a single cellular white face. The graph G and its embedding are shown schematically in Figure 10.

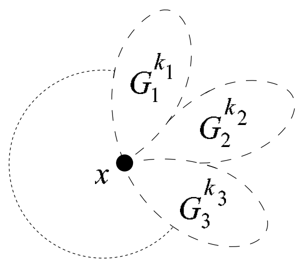

Figure 10: Schematic representation of G.

We choose the values of $k_i = 1$ or 2 in such a way that the directed sum of the boundary labels on the single white face of G is coprime with n. For example, if we suppose that (without loss of generality) $\beta_1 = n-1$, so that $c_1 = n-1$, then we can select $k_1 = 2$ and $k_i = 1$ for $i = 2, 3, \ldots, (n-1)/6$. This gives the directed sum of the boundary labels on the single white face of G as $\pm(a_1 + b_1 + c_1) \equiv \pm 2c_1 \equiv \pm 2 \pmod{n}$. However the choice is made, it is necessary to use at least one value of k_i which equals 2, and so the embedding of G is nonorientable.

We now lift the graph G using the edge labels to form a voltage assignment in the group Z_n. The resulting black faces are all triangular with vertex sets of the form $\{x_k, x_{k+a_i}, x_{k+a_i+b_i}\} = \{x_k, x_{k+\alpha_i}, x_{k+\beta_i}\}$ for $k = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, (n-1)/6$, and so form an isomorphic copy of the Steiner triple system S. The single white face of G lifts to a single white face having all $n(n-1)/2$ edges of the lifted graph for its boundary. The lifted graph and embedding have a cyclic automorphism and Theorem 3.2 ensures that the corresponding surface is nonorientable. It follows that the lifted graph and embedding give a nonorientable upper embedding of S. This completes the proof in the case $n \equiv 1 \pmod 6$.

If $n \equiv 3 \pmod 6$, we modify the construction of G as follows. One of the orbit starters may be taken as $\{0, n/3, 2n/3\}$. Corresponding to this starter we do not use a copy of G_i^1 or G_i^2 but instead a black loop G^* with a single vertex z^* and its edge labelled $n/3$. The remainder of the argument is similar to that previously given, with obvious modifications. □

Corollary 4.2.1 *If $n \equiv 1$ or 3 (mod 6), $n > 3$ and $k|n$, then every cyclic STS(n) has a nonorientable upper embedding having an automorphism which is the product of disjoint cycles of length k.*

Proof: If ϕ is the cyclic automorphism guaranteed by Theorem 4.2, and if $kl = n$, then ϕ^l is a product of disjoint cycles of length k. □

5 Automorphisms with one fixed point

Our first Theorem in this Section deals with the orientable case and shows that the situation regarding automorphisms with a single fixed point is quite different to that described in Theorem 3.2 for automorphisms without a fixed point.

Theorem 5.1 *Suppose that S is an STS(n) with an automorphism ϕ having a single fixed point and l cycles each of length k, where k is odd and $n = kl + 1$. Then there exists an orientable upper embedding of S having ϕ as an automorphism.*

Proof: Suppose that the points of S are ∞, α_p $(1 \leq \alpha \leq l,\ 0 \leq p \leq k-1)$, and that

$$\phi = (\infty)(1_0\ 1_1\ \ldots\ 1_{k-1})(2_0\ 2_1\ \ldots\ 2_{k-1})\ \ldots\ (l_0\ l_1\ \ldots\ l_{k-1}).$$

The set of points $\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ will be described as the α^{th} group. Since $n = kl+1$ and k, n are odd, l must be even. There can be no triple $\{\infty, \alpha_p, \alpha_q\}$ in S because such a triple would imply the existence of a triple $\{\infty, \alpha_q, \alpha_{2q-p}\}$ also in S, giving $2q - p \equiv p \pmod{k}$, a contradiction. Thus all triples in S containing ∞ must contain points from two distinct groups. Without loss of generality, we may assume that a set of starters for such triples is

$$\{\{\infty, 1_0, 2_0\}, \{\infty, 3_0, 4_0\}, \ldots, \{\infty, (l-1)_0, l_0\}\}.$$

Corresponding to these starters, we construct a planar embedding of a graph G having vertices $\infty, 1, 2, \ldots, l$ as shown in Figure 11. We label the edges of G which are not incident with ∞ with the value 0.

