13. Embeddings and Designs

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When a graph is embedded in a surface, the faces that result can be regarded as the blocks of a combinatorial design. The resulting design may be thought of as being embedded in the surface. This perspective leads naturally to a number of fascinating questions about embeddings, in particular about embeddings of Steiner triple systems and related designs. Can every Steiner triple system be embedded, can every pair of Steiner triple systems be biembedded, and how many embeddings are there of a given type?

1 Introduction

In this section we define the terminology taken from combinatorial design theory and summarize some of the basic results. Before doing this, we remark that the study of the relationship between block designs and graph embeddings dates back to Heffter, who in 1891 realized the connection between twofold triple systems and surface triangulations. Later work in this field was done by Emch [27], Alpert [2], White [57], Anderson and White [6], Anderson [4], [5], Jungerman, Stahl and White [40], Rahn [54], and more recently White [58]. These authors considered various aspects of the above relationship, including embeddings into closed surfaces, pseudosurfaces and generalized pseudosurfaces, embeddings of balanced incomplete block designs (BIBDs) with block size greater than 3, and the symmetry properties of the resulting embeddings. However, the material we survey in this chapter is mainly, although not exclusively, concerned with embeddings of Steiner triple systems in both orientable and nonorientable surfaces. Embeddings in pseudosurfaces and generalized pseudosurfaces will not be considered.

The reader is assumed to be familiar with embeddings of graphs in both orientable and nonorientable surfaces, and the description of such embeddings by means of rotation schemes. Some familiarity with the use of current and voltage graphs in the construction of embeddings is assumed (see Chapters 1 and 2). When referring to the number of embeddings (or other combinatorial objects), we mean the number of nonisomorphic embeddings (or objects) of the specified type.

In order to reduce the number of references, particularly for results in design theory, we give as secondary sources the books *Triple Systems* by Colbourn and Rosa [21] and *The CRC Handbook of Combinatorial Designs* edited by Colbourn and Dinitz [20].

The principal item required from design theory is the following definition. A *Steiner triple system of order* n is a pair $(V, \mathcal{B})$ where V is an n-element set (the *points*) and $\mathcal{B}$ is a collection of 3-element subsets (the *blocks*) of V such that each 2-element subset of V is contained in exactly one block of

$\mathcal{B}$. It is well known that a Steiner triple system of order n (briefly STS(n)) exists if and only if $n \equiv 1$ or 3 (mod 6). If, in the definition, the words "exactly one block" are replaced by "exactly two blocks", then we have a *twofold triple system of order* n, TTS(n) for short. A twofold triple system of order n exists if and only if $n \equiv 0$ or 1 (mod 3). If a TTS(n) has no repeated blocks, it is said to be *simple*. A (possibly non-simple) TTS(n) may be obtained by combining the block sets of two STS(n)s which have a common point set. An STS(n) can be considered as a decomposition of the complete graph K_n into triangles (copies of K_3); likewise a TTS(n) can be considered as a decomposition of the twofold complete graph $2K_n$ (in which there are two edges between each pair of vertices) into triangles.

Up to isomorphism, there is just one STS(n) for $n = 3, 7, 9$, while there are two for $n = 13$, precisely one of which is cyclic (that is, has an automorphism of order 13). There are 80 STS(15)s, of which two are cyclic, and there are 11 084 874 829 STS(19)s [41], of which four are cyclic. The number of nonisomorphic STS(n)s is $n^{n^2(\frac{1}{6}+o(1))}$ as $n \to \infty$ [59] and, again speaking asymptotically, almost all of these have only the trivial automorphism group [7].

A *transversal design of order* n *and block size* 3 is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into 3 parts (the *groups*) each of size n, and $\mathcal{B}$ is a collection of 3-element subsets (the *blocks*) of V such that each 2-element subset of V is either contained in exactly one block of $\mathcal{B}$ or in exactly one group of $\mathcal{G}$, but not both. A transversal design of order n and block size 3 is denoted by TD$(3, n)$; since we consider only block size 3, we will simply speak of a transversal design of order n. A TD$(3, n)$ may be considered as a decomposition of a complete tripartite graph $K_{n,n,n}$ into triangles, with the tripartition defining the groups of the design. A TD$(3, n)$ is equivalent to a Latin square of side n in which the triples are given by (row, column, entry).

A *Mendelsohn triple system of order* n is defined in a similar fashion to an STS(n) except that triples and pairs are taken to be ordered, so that the cyclically ordered triple (a, b, c) "contains" the ordered pairs $(a, b), (b, c)$ and (c, a). A Mendelsohn triple system of order n, MTS(n) for short, exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 6$. An MTS(n) may be considered as a decomposition of a complete directed graph on n vertices into directed 3-cycles. If the directions are ignored, then an MTS(n) gives a TTS(n).

To see the connection between design theory and graph embeddings, consider the case of an embedding of a complete graph K_n in an orientable surface in which all the faces are triangles. Taking these triangles with a consistent orientation to form a set of blocks, the faces of the embedding yield a Mendelsohn triple system of order n. Similarly, a triangulation of K_n in a nonorientable surface gives a twofold triple system of order n.

The precise correspondence between such systems and triangulations is given in [48] and involves pseudosurfaces. Our interest is in the questions: which Mendelsohn (respectively, twofold) triple systems occur as triangulations of K_n in orientable (respectively, nonorientable) surfaces? The following answer is given in [24]. Let $(V, \mathcal{B})$ be a TTS(n). For each $x \in V$, define a *neighbourhood graph* G_x: its vertex set is $V \setminus \{x\}$, and two vertices y and z are joined by an edge if $\{x, y, z\} \in \mathcal{B}$. Clearly, G_x is a union of disjoint cycles. A TTS(n) occurs as a triangulation of a nonorientable surface if and only if every neighbourhood graph consists of a single cycle. If the blocks of the TTS(n) can be ordered to form an MTS(n), then the surface is orientable. We now move on to the much more interesting relationship between embeddings and Steiner triple systems.

