Hamiltonian embeddings from triangulations

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Abstract

A *Hamiltonian embedding* of K_n is an embedding of K_n in a surface, which may be orientable or non-orientable, in such a way that the boundary of each face is a Hamiltonian cycle. Ellingham and Stephens recently established the existence of such embeddings in non-orientable surfaces for $n = 4$ and $n \geq 6$. Here we present an entirely new construction which produces Hamiltonian embeddings of K_n from triangulations of K_n when $n \equiv 0$ or 1 (mod 3). We then use this construction to obtain exponential lower bounds for the numbers of nonisomorphic Hamiltonian embeddings of K_n.

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1 Introduction

A *Hamiltonian embedding* of K_n, the complete graph of order n, is an embedding of K_n in a surface, which may be orientable or non-orientable, in such a way that the boundary of each face is a Hamiltonian cycle. The recent paper by Ellingham and Stephens [5] established the existence of such embeddings in non-orientable surfaces for $n = 4$ and $n \geq 6$. In this paper we present an entirely new construction which, by surgery on a surface triangulation of K_n, generates a Hamiltonian embedding of K_n on a surface of higher genus. This novel construction may be used to establish the existence of $2^{an-o(n)}$ nonisomorphic Hamiltonian embeddings of K_n for $n \equiv 0$ or $1 \pmod 3$, where $a > 0$ is a constant. For certain residue classes of n this lower bound may be improved to $2^{an^2-o(n^2)}$.

Ringel, Youngs and others established the existence of triangulations of K_n for $n \equiv 0$ or $1 \pmod 3$ in the course of proving the Heawood map colouring conjecture [11]. To elaborate briefly: if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ then there is a triangulation of K_n in an orientable surface, while if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 3, 4$ or 7, then there is a triangulation of K_n in a non-orientable surface. In a triangulation, each face is as small as possible. At the opposite extreme, for every n there exists an embedding of K_n having a single face (see, for example, [9]). Around this single face every vertex appears $n - 1$ times. The problem of constructing Hamiltonian embeddings of K_n is intermediate between the two extremes - the face lengths are as large as possible subject to the restriction that no vertex is repeated on the boundary of any face.

In a Hamiltonian embedding of K_n, the number of faces is $n - 1$. In the non-orientable case, Euler's formula gives the genus as $\gamma = (n - 2)(n - 3)/2$. In the orientable case, the genus is $g = (n - 2)(n - 3)/4$, which implies that $n \equiv 2$ or $3 \pmod 4$ is a necessary condition for the embedding.

We assume that the reader is familiar with the basic terminology and methods of topological graph theory, such as may be found in [9, 11], in particular the representation of embeddings by rotation schemes.

2 The construction

As remarked above, for $n \equiv 0$ or $1 \pmod 3$, there exists a triangulation of K_n in a surface. Our construction starts with any such triangulation of K_n; whether the triangulation is in an orientable or non-orientable surface is immaterial. To avoid trivial cases we assume that $n \geq 4$.

Construction 2.1

Take a triangulation of K_n on the vertex set $\{\infty, a_1, a_2, \ldots, a_{n-1}\}$ and, without loss of generality, take the rotation scheme to have the following form.

$$\begin{array}{ccccccccc}
\infty & : & a_1 & a_2 & a_3 & a_4 & \ldots & a_{n-2} & a_{n-1} \\
a_1 & : & \infty & a_2 & b_{1,1} & b_{1,2} & \ldots & b_{1,n-4} & a_{n-1} \\
a_2 & : & \infty & a_3 & b_{2,1} & b_{2,2} & \ldots & b_{2,n-4} & a_1 \\
\vdots & & & & & & \vdots & & \vdots \\
a_i & : & \infty & a_{i+1} & b_{i,1} & b_{i,2} & \ldots & b_{i,n-4} & a_{i-1} \\
\vdots & & & & & & \vdots & & \vdots \\
a_{n-1} & : & \infty & a_1 & b_{n-1,1} & b_{n-1,2} & \ldots & b_{n-1,n-4} & a_{n-2}
\end{array}$$

where, for each $i = 1, 2, \ldots, n-1$, $(b_{i,1}\ b_{i,2}\ \ldots\ b_{i,n-4})$ is some permutation of $\{a_1, a_2, \ldots, a_{n-1}\} \setminus \{a_{i-1}, a_i, a_{i+1}\}$, with subscript arithmetic modulo $n-1$.