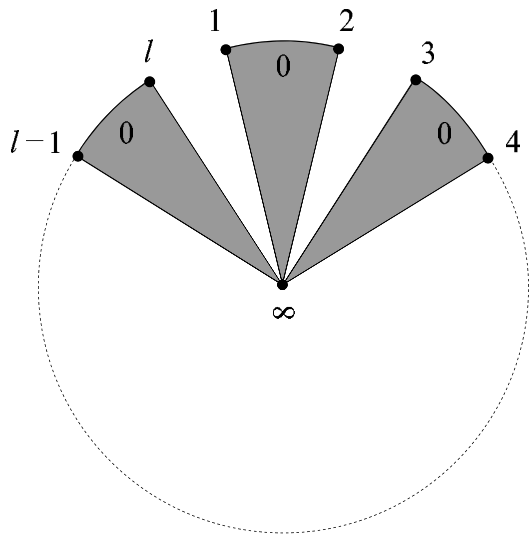

Figure 11: Planar embedding of G.

Now add the remaining starters for S, one at a time, to this embedding in the following manner. Suppose firstly that $\{\alpha_p, \beta_q, \gamma_r\}$ is a starter for a full orbit (i.e. an orbit of length k). If α, β, γ are distinct then we add the triangle (α, β, γ) (or (α, γ, β) in the manner described in the proof of Theorem 2.1. If $\alpha = \beta \neq \gamma$, the insertion method described in Theorem 2.1 is adjusted by identifying α and β as shown in Figure 12.

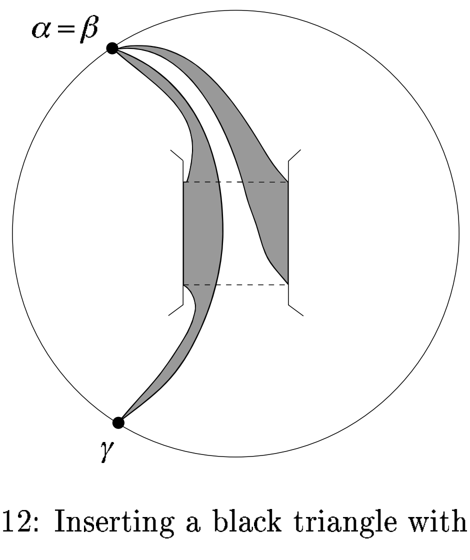

Figure 12: Inserting a black triangle with $\alpha = \beta$.

We proceed in a similar fashion if $\alpha = \gamma \neq \beta$ or if $\beta = \gamma \neq \alpha$. If $\alpha = \beta = \gamma$, the insertion method is further adjusted by identifying all three vertices α, β and γ (see the embedding of G_i^1 in Figure 9). In each of these cases the (orientable) genus of the embedding is increased by 1 and we label the directed edges of the black triangle with elements of the group Z_k as shown in Figure 13.

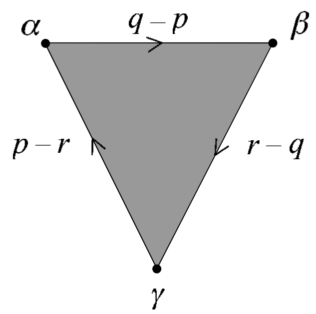

Figure 13: Edge labels on the triangle (α, β, γ).

Secondly consider a starter for a one-third orbit (i.e. an orbit of length $k/3$) - this is only possible if $3|k$. If such a starter is present, it may be assumed to have the form $\{\alpha_0, \alpha_{k/3}, \alpha_{2k/3}\}$. Corresponding to this we insert a black loop rooted at α into the white face and label the edge of this loop with the value $k/3$. The addition of such a loop leaves the genus of the embedding unaltered.

After all the starters have been added to G, we obtain a multigraph H having a cellular face 2-coloured embedding whose faces are black triangles, (possibly) black loops, and one large white face incident with all the edges of H. Apart from those edges incident with ∞, all the edges of H are labelled with elements of Z_k.