2 Steiner triple systems and triangulations

Let $(V, \mathcal{B})$ be an STS(n) and let K_n be the complete graph with vertex set V. By an *embedding of* $(V, \mathcal{B})$ *in a surface* S (which may be orientable or nonorientable) we mean any embedding $\phi: K_n \to S$ with the property that for each $\{u, v, w\} \in \mathcal{B}$, the 3-cycle (uvw) constitutes the boundary of some face of ϕ. For the sake of convenience, we abbreviate the above definition by just saying that in the embedding ϕ, *every triple of* $\mathcal{B}$ *is facial.* Since each edge of K_n belongs to precisely one facial triple, the faces of ϕ can be properly 2-coloured. Usually we colour the facial triples of $\mathcal{B}$ *black* and the remaining faces *white*.

Conversely, let $\psi: K_n \to S$ be an embedding whose faces can be properly 2-coloured (black and white) with all black faces bounded by 3-cycles. Then ψ is an embedding of some STS(n). Indeed, let $\mathcal{B}$ be the collection of the 3-subsets of $V = V(K_n)$ that correspond to the boundary triangles of black faces. Since our face colouring is proper, no edge is on the boundary of just one face. Thus, each edge of K_n is incident to precisely one black face, and so each pair of elements of V belongs to precisely one 3-subset of $\mathcal{B}$. Hence $(V, \mathcal{B})$ is an STS(n).

A particularly interesting case occurs when the family of all *white* faces constitutes an STS as well. Let $(V, \mathcal{B})$ and $(V, \mathcal{B}')$ be two STSs on a common point set V with $|V| = n$. We say that the pair $\{\mathcal{B}, \mathcal{B}'\}$ is *biembeddable* (or that the two STSs are biembeddable) in some surface S if there is an embedding ϕ of $(V, \mathcal{B})$ whose white faces are 3-cycles constituting the blocks of an STS isomorphic to $(V, \mathcal{B}')$. In such circumstances, ϕ is called a *biembedding*. Briefly, in a biembedding ϕ of the pair $\{\mathcal{B}, \mathcal{B}'\}$, the facial triples of $\mathcal{B}$ are black while those corresponding to $\mathcal{B}'$ are white. Necessarily, the biembedding ϕ is then a triangular embedding of a complete graph on n vertices and the surface has minimum genus. Conversely, each triangular embedding $\psi : K_n \to S$ whose faces can be properly 2-coloured induces a biembedding of a pair of STSs. In the orientable case we must then have $n \equiv 0, 3, 4$ or 7 (mod 12) (see Chapter 1) and combining this with the existence condition for STSs, we see that a pair of STSs on n points can have an orientable biembedding *only if* $n \equiv 3$ or 7 (mod 12). A similar argument in the nonorientable case shows that we must then have $n \equiv 1$ or 3 (mod 6).

We illustrate these concepts in Fig. 1, which depicts a biembedding of a pair of isomorphic STS(7)s. Specifically, $\mathcal{B} = \{013, 124, 235, 346, 450, 561, 602\}$ and $\mathcal{B}' = \{023, 134, 245, 356, 460, 501, 612\}$.

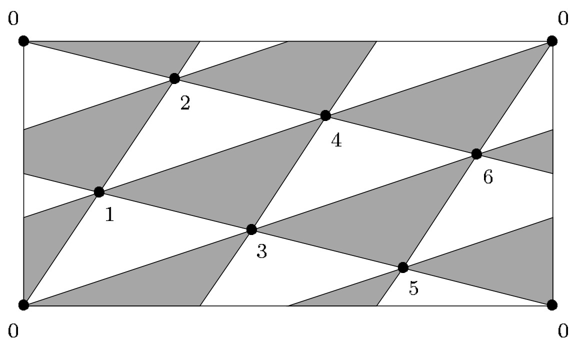

Fig. 1. A biembedding of STS(7) in the torus.

Because of the connection between biembeddings of STSs and face 2-colourable triangular embeddings of complete graphs, we recall a few facts about the latter. Constructions of minimum genus embeddings (which include triangulations) of complete graphs in orientable and nonorientable surfaces have a rich history. They form the essential part of the solution of the famous Heawood problem of determining the chromatic number of a surface, or, equivalently, determining the genus of a complete graph. Most of the solution (which also gave birth to modern topological graph theory as treated in [38]) is due to Ringel and Youngs; we recommend Chapter 1 or Ringel's book [56] for details. However, the majority of the known minimum genus embeddings of complete graphs are *not* face 2-colourable.

In the case when $n \equiv 3 \pmod{12}$, the orientable embeddings of K_n found in [56] are indeed face 2-colourable. The proof technique there uses the theory of current graphs. However, no information is yielded concerning the STS(n)s that have been biembedded. Below, in Theorem 2.1, we outline a proof of this result using exclusively design-theoretic methods. To our mind this is not only simpler and more transparent, but it also positively identifies the STS(n)s so embedded. They are those obtained from the well-known Bose construction based on a Latin square constructed as the square-root Cayley table of an odd-order cyclic group [18].

Theorem 2.1 *If $n \equiv 3$ (mod 12), then there exists a pair of biembedded Steiner triple systems of order n in some orientable surface.*

Outline of Proof. Take the additive group $\boldsymbol{Z}_{4s+1}$ and define on it the operation $\circ$ by $i \circ j = (i+j)/2 = (2s+1)(i+j)$. Use the classical Bose construction [18] to build an STS $(V, \mathcal{B})$ on the point set $V = \boldsymbol{Z}_{4s+1} \times \boldsymbol{Z}_3$. The block set $\mathcal{B}$ consists of $4s+1$ triples of the form $(i,0),(i,1),(i,2)$, $i \in \boldsymbol{Z}_{4s+1}$, together with $3 \times (4s+1)2s$ triples of the form $(i,k),(j,k),(i \circ j, k+1)$, where $i, j \in \boldsymbol{Z}_{4s+1}$, $i \neq j$ and $k \in \boldsymbol{Z}_3$.