From the n lines of the rotation scheme, create $n-1$ Hamiltonian cycles by discarding the first line and, for each i, replacing the line corresponding to a_i by the cycle $A_i = (\infty a_i a_{i+1} b_{i,1} b_{i,2} \ldots b_{i,n-4} a_{i-1})$. It is easy to see that these cycles form a Hamiltonian decomposition of $2K_n$. The Hamiltonian face corresponding to A_i is formed from the triangular faces that comprise the rotation at a_i in the original triangulation, with the triangle $(\infty\ a_i\ a_{i+1})$ removed. It remains to show that these Hamiltonian faces may be sewn together along common edges to produce a Hamiltonian embedding of K_n. In order to prove this, it is only necessary to prove that the resulting rotation about any vertex comprises a single cycle of length $n-1$, rather than a set of shorter cycles with total length $n-1$. Note that a section of a cycle boundary $(\ldots\ a\ b\ c\ \ldots)$ gives rise to a part of the rotation about b having the form $b\ :\ \ldots\ c\ a\ \ldots$ (of course, the direction of the rotation at b is not determined).

Consider first the point ∞. The rotation about this point obtained from the Hamiltonian cycles is

$$\infty \quad : \quad a_1 \quad a_2 \quad \ldots \quad a_{n-1}$$

which is a single cycle of length $n-1$.

Consider next any of the remaining points, say a_i. In the original triangulation, the rotation

$$a_i \quad : \quad \infty \quad a_{i+1} \quad b_{i,1} \quad b_{i,2} \quad \ldots \quad b_{i,n-4} \quad a_{i-1}$$

implies that, in this triangulation, the rotations about $a_{i+1}, b_{i,1}, b_{i,2}, \ldots, b_{i,n-5}$, $b_{i,n-4}, a_{i-1}$ contain the following sequences.

$$\begin{array}{lclccc}
a_{i+1} & : & \ldots & b_{i,1} & a_i & \infty & \ldots \\
b_{i,1} & : & \ldots & b_{i,2} & a_i & a_{i+1} & \ldots \\
b_{i,2} & : & \ldots & b_{i,3} & a_i & b_{i,1} & \ldots \\
\vdots & & & & \vdots & & \\
b_{i,n-5} & : & \ldots & b_{i,n-4} & a_i & b_{i,n-6} & \ldots \\
b_{i,n-4} & : & \ldots & a_{i-1} & a_i & b_{i,n-5} & \ldots \\
a_{i-1} & : & \ldots & \infty & a_i & b_{i,n-4} & \ldots
\end{array}$$

These sequences for $a_{i+1}, b_{i,1}, b_{i,2}, \ldots, b_{i,n-4}$ appear in the corresponding Hamiltonian cycles, while for A_{i-1} and A_i we have

$$\begin{aligned} A_{i-1} &= (\ldots\ a_{i-1}\ a_i\ b_{i-1,1}\ \ldots) = (\ldots\ a_{i-1}\ a_i\ b_{i,n-4}\ \ldots), \\ A_i &= (\ldots\ \infty\ a_i\ a_{i+1}\ \ldots). \end{aligned}$$

These sequences enable us to construct the rotation about a_i in the embedding of the Hamiltonian cycles. For n even it is

$$a_i \quad : \quad a_{i+1}\ \infty\ b_{i,1}\ b_{i,3}\ b_{i,5}\ \ldots\ b_{i,n-5}\ a_{i-1}\ b_{i,n-4}\ b_{i,n-6}\ \ldots\ b_{i,2},$$

while for n odd it is

$$a_i \quad : \quad a_{i+1}\ \infty\ b_{i,1}\ b_{i,3}\ b_{i,5}\ \ldots\ b_{i,n-4}\ a_{i-1}\ b_{i,n-5}\ b_{i,n-7}\ \ldots\ b_{i,2}.$$

In either case, this is a cycle of length $n-1$, and this completes the verification of the construction. □

To consider the question of orientability, delete the point ∞ and the edges incident with ∞ from the embedding to obtain a single face embedding of K_{n-1} with boundary

$$(a_1 a_2 b_{1,1} b_{1,2} \ldots b_{1,n-4} a_{n-1} a_1 b_{n-1,1} b_{n-1,2} \ldots b_{n-1,n-4} a_{n-2} a_{n-1} \ldots b_{2,n-4}).$$