Consider the rotation at ∞ as shown in Figure 11 above. Insert additional edges $23, 45, \ldots, l1$, thereby forming $l/2$ white triangles and shortening the boundary of the large white face. Then delete the point ∞, all (open) edges incident with ∞ and all (open) triangles, both black and white, incident with ∞, from the embedding. In effect these operations remove a cap from the surface embedding of H, leaving a hole whose boundary is the l-cycle $(123 \cdots l)$. It is convenient to consider this hole as being filled by an l-gon in a third colour, say grey. Choose any one unlabelled directed edge of this l-gon and give this edge the label ± 1; the choice of sign is determined below. For the sake of definiteness in subsequent diagrams we will assume that this edge is $l1$ and is labelled $+1$. Label all the remaining unlabelled edges with 0. If H' denotes the resulting graph, then Figure 14 shows part of the corresponding embedding of H' which has been constructed.

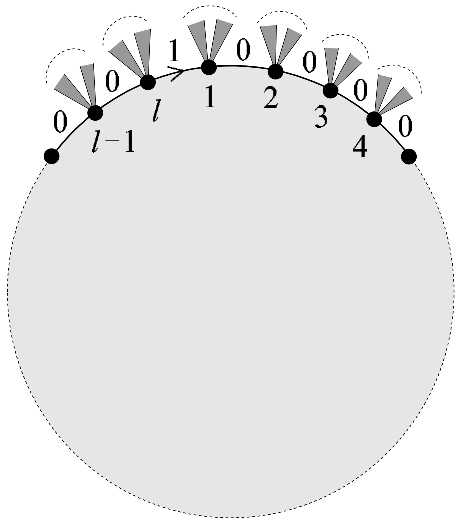

Figure 14: The embedding of H'.

The vertices of H' are the points $1, 2, \ldots, l$. The number of edges of H', $|E(H')|$, is given by the number of pure and mixed differences on the set $\{\{1_0, 1_1, \ldots, 1_{k-1}\}, \{2_0, 2_1, \ldots, 2_{k-1}\}, \ldots, \{l_0, l_1, \ldots, l_{k-1}\}\}$, plus $l/2$. This gives

$$|E(H')| = \frac{(k-1)l}{2} + \frac{kl(l-1)}{2} + \frac{l}{2} = \frac{kl^2}{2}.$$

With appropriate directions, the sum modulo k of the edge labels around each black triangle is 0, around any black loop is $k/3$, around the grey face is 1, and around the white face is $\pm 1 + km/3$ where the integer m depends on the number of loops. The $\pm$ sign is determined by the requirement that 3 should not be a divisor of the sum of the edge labels around the white face, so that this sum is coprime with k.

We now lift the graph H' and its embedding using the edge labels to form a voltage assignment in the group Z_k. The resulting black faces are all triangles and correspond to those triples of s which do not contain the point ∞. The white face lifts to a white face incident with all $k^2l^2/2$ edges of the lifted graph and the grey face lifts to a grey kl-gon as illustrated in Figure 15.

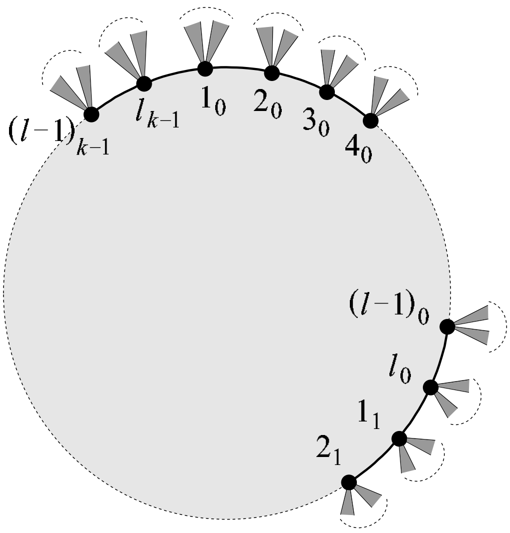

Figure 15: The grey kl-gon.