Let $n = 12s+3$. We define two STSs $(\boldsymbol{Z}_n, \mathcal{B}_0)$ and $(\boldsymbol{Z}_n, \mathcal{B}_1)$, both isomorphic to $(V, \mathcal{B})$, using the bijections $f_m:\ V \to \boldsymbol{Z}_n$, $m = 0, 1$, given by $f_m(i,k) = 3i + (-1)^m kt$ where $t = 6s+1$; naturally, $\mathcal{B}_m = f_m(\mathcal{B})$. On the right side of the equation for $f_m(i,k)$ we have $i \in \{0, 1, \ldots, 4s\}$, $k \in \{0, 1, 2\}$, and the addition is modulo n. It can easily be checked that the two STSs are disjoint.

It remains only to check that the pair $\{\mathcal{B}_0, \mathcal{B}_1\}$ is biembeddable in an orientable surface. This is routine but somewhat tedious, and we refer the reader to the original paper [35]. ■

We note here that a similar approach (that is, constructing triangular embeddings of K_n using the Bose construction) can also be found in [23]. However, the proof given there, which applies to all $n \equiv 3 \pmod 6$, $n \geq 9$, always produces an embedding in a *nonorientable* surface.

In the case $n \equiv 7 \pmod{12}$, there are the toroidal embedding of K_7 given above and the face 2-colourable triangular embedding of K_{19} given in [57] (see also [49]). Youngs [61] produced orientable triangular embeddings of K_n by means of current assignments on ladder graphs. Amongst the variety of ladder graphs used in [61], it is possible to find, for each $n \equiv 7 \pmod{12}$, one which is bipartite [61, pp. 39–44]. Anderson [5] points out the significance of a bipartition; for our purposes this ensures that the corresponding triangular embedding is face 2-colourable. Thus it is known that there are orientable biembeddings for all $n \equiv 7 \pmod{12}$. But here also, no information is produced about the STSs that have been embedded.

In Section 3 we describe a recursive construction, a topological analogue of a well-known design-theoretic construction, which shows that such embeddings exist for half of the residue class $n \equiv 7 \pmod{12}$. This particular method has the additional advantage that it produces a large number of new embeddings, as will also be described in the next section.

3 Recursive constructions

The recursive construction that appears in Theorems 1 and 2 of [34] in a topological form, and again in Theorems 2 and 3 of [35] in a design-theoretical form, takes a biembedding of two STS(n)s and produces a biembedding of two STS($3n-2$)s. Here we give an informal description of this construction and then discuss extensions and related constructions.

The construction commences with a given biembedding of two STS(n)s, which is equivalent to a face 2-colourable triangulation of K_n. We fix a particular vertex z^* of K_n and, from the embedding, we delete z^*, all edges incident with z^* and all the triangular faces incident with z^*. The resulting surface S now has a hole whose boundary is an oriented Hamiltonian cycle in $G = K_n - z^* \cong K_{n-1}$. We next take three disjoint copies of the surface S, all with the same colouring and, in the orientable case, the same orientation; we denote these by S^0, S^1 and S^2, and use superscripts in a similar way to identify corresponding points on the three surfaces. For each white triangular face (uvw) of S, we "bridge" S^0, S^1 and S^2 by gluing a torus to the three triangles $(u^iv^iw^i)$ in the following manner. We take a face 2-colourable triangulation in a torus of the complete tripartite graph $K_{3,3,3}$ with the three vertex parts $\{u^i\}, \{v^i\}$ and $\{w^i\}$ and with black faces $(u^iw^iv^i)$, for $i = 0, 1, 2$ (see Fig. 2). In the orientable case, the orientation of the torus must induce the opposite cyclic permutation of $\{u^i, v^i, w^i\}$ to that induced by the surfaces S^i; this is important for the integrity of the gluing operation where black faces $(u^iw^iv^i)$ on the torus are glued to the white faces $(u^iv^iw^i)$ on S^0, S^1, and S^2, respectively.

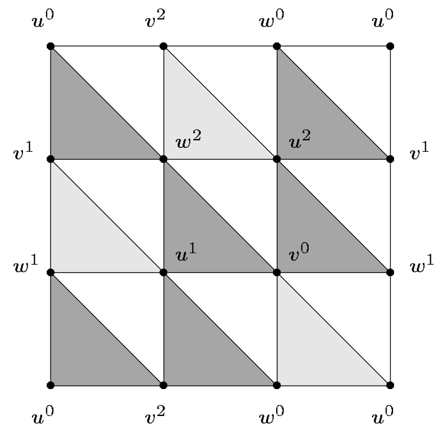

Fig. 2. Toroidal embedding of $K_{3,3,3}$.

After all the white triangles have been bridged we are left with a new connected triangulated surface with a boundary. We denote this surface by Σ. It has $3n-3$ vertices and the boundary comprises three disjoint cycles, each of length $n-1$. In order to complete the construction to obtain a face 2-colourable triangulation of K_{3n-2} (which gives a biembedding of two STS$(3n-2)$s), we must construct an auxiliary triangulated bordered surface $\bar{S}$ and paste it to Σ so that the three holes of Σ are capped. To do this, suppose that $D = (u_1 u_2 \ldots u_{n-1})$ is our oriented Hamiltonian cycle in $G = K_n - z^*$. Since n is odd, each alternate edge of D is incident with a white triangle in S; let these edges be $u_2u_3, u_4u_5, \ldots, u_{n-1}u_1$.

The surface $\bar{S}$ has, as vertices, the points u_j^i for $i = 0,1,2$ and $j = 1,2,\ldots,n-1$ together with one additional point which we call ∞. Suppose initially that $n \equiv 3 \pmod{12}$. We may then construct $\bar{S}$ from the oriented triangles listed in Table 1. The reason for the classification of the triangles into types 1 and 2 will become apparent shortly. Precisely how $\bar{S}$ is constructed is described in more detail in [34], where it is also proved that the final graph that triangulates the final surface is indeed K_{3n-2}.