If, in the order given, any subsequence of the form $a_j a_{j+1}$ appears twice in this boundary then the embedding of K_{n-1}, and hence that of K_n, must be non-orientable. When the original triangulation of K_n is orientable this will happen for <u>every</u> $j = 1, 2, \ldots, n-1$. This is because each directed edge $a_j a_{j+1}$ must appear precisely once in one of the rotations $a_i\ :\ \infty\ a_{i+1}\ b_{i,1}\ b_{i,2}\ \ldots\ b_{i,n-4}\ a_{i-1}$. Thus an orientable triangulation of K_n will, by this construction, produce a non-orientable Hamiltonian embedding of K_n. Although it appears conceivable that a non-orientable triangulation might produce an orientable Hamiltonian embedding of K_n for $n \equiv 3, 6, 7$ or $10 \pmod{12}$, we have no examples of this and such situations seem likely to be rare.

3 Exponential lower bounds

In this section we establish lower bounds for the numbers of nonisomorphic Hamiltonian embeddings of K_n for n lying in certain residue classes. The supporting surface may be either orientable or nonorientable. We start with a lemma.

Lemma 3.1 *A Hamiltonian embedding of K_n, $n \equiv 0$ or $1 \pmod 3$, can be obtained from at most $2n$ distinct triangulations of K_n by means of Construction 2.1.*

Proof. Given a Hamiltonian embedding of K_n on a fixed set of n points, we check each point in turn to see if it can play the role of the point ∞ in the construction, and we show that, for each point, this can happen in at most two ways. So, take a point h^* and suppose that the rotation at h^* in the Hamiltonian embedding is

$$h^* \ : \ h_1 \ h_2 \ \ldots \ h_{n-1}$$

Then the Hamiltonian cycles may be taken as

$$H_i = (h^* \ h_i \ j_{i,1} \ j_{i,2} \ \ldots \ j_{i,n-3} \ h_{i-1})$$

where $(j_{i,1} \ j_{i,2} \ \ldots \ j_{i,n-3})$ is some permutation of $\{h_1, h_2, , \ldots, h_{n-1}\} \setminus \{h_i, h_{i-1}\}$.

In order to be derived from the construction, we must either have $j_{i,1} = h_{i+1}$ for every $i = 1, 2, \ldots, n-1$, or $j_{i,n-3} = h_{i-2}$ for every $i = 1, 2, \ldots, n-1$. In the former case, the rotations in the triangulation (assuming it exists) are determined as

$$\begin{array}{lcll} h^* & : & h_1 \ h_2 \ \ldots \ h_{n-1} & \\ h_i & : & h^* \ h_{i+1} \ j_{i,2} \ j_{i,3} \ \ldots \ j_{i,n-3} \ h_{i-1} & (i = 1, 2, \ldots, n-1). \end{array}$$

In the latter case the rotations in the triangulation (assuming it exists) must be

$$\begin{array}{lcll} h^* & : & h_1 \ h_2 \ \ldots \ h_{n-1} & \\ h_i & : & h^* \ h_{i-1} \ j_{i+1,n-4} \ j_{i+1,n-5} \ \ldots \ j_{i+1,1} \ h_{i+1} & (i = 1, 2, \ldots, n-1). \end{array}$$

The result now follows. □

We now prove a result from which lower bounds may easily be deduced.

Theorem 3.1 *If there exist M nonisomorphic triangulations of K_n, $n \equiv 0$ or 1 (mod 3), then there exist at least $M/4n^2(n-1)$ nonisomorphic Hamiltonian embeddings of K_n.*

Proof. From M nonisomorphic triangulations of K_n, it is possible to construct at least $Mn!/2n(n-1)$ distinct triangulations of K_n on a common point set by applying all possible $n!$ permutations of the points and noting that the largest possible order of an automorphism group of such a triangulation is $2n(n-1)$. From each of these distinct triangulations we may construct a Hamiltonian embedding of K_n using Construction 2.1. By Lemma 3.1, each such embedding can be obtained from at most $2n$ distinct triangulations. Hence there are at least $Mn!/4n^2(n-1)$ distinct Hamiltonian embeddings on a common point set. The largest possible size of an isomorphism class for such an embedding is $n!$. Hence there are at least $M/4n^2(n-1)$ nonisomorphic Hamiltonian embeddings of K_n. □