We next insert a vertex ∞ into the interior of this face and join it by kl new edges to all the boundary vertices. We then remove the $kl/2$ alternate edges $2_0 3_0, 4_0 5_0, \ldots, l_{k-1} 1_0$ to form a graph K with an associated embedding. Figure 16 shows part of the embedding of K. The triangles $(\infty, 1_0, 2_0), (\infty, 3_0, 4_0), \ldots, (\infty, (l-1)_{k-1}, l_{k-1})$ are coloured black and that part of the surface previously lying in the triangles $(\infty, 2_0, 3_0), (\infty, 4_0, 5_0)$, $\ldots, (\infty, l_{k-1}, 1_0)$ is re-coloured white.

It is easy to see that K is a complete graph on $kl + 1 = n$ vertices and that the black triangular faces of the embedding correspond precisely to the triples of S. The boundary of the white face contains every edge of K exactly once and the face itself is cellular. Thus K and the associated embedding form an orientable upper embedding of S.

We claim that ϕ is an automorphism of this embedding. In all cases, a cellular face 2-coloured embedding in an orientable surface is completely determined by the oriented faces of that embedding. From the construction given above it is clear that ϕ preserves the black faces of the embedding, including their orientations. To prove that ϕ is an automorphism of the embedding, it is therefore sufficient to establish that ϕ preserves the white face and its orientation. It is certainly the case that in the lift of the embedding

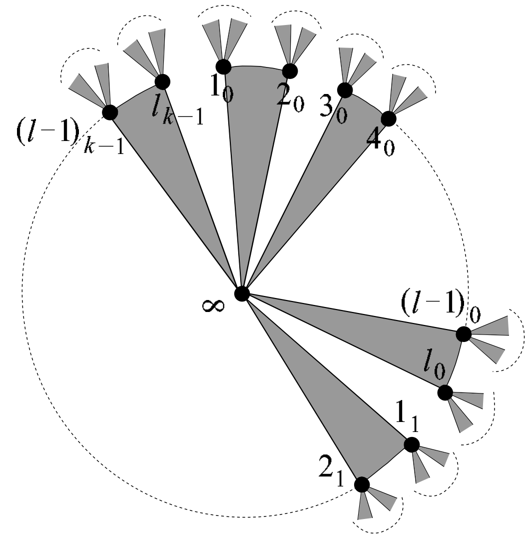

Figure 16: The embedding of K.

of H', ϕ preserves the white face of that embedding, including its orientation. It is then easy to see that the modified white face in the embedding of K, obtained by deletion of the edges $2_0 3_0, 4_0 5_0, \ldots, l_{k-1} 1_0$ and their replacement by the edges $2_0\infty, \infty 3_0, 4_0\infty, \infty 5_0, \ldots, l_{k-1}\infty, \infty 1_0$ as illustrated in Figure 16, is still invariant under ϕ. □

The second Theorem in this Section extends the previous result to the nonorientable case.

Theorem 5.2 *Suppose that S is an STS(n) with an automorphism ϕ having a single fixed point and l cycles each of length k, where k is odd and $n = kl + 1$. Then there exists a nonorientable upper embedding of S having ϕ as an automorphism.*

Proof: The proof follows closely that of Theorem 5.1. The significant change is that one of the full orbit starters is selected for special treatment. Without loss of generality we may take this full orbit starter to be $\{1_0, 2_1, \gamma_r\}$ and corresponding to this starter we add the triangle $(1, 2, \gamma)$ or $(2, 1, \gamma)$ using two crosscaps rather than one handle. If $\gamma \neq 1, 2$, we do this using the method illustrated in Figure 4, replacing u, v, w by $1, 2, \gamma$ or $2, 1, \gamma$ as appropriate. If $\gamma = 1$ or 2, then identify the corresponding point u or v in Figure 4 with the point w. The directed edges of this special triangle are still labelled with elements of the group Z_k as in Figure 13.

The special triangle contributes $\pm 2 \pmod{k}$ to the sum of the edge labels around the white face. With other edge labels assigned as before and with an appropriate choice of direction around each face, the directed sum modulo k of the edge labels around each black triangle is 0, around any black loop is $k/3$, around the grey face is 1, and around the white face is $\pm 1 \pm 2 + km/3$ where the integer m depends on the number of black loops. The $\pm$ sign on the ± 1 term is again selected so that 3 is not a divisor of the sum of the edge labels around the white face, so that this sum is coprime with k.

The remaining changes to the proof of Theorem 5.1 necessary to establish Theorem 5.2 are straightforward and mainly consist of modifying references to orientability. □

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