Type 1 oriented triangles ($j = 1, 3, 5, \ldots, n-2$)		
White		**Black**
$(u_j^0 u_{j+1}^0 u_{j+1}^2)$	$(u_j^0 u_{j+1}^1 \infty)$	$(u_j^0 u_j^2 u_{j+1}^1)$
$(u_j^1 u_{j+1}^1 u_{j+1}^0)$	$(u_j^1 u_{j+1}^2 \infty)$	$(u_j^1 u_j^0 u_{j+1}^2)$
$(u_j^2 u_{j+1}^2 u_{j+1}^1)$	$(u_j^2 u_{j+1}^0 \infty)$	$(u_j^2 u_j^1 u_{j+1}^0)$
$(u_j^0 u_j^1 u_j^2)$		$(u_{j+1}^0 u_{j+1}^1 u_{j+1}^2)$
Type 2 oriented triangles ($j = 1, 3, 5, \ldots, n-2$)		
		Black
		$(u_{j+1}^0 u_{j+2}^0 \infty)$
		$(u_{j+1}^1 u_{j+2}^1 \infty)$
		$(u_{j+1}^2 u_{j+2}^2 \infty)$

(All subscripts are modulo $n-1$.)

Table 1.

The importance of the condition $n \equiv 3 \pmod{12}$ is that it ensures that the resulting surface is a closed surface and not a pseudosurface. Equivalently, it ensures that the neighbourhood graph of the point ∞ comprises a single cycle of $3n-3$ points rather than a union of shorter cycles. As given above, the construction does not work for $n \equiv 7 \pmod{12}$; however we can modify it by taking a single value of $j \in \{1, 3, 5, \ldots, n-2\}$ and applying a "twist" to the type 1 triangles associated with this value of j. To do this, we replace them by those shown in Table 2.

Oriented triangles		
White		**Black**
$(u_j^0 u_{j+1}^0 u_{j+1}^1)$	$(u_j^0 u_{j+1}^2 \infty)$	$(u_j^0 u_j^1 u_{j+1}^2)$
$(u_j^1 u_{j+1}^1 u_{j+1}^2)$	$(u_j^1 u_{j+1}^0 \infty)$	$(u_j^1 u_j^2 u_{j+1}^0)$
$(u_j^2 u_{j+1}^2 u_{j+1}^0)$	$(u_j^2 u_{j+1}^1 \infty)$	$(u_j^2 u_j^0 u_{j+1}^1)$
$(u_j^0 u_j^2 u_j^1)$		$(u_{j+1}^0 u_{j+1}^2 u_{j+1}^1)$

Table 2.

Again, for an explanation of why this works, see [34]. It is also there remarked that we may apply any number k of such twists provided that $k \equiv 0$ or 1 (mod 3) if $n \equiv 3$ (mod 12), and $k \equiv 1$ or 2 (mod 3) if $n \equiv 7$ (mod 12).

We now make two observations about the construction that enable us to extend it. The proof of the original construction given in [34] continues to hold good for the extended version with obvious minor modifications.

Firstly, the toroidal embedding of $K_{3,3,3}$ given in Fig. 2 may be replaced by one in which the cyclic order of the three superscripts is reversed. The reversed embedding of $K_{3,3,3}$ is isomorphic to the original but is labelled differently (see Fig. 3). For each white triangular face (uvw) of S we may carry out the bridging operation across S^0, S^1 and S^2 using either the original $K_{3,3,3}$ embedding or the reversed embedding. The choice of which of the two to use can be made *independently* for each white triangle.

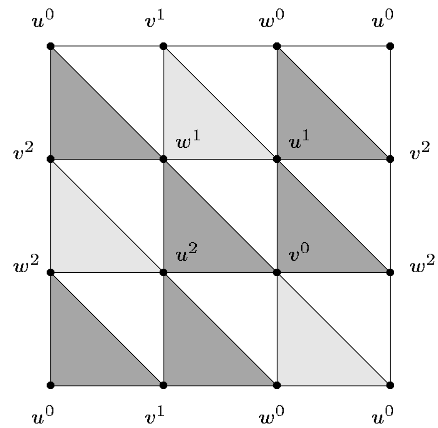

Fig. 3. Reversed toroidal embedding of $K_{3,3,3}$.

Secondly, it is not necessary for S^0, S^1 and S^2 to be three copies of the same surface S. All that the construction requires is that the three surfaces have the "same" white triangular faces and the "same" cycle of $n-1$ points around the border, all with the "same" orientations. To be more precise, by the term "same" we mean that there are mappings from the vertices of each surface onto the vertices of each of the other surfaces that preserve the white triangular faces, the border and the orientation. The sceptical reader may feel dubious that we can satisfy this requirement without in fact having three identically labelled copies of a single surface S. However, we shall see that not only is it possible to arrange this by other means, but it can often be done in many ways.

The main result of [17] is the following:

Theorem 3.1 *For* $n \equiv 7$ *or* 19 $(mod\ 36)$, *there are at least* $2^{n^2/54-O(n)}$ *nonisomorphic face 2-colourable triangulations of the complete graph* K_n, *and hence biembeddings of* STS(n)*s, in an orientable surface.*

The basic idea of the proof is to use the construction described above with three fixed copies of the same embedding of K_n and a fixed auxiliary surface $\bar{S}$, but varying the toroidal bridges. Since there are $(n-1)(n-3)/6$ bridges and two choices for each bridge, we may construct $2^{(n-1)(n-3)/6}$ differently labelled embeddings of K_{3n-2}. From this it is possible to prove that, if $\bar{S}$ is suitably chosen, these embeddings are actually nonisomorphic. Thus the number of nonisomorphic embeddings of K_{3n-2} is at least $2^{n^2/6-O(n)}$, and replacing $3n-2$ by n gives the result cited above.

We now make an observation about the black triangles of the embeddings generated as described in Theorem 3.1. In any two such embeddings, the black triangles are identical and have the same orientations. To see this, note that the black triangles come from three sources. Those lying on the surfaces S^0, S^1 and S^2 are unaltered during the construction and therefore are common to both embeddings. Those lying on the $K_{3,3,3}$ bridges are the same whether or not the bridges are reversed (see Figs. 2 and 3). Those lying on the surface $\bar{S}$ are common to both embeddings. It follows that the $2^{(n-1)(n-3)/6}$ nonisomorphic embeddings of K_{3n-2} all contain identical black triangles with the same orientations. In particular, the STS$(3n-2)$ defined by the black triangles is identical for each of these nonisomorphic embeddings.