Corollary 3.1.1 *For $n \equiv 0$ or 1 (mod 3) there are at least $2^{n/6-o(n)}$ nonisomorphic Hamiltonian embeddings of K_n.*

Proof. For $n \equiv 0$ or 1 (mod 3), Korzhik and Voss [10] established that there are at least $2^{n/6-o(n)}$ nonisomorphic triangulations of K_n. The result follows immediately from this and the Theorem. □

Corollary 3.1.2 *For $n \equiv 1, 7$ or 9 (mod 18) there are at least $2^{n^2/54-o(n^2)}$ nonisomorphic Hamiltonian embeddings of K_n.*

Proof. Firstly we note that for each $n \equiv 3 \pmod 6$ there is a face 2-colourable triangulation of K_n having a parallel class of faces (that is, a set of faces covering all n vertices, each precisely once) in each colour class.

The orientable triangulations of K_n, $n \equiv 3 \pmod{12}$, given by Ringel [11] are face 2-colourable because the current graphs employed to construct these embeddings are bipartite. The Steiner triple systems involved in these embeddings are those produced by the Bose construction (see, for example, [3]) from the group $(\mathcal{Z}_{n/3}, +)$, and indeed a direct construction of the embeddings from these Steiner systems is given in [6]. The Bose construction produces Steiner triple systems having a parallel class, and so these orientable triangulations each contain a parallel class of faces in each colour class. Similarly, the nonorientable triangulations of K_n, $n \equiv 9 \pmod{12}$, also given by Ringel [11] are face 2-colourable since the cascades used to construct them are bipartite. As shown in [1], the Steiner triple systems involved here are also copies of Bose systems and hence the embeddings again have a parallel class of faces in each colour class. In fact, Ducrocq and Sterboul [4] also give a direct construction producing face 2-colourable triangulations of K_n in nonorientable surfaces for all $n \equiv 3 \pmod 6$, $n \geq 9$, with the Steiner triple systems involved being copies of Bose systems.

Secondly we note that for $n \equiv 1 \pmod 6$, Grannell and Korzhik [8] proved that there is a face 2-colourable triangulation of K_n in a nonorientable surface. And we also remark that for all $n \equiv 7 \pmod{12}$, Youngs [12] gives a variety of embeddings, including face 2-colourable triangulations.

Having made these prelimnary observations, we can now use two recursive constructions to produce our lower bounds.

Applying the $n \to 3n - 2$ construction for triangular embeddings given in [2] establishes that for $n \equiv 1$ or $7 \pmod{18}$ there are at least $2^{n^2/54-o(n^2)}$ nonisomorphic triangulations of K_n.

The paper [7] gives an $n \to mn$ construction for orientable triangulations but, as remarked in that paper, the method is easily extended to the nonorientable case. In the case $m = 3$ it requires a face 2-colourable triangulation of K_9 and a parallel class of faces in one of the colour classes of the original K_n triangulation. This construction then establishes the same lower bound $(2^{n^2/54-o(n^2)})$ on the number of triangulations of K_n for $n \equiv 9 \pmod{18}$.

Again the result follows immediately from these estimates and the Theorem.

□

Corollary 3.1.3 *The constant $1/54$ that appears in the exponent in Corollary 3.1.2 may be improved to $2/81$ for $n \equiv 1, 19, 25$ or 27 (mod 54)*

Proof. This follows by reapplying the $n \to 3n - 2$ and $n \to 3n$ recursive constructions for triangulations as indicated in [2] and [7]. □

Remarks.

A new recursive construction for triangulations by two of the present authors, as yet unpublished, takes a face 2-colourable triangulation of K_n and produces a face 2-colourable triangulation of K_{3n} without the need for the original triangulation to have a parallel class. This enables us to extend the result of Corollary 3.1.2 to include $n \equiv 3 \pmod{18}$ and, by reapplication, to extend the result of Corollary 3.1.3 to include $n \equiv 3, 7, 9$ and 21 (mod 54).

It is also possible to use some of the other constructions given in [7] to obtain lower bounds of the form $2^{an^2 - o(n^2)}$ for the number of Hamiltonian embeddings of K_n for certain values of n within the remaining residue classes.

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