We next show how, by reapplying the construction, we can improve the exponent in Theorem 3.1 for some residue classes. In each of the embeddings of K_{3n-2}, reverse the colours. Also, from each, delete the point ∞ together with all its incident edges and triangular faces. This produces a plentiful supply of nonisomorphic surfaces S^i on which to base a reapplication of the construction in order to produce embeddings of K_{9n-8}. All of these surfaces

S^i have the "same" white triangles and the "same" Hamiltonian cycle of points forming the border, all with the "same" orientation. We can select three different surfaces from this collection to form S^0, S^1 and S^2 (in some order) in $\binom{N}{3}$ ways, where $N = 2^{(n-1)(n-3)/6}$. We again use a suitable fixed auxiliary surface $\bar{S}$. The $K_{3,3,3}$ bridges may be selected in $2^{(3n-3)(3n-5)/6}$ different ways. Any two of the resulting embeddings of K_{9n-8} (obtained by varying the surfaces S^0, S^1 and S^2, and the $K_{3,3,3}$ bridges, but with a fixed $\bar{S}$) are nonisomorphic. These results lead to the next theorem.

Theorem 3.2 *For $n \equiv 19$ or 55 (mod 108), there are at least $2^{2n^2/81-O(n)}$ nonisomorphic face 2-colourable triangulations of the complete graph K_n, and hence biembeddings of* STS(n)*s, in an orientable surface.*

It is also shown in [17] that each of these embeddings has only the trivial automorphism.

Similar results may be obtained in the nonorientable case. We form S^0, S^1 and S^2 from three face 2-colourable embeddings (having the "same" white triangles and the "same" cycle of points around ∞) of K_n in a nonorientable surface. The white triangles are bridged using the toroidal embeddings given in Figs. 2 and 3. The construction is completed, to form a face 2-coloured triangular embedding of K_{3n-2} in a nonorientable surface, by forming a cap $\bar{S}$ with k twists, in the manner previously described. We must select $k \equiv 1$ or $2 \pmod 3$ if $n \equiv 1 \pmod 6$, and $k \equiv 0$ or $1 \pmod 3$ if $n \equiv 3 \pmod 6$.

It was stated in [17] that there is a face 2-colourable triangular embedding of K_n in a nonorientable surface for each $n \equiv 1$ or $3 \pmod 6$ with $n \geq 9$. At the time that paper was published, this does not seem to have been proved. However, the error was made good in [37], where the result was established. This enables us to state the following theorem from [17].

Theorem 3.3 *If $n \equiv 1$ or 7 (mod 18) and $n \geq 25$, then there are at least $2^{n^2/54-O(n)}$ nonisomorphic face 2-colourable triangulations of the complete graph K_n, and hence biembeddings of* STS(n)*s, in a nonorientable surface.*

Once again we can make a colour reversal and then reapply the construction to form a face 2-colourable triangular embedding of K_{9n-8} in a nonorientable surface. Similar arguments to those given previously lead to the following theorem, again an amended version of a result of [17].

Theorem 3.4 *If $n \equiv 1$ or 19 (mod 54) and $n \geq 73$, then there are at least $2^{2n^2/81-O(n)}$ nonisomorphic face 2-colourable triangulations of the complete graph K_n, and hence biembeddings of* STS(n)*s, in a nonorientable surface.*

Again, all of the embeddings of Theorem 3.4 are automorphism-free.

To conclude in this section we refer to some of the results given in [36]. One result generalizes the construction given above, and two other recursive constructions are presented. The generalization extends the earlier construction in suitable circumstances to produce biembeddings of two STS($m(n-1)+1$)s from those of STS(n)s. Another construction parallels a well-known product construction for Steiner triple systems to produce a biembedding of STS(mn)s from one of STS(n)s, again subject to certain conditions. The third construction deals with biembeddings of transversal designs, defined in an analogous manner to biembeddings of Steiner triple systems, so that suitable face 2-colourable triangulations of $K_{n,n,n}$ and $K_{m,m,m}$ yield face 2-colourable triangulations of $K_{mn,mn,mn}$. All three constructions can be employed to give estimates, of a similar form to those above, of the numbers of nonisomorphic embeddings of K_n and of $K_{n,n,n}$ for values of n in certain residue classes. It is worth noting that face 2-colourable triangulations of $K_{n,n,n}$, equivalent to biembeddings of transversal designs TD$(3, n)$, are necessarily orientable as a consequence of the tripartition. For face 2-colourable triangulations of K_n, the estimates apply to both the orientable and the nonorientable cases.

4 Small systems

In this section we briefly summarize the current state of knowledge about biembeddings of STS(n)s for $n = 3, 7, 9, 13$, and 15. We re-emphasize that when referring to the number of biembeddings, we mean the number of nonisomorphic biembeddings of the specified type. When speaking of automorphisms, we include those that exchange the colour classes and, in the orientable case, those that reverse the orientation. The case $n = 3$ is trivial, since there is only one biembedding: it is orientable and has the automorphism group S_3. The case $n = 7$ is less trivial, but well known. There is again only one biembedding: it is orientable and its automorphism group is the affine general linear group $AGL(1, 7)$ which has order 42. In the realization shown in Fig. 1, this is $\langle z \to az + b,\ a, b \in GF(7), a \neq 0\rangle$. The automorphisms of even order exchange the colour classes but preserve the orientation.

For $n = 9$ and $n = 13$, all biembeddings are necessarily nonorientable. The results for $n = 9$, $n = 13$, and some of those for $n = 15$ were obtained by computer search. Two fixed systems are selected, one forming the black system and permutations of the other giving potential white systems. Since the black and the white systems cannot have a common triple, permutations giving rise to a common triple are discarded. This approach facilitated exhaustive searches in the cases $n = 9$ and $n = 13$, and partial searches in the case $n = 15$. The results cited below come from [28], [10], [11], [13] and [14], and most also appear in [8].

For $n = 9$, the biembedding is unique and has the automorphism group $C_3 \times S_3$ of order 18. A realization is obtained by taking one system with block set $\{012, 345, 678, 036, 147, 258, 048, 156, 237, 057, 138, 246\}$ and the other obtained from this by applying the permutation $\pi = (0\ 1)(2\ 6)(4\ 7)(3)(5)(8)$. In this realization, the permutations π and $(0\ 6\ 7)(1\ 8\ 4\ 3\ 2\ 5)$ generate the automorphism group. The automorphisms of even order exchange the colour classes.

There are two STS(13)s, one is cyclic and the other is not. We refer to these here as C and N, respectively. There are 615 biembeddings of C with C, of which 36 have an automorphism group of order 2 and four have an automorphism group of order 3; the rest have only the trivial automorphism. There are 8539 biembeddings of C with N, of which ten have an automorphism group of order 3 and the rest have only the trivial automorphism. Finally, there are 29454 biembeddings of N with N, of which 238 have an automorphism group of order 2 and the rest have only the trivial automorphism. In each case, automorphisms of order 2 exchange the colour classes. We also note a paper of Ellingham and Stephens [25] in which they determine all nonorientable triangulations of K_{12} and K_{13}, the latter including all face 2-colourable triangulations, that is, biembeddings of STS(13)s. The numbers are 182 200 for K_{12} and 243 088 286 for K_{13}. For K_{12} there are, in addition, 59 orientable triangulations, [3].

There are 80 nonisomorphic STS(15)s; a standard numbering and some of their structural features are given in [51]. A computer search has shown that each pair may be biembedded nonorientably [14], so there are at least 3240 nonorientable biembeddings (actually, far more). Almost all of those found have only the trivial automorphism group.

Turning to orientable biembeddings of the STS(15)s, we firstly observe that there are precisely three systems with an automorphism of order 5. Each of these systems has an embedding with itself having an automorphism group of order 10. One of these was originally given by Ringel [56] and can also be obtained from Theorem 2.1. The other two may be obtained by Ringel's method from current graphs [11]. In each case the automorphism of order 2 has a single fixed point, exchanges the colour classes, but preserves the orientation. In [15] a computer search for biembeddings of the 80 systems, each with itself, was based on examining all possible automorphisms of order 2 with a single fixed point and exchanging the colour classes. As a result, it was shown that 78 of the 80 systems have orientable biembeddings of this type. The exceptions are the systems numbered #2 and #79 in the standard listing. In the case of #2, it was further shown in [15] not to have an orientable biembedding with itself, and in [13] it was also shown that #1 cannot be orientably biembedded with #2. Again in [15], it was shown that if #79 biembeds with itself, then the biembedding can only have the trivial automorphism group. However more recent and, at the time of writing, unpublished work by the present authors and Martin Knor has disposed of this possibility. Hence we can state the following theorem.

Theorem 4.1 *Of the* 80 *nonisomorphic* STS(15)*s,* 78 *have a biembedding with themselves in an orientable surface. The two exceptions which have no such biembedding are* #2 *and* #79 *in the standard listing.*

An orientable biembedding of system #79 with system #77 having an automorphism of order 3 is also given in [15] and is the first known example of a biembedding of a pair of nonisomorphic STS(15)s, though as described in Section 5, there are already many known biembeddings of pairs of nonisomorphic STS(n)s for $n = 19$ and $n = 31$. Again, with Martin Knor, we have established a programme to find further such biembeddings. Of particular interest is whether there exists a biembedding of system #2 with some other system. In fact we have discovered such a biembedding and hence can state another theorem.

Theorem 4.2 *Each of the* 80 *nonisomorphic* STS(15)*s has a biembedding with some* STS(15) *in an orientable surface.*

5 Cyclic embeddings

A triangular embedding of a graph in a surface may be described by means of a rotation scheme (see Chapter 1). Given a vertex x of the graph, the rotation about x comprises the cyclically ordered list of the vertices adjacent to x, taken in the order in which they appear around x in the embedding. The *rotation scheme* for the embedding is the set of all the vertices together with their rotations. In the orientable case, the rotations may be taken with a consistent orientation, that is, all clockwise or all anticlockwise. A rotation scheme is *cyclic* or *of index 1* if we can denote the vertices by $0, 1, \ldots, n-1$ in such a way that the rotation about x is obtained by adding $x \pmod{n}$ to the rotation about 0. As we observed earlier, an orientable biembedding of two STS(n)s corresponds to a face 2-colourable triangular embedding of the complete graph K_n in an orientable surface, and it requires that $n \equiv 3$

or 7 (mod 12). In the case where $n \equiv 3$ (mod 12), a cyclic STS(n) contains a unique short orbit and consequently there can be no cyclic biembeddings.

In [61], Youngs gives a cyclic orientable biembedding for all $n \equiv 7$ (mod 12), and it is this case that we consider here. We take as our starting point the result of [56] that every cyclic orientable embedding of K_{12s+7} can be derived from an appropriate *current graph* with $4s + 2$ vertices. In our context, a current graph is a graph with *directions* (clockwise or anticlockwise) assigned at each vertex and whose edges are assigned both a direction (in the ordinary sense of the word) and a *current*, the current being a non-zero element of the group $\boldsymbol{Z}_{12s+7}$. An example for $s = 2$ is shown in Fig. 4.

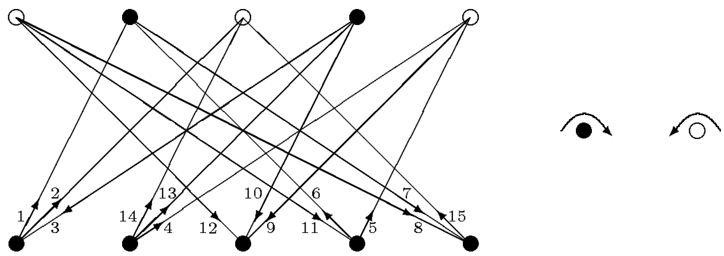

Fig. 4. A current graph for $s = 2$.

The rotation about 0 in the resulting embedding of K_{31} is obtained by traversing the graph, recording the (directed) currents encountered on each edge, and taking the clockwise or anticlockwise exit from each edge as indicated at that vertex. Thus we obtain the permutation

1 7 15 29 28 10 22 26 20 8 24 25 5 27 14 16 23 12 21 18 4 9 19 11 6 30 2 17 13 3.

The rotation about the vertex x is then obtained by adding x (mod 31) to each entry in this permutation. (A full explanation of current graphs is given in [38].) In the case where we are seeking an orientable biembedding of two STS($12s + 7$)s the current graph must have the following properties.

(i) Each vertex has degree 3. (The graph is cubic.)

(ii) At each vertex, the sum of the directed currents is 0 (mod $12s+7$) (*Kirchoff's current law*).

(iii) Each of the elements $1, 2, \ldots, 6s+3$ of $\boldsymbol{Z}_{12s+7}$ appears exactly once as a current on one of the edges and each edge has exactly one of these currents.

(iv) The directions (clockwise or anticlockwise) assigned to each vertex are such that a *complete circuit* is formed, that is, one in which every edge of the graph is traversed in each direction exactly once.

(v) The graph is bipartite.

Properties (i) and (ii) ensure that the embedding is a triangulation, while properties (iii) and (iv) ensure that it is cyclic (see [56] and [38] for further details). Property (v) ensures that the embedding is face 2-colourable and therefore represents a biembedding of two STS($12s+7$)s. Consideration of the degree and the currents shows that these current graphs have $4s+2$ vertices. Furthermore, there can be no loops and (save for the exceptional case $s=0$) no multiple edges. This last fact follows from consideration of the configuration shown in Fig. 5.

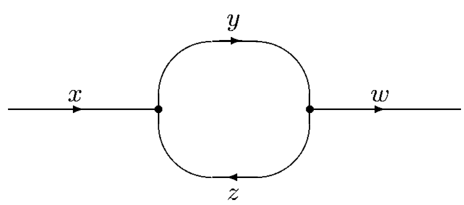

Fig 5. A possible multiple edge.

If this forms part of a current graph then $w \equiv x$ and so the whole current graph comprises two vertices with a triply repeated edge.

There is a close connection between current graphs and solutions of *Heffter's first difference problem* (HDP). In 1897 Heffter [39] asked whether the integers $1, 2, \ldots, 3k$ can be partitioned into k triples $\langle a, b, c \rangle$ such that, for each triple, $a + b \pm c \equiv 0 \pmod{6k+1}$. Examination of the triples formed by the directed currents at each vertex in either of the two vertex sets of a bipartite current graph shows that they form a solution to HDP for $k = 2s + 1$.

In view of the above observations, the problem of constructing cyclic orientable biembeddings of STS($12s + 7$)s, $s > 0$, may be reduced to three steps.

- Identify cubic bipartite simple connected graphs having $4s+2$ vertices.
- Assign directions (clockwise or anticlockwise) at each of the vertices which then give rise to a complete circuit.
- Take two solutions of HDP and label the edges of the graph in such a way that the triples arising from each of the vertex sets of the bipartition correspond to these two solutions.

These three steps have a large measure of independence from one another. However, we cannot exclude the possibility that for a particular graph it may be impossible to assign vertex directions to give a complete circuit, and, even if this is possible, it may not be possible to assign the HDP solutions to the edges. A test for the existence of a complete circuit in a graph G was given by Xuong [60]. It asserts the existence of such a circuit (equivalent to a one-face orientable embedding of G) if and only if G

has a spanning tree whose co-tree has no component with an odd number of edges.

Before proceeding further, we recall how Steiner triple systems arise from solutions to HDP. Given a difference triple $\langle a, b, c\rangle$ with $a + b \pm c \equiv 0 \pmod{6k+1}$, we can form a cyclic orbit by developing the starter $\{0, a, a + b\}$ or the starter $\{0, b, a+b\}$. By taking all the difference triples from a solution of HDP and forming a cyclic orbit from each, a cyclic STS$(6k+1)$ is obtained. The converse is also true: given a cyclic STS$(6k + 1)$, we can obtain a solution to HDP by taking from each orbit a block $\{0, \alpha, \beta\}$ and forming the difference triple $\langle \hat{\alpha}, \widehat{\beta - \alpha}, \hat{\beta}\rangle$, where

$$\hat{x} = \begin{cases} x & \text{if} \quad 1 \le x \le 3k, \\ 6k + 1 - x & \text{if} \quad 3k + 1 \le x \le 6k. \end{cases}$$

Each solution to HDP produces 2^k different STS$(6k+1)$s; however, there may be isomorphisms between these systems. In addition, for a given value of k, there will generally be many distinct solutions to HDP. For example, in [19] it is shown that for $k = 3$ there are four solutions to HDP, and these produce 4×2^3 distinct STS(19)s which lie in four isomorphism classes.

For $n = 19$, all the computations may be done by hand. The only cubic bipartite graph on six vertices is $K_{3,3}$. Fixing the rotation about one vertex of $K_{3,3}$, there are twelve ways of assigning vertex directions to produce a complete circuit. It is also easy to show that from the four solutions to HDP, it is only possible to obtain (up to isomorphism), one pair of solutions with which to label the edges of $K_{3,3}$ as described above. The resulting cyclic orientable biembeddings of STS(19)s are then found to lie in just eight isomorphism classes. The four cyclic STS(19)s are cyclically biembeddable, but none is cyclically biembeddable with itself. These embeddings were first given in [35] and further details of the argument sketched here appear in [12].

For $n = 31$, the computations require a computer. There are two cubic bipartite graphs on 10 vertices and they may be obtained from $K_{5,5}$ by either removing a single 10-cycle, or a 6-cycle and a 4-cycle. Fixing the direction at one vertex gives a total of 160 sets of vertex directions in the former case and 128 sets of vertex directions in the latter case which result in complete circuits. Using the list of all solutions of HDP for $k = 5$ given in [19], we find 2408 isomorphism classes for cyclic orientable biembeddings of STS(31)s. There are 80 cyclic STS(31)s (see [22]), of which 76 are cyclically biembeddable. Of the 2408 isomorphism classes, 64 represent biembeddings of a system with itself and these involve 44 distinct systems. These were first given in [9] and further details of the argument again appear in [12].

For $n = 43$, there are 13 cubic bipartite graphs on 14 vertices to consider [55]. Of these, two have edge-connectivity 2, and so cannot have currents assigned along their edges that are different as required by property (iii) above. This is because the current in one of the two edges of the cutset would have to be equal (but opposite in direction) to that in the other. The 11 remaining graphs admit direction and current assignments. Further details are given in [12].

Before leaving this section we remark that [12] gives theoretical reasons, based on the above analysis, why certain pairs of cyclic STS(n)s cannot be *cyclically* biembedded together in an orientable surface. These are sufficient to give a complete explanation of cyclic biembeddability for $n = 19$ and 31, but not in general.

6 Concluding remarks

In this final section, we note a variety of results related to our main theme of embedding Steiner triple systems. We also review some open problems.

A particular interest of design theorists is the concept of a *trade*. Informally, this is a set T_1 of blocks of an STS(n) for which it is possible to find a *disjoint* set T_2 of blocks (not lying in the system) which cover exactly the same pairs of points. The original STS(n) may then be transformed to a different, but possibly isomorphic, STS(n) by replacing T_1 by T_2. In [33] and [16], the authors investigate the analogous topological equivalent, where one set of triangles is replaced by a different set covering the same edges. In the recursive construction of Section 2, the replacement of a toroidal bridge (Fig. 2) by the reversed bridge (Fig. 3) is an example of such a topological trade. Indeed, Theorem 3.1 may be considered as a result about topological trades. Minimal trades between two orientable, between two nonorientable, and between orientable and nonorientable triangulations of K_n are determined in [33]. All topological trades involving m triangular faces for $m \leq 6$ are described in [16]; there are none for $m = 1, 2, 3$ and 5, one for $m = 4$ and four for $m = 6$, each of which must lie in a limited number of geometrical patterns.

For n lying in certain residue classes, Theorem 3.1 gives a lower bound of the form 2^{an^2} for the number of face 2-colourable triangulations of K_n in both orientable and nonorientable surfaces. Can this bound be extended to all possible residue classes? Is this the true order of growth? In a series of papers Korzhik and Voss ([45], [46], [47] and [43]) prove that for sufficiently large n, the number of orientable and nonorientable minimum genus embeddings of K_n is at least $c2^{\beta n}$, where $c > 0$ and $\beta > \frac{1}{12}$ are constants.

Minimum genus embeddings of K_n are triangulations when $n \equiv 0$ or 1 (mod 3) in the nonorientable case, and when $n \equiv 0, 3, 4$ or 7 (mod 12) in the orientable case. In the remaining cases, minimum genus embeddings of K_n are near-triangulations, having a small number of non-triangular faces. The basic technique employed in these papers relies on the use of appropriate current graphs. Although these results cover all residue classes, the bound is a long way from 2^{an^2}. In a more recent development, Korzhik and Kwak [44] combined the current graph approach with the cut-and-paste technique of Theorem 3.1 to prove that if $12s+7$ is prime and if $n = (12s+7)(6s+7)$, then the number of nonorientable triangulations of K_n is at least $2^{n^{3/2}(\sqrt{2}/72+o(1))}$.

Mao, Liu and Tian [50] give formulas for the numbers of embeddings of K_n in an orientable and in a nonorientable surface. However, no information is obtained about facial properties of the embeddings. We can obtain an upper estimate of the number of face 2-colourable triangulations by using the known upper bound for the number of labelled Steiner triple systems of order n, namely $(e^{-1/2}n)^{n^2/6}$, [59]. Each labelled face 2-colourable orientable triangulation of K_n gives rise to a pair of labelled STS(n)s, "white" and "black". There are $2^{n(n-1)/6}$ possible choices for the orientations of the white triangles (that is, the blocks of the white system). Each such choice determines the orientation of the corresponding black triangles, so the number of labelled face 2-colourable orientable triangulations of K_n is at most $(e^{-1/2}n)^{n^2/6}(e^{-1/2}n)^{n^2/6}2^{n(n-1)/6} < n^{n^2/3}$. Consequently, the number of nonisomorphic face 2-colourable orientable triangulations of K_n is less than $n^{n^2/3}$. A similar argument may be applied in the nonorientable case. Unfortunately, there seems to be no simple way of using this type of argument to establish a lower bound, since an arbitrary pair of labelled STS(n)s is not, in general, biembeddable as the black and white systems of a face 2-colourable orientable triangulation of K_n, no matter what orientations are chosen for the blocks (for example, the systems may have a common triple). If the rate of growth of the number of nonisomorphic face 2-colourable triangulations of K_n were of the order 2^{an^2}, then this would imply that almost all STS(n)s are *not* biembeddable either orientably or nonorientably. Evidence culled from investigations of the 80 STS(15)s emboldens us to conjecture that, for $n \geq 9$, each pair of STS(n)s is biembeddable in a nonorientable surface. However, it seems that the same may not be true in the orientable case, even with a finite number of exceptions; indeed, it is unclear whether, for each $n \equiv 3$ or 7 (mod 12), each STS(n) has an orientable biembedding with *some* other STS(n).

Lastly, one might reasonably consider variants of the above problems. For instance, embeddings of K_n could be sought in which the faces are all m-gons for some $m > 3$. In the case $m = n$, the faces are Hamiltonian cycles, and Ellingham and Stephens [26] have shown that such Hamiltonian embeddings exist in a nonorientable surface for $n \geq 4, n \neq 5$, and that for n odd the embeddings may be taken to be face 2-colourable. For graphs other than K_n a variety of results have been obtained. In particular, triangulations of $K_{n,n,n}$ and related Hamiltonian embeddings of $K_{n,n}$ are given in [1], [29], [30], [32] and [42], while triangulations related to biembeddings of symmetric configurations of triples (see [20] for terminology) appear in [52], [53] and [31].

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