New type B colourable $S(2,4,v)$ designs

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Abstract

An $S(2,4,v)$ design has a type B χ-colouring if it is possible to assign one of χ colours to each point such that each block contains three points of one colour and one point of a different colour, and all χ colours are used. In this paper we describe the constructions of type B χ-colourable $S(2,4,v)$s for $(v,\chi) = (61,3)$, $(100,2)$ and $(109,3)$, and we give a new general construction.

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1 Introduction

A *Steiner system*, $S(t, k, v)$, is a pair $(V, \mathcal{B})$ where V is a set of cardinality v of *elements*, or *points*, and $\mathcal{B}$ is a collection of k-subsets of V, also called *blocks*, which has the property that every t-element subset of V occurs in precisely one block. In this paper we are concerned only with the cases $t = 2$ and $k = 3$ or 4. An $S(2, 3, v)$ is usually called a *Steiner triple system of order* v, or STS(v) for short. An STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6) [5], and an $S(2, 4, v)$ exists if and only if $v \equiv 1$ or 4 (mod 12) [4]. Note that $v = 1$ is admissible in both cases—the STS(1) and the $S(2, 4, 1)$ each consists of a single point and an empty set of blocks. A resolvable Steiner triple system is an STS(v) whose blocks can be partitioned into $(v - 1)/2$ *resolution classes* $\mathcal{B}_i$, $i = 1, 2, \ldots, (v - 1)/2$, where $|\mathcal{B}_i| = v/3$ and $\mathcal{B}_i$ covers the entire point set. A *Kirkman triple system of order* v, KTS(v), is a resolvable STS(v) with a specified partition into resolution classes.

In this paper, we are interested in colourings of Steiner systems. A χ-*colouring* of a Steiner system $(V, \mathcal{B})$ is a surjection $\phi : V \to \Gamma$ where Γ is a set of cardinality χ whose elements are called *colours*. In the case of a Steiner triple system, each block will have one of three colour patterns: $\{a, a, a\}$ (type A), $\{a, a, b\}$ (type B), or $\{a, b, c\}$ (type C). Let $X \subseteq \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$. A χ-colouring of type X is a colouring as defined above in which each block is of type I for some $I \in X$. We will also require that every block type of X must occur. Thus there are eight possible combinations of colourings although some of these are trivial. There exists an extensive literature on colourings of Steiner triple systems.

In [7], the authors extend the above ideas to colourings of Steiner systems $S(2, 4, v)$. Here there are five colour patterns: $\{a, a, a, a\}$ (type A), $\{a, a, a, b\}$ (type B), $\{a, a, b, b\}$ (type C), $\{a, a, b, c\}$ (type D) and $\{a, b, c, d\}$ (type E), and consequently 32 possible combinations of colourings. Of these, perhaps the most natural are those in which the set X consists of just one type. The cases where $X = \{\mathrm{A}\}$ or $X = \{\mathrm{E}\}$ are trivial. Moreover, it was shown in [7] that an $S(2, 4, v)$ has a χ-colouring of type $X = \{\mathrm{C}\}$ if and only if $v = 4$ and $\chi = 2$. That leaves colourings where $X = \{\mathrm{B}\}$ or $X = \{\mathrm{D}\}$. The former case is of particular interest and we will refer to such colourings simply as type B χ-colourings.

In the next section we review some properties of $S(2, 4, v)$ systems and their type B χ-colourings. If the colour set is $\Gamma = \{\Gamma_1, \Gamma_2, \ldots \Gamma_\chi\}$, we let $\gamma_i = |\phi^{-1}(\Gamma_i)|$, $i = 1, 2, \ldots, \chi$, the colour class sizes. However, in describing constructions it is more convenient to use unsubscripted letters for the elements of Γ, in which case we would, for instance, refer to the members of $\phi^{-1}(X)$ as 'X points'.

2 Type B χ-colourable $S(2,4,v)$ systems

The first lemma, which is inherent in [7] and is easy to prove, provides important information on the structure of Steiner systems $S(2,4,v)$ with type B χ-colourings.

Lemma 2.1 *Suppose $S = (V, \mathcal{B})$ is an $S(2,4,v)$ with a type B χ-colouring $\phi : V \to \Gamma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_\chi\}$. For $i = 1, 2, \ldots, \chi$, let $V_i = \phi^{-1}(\Gamma_i)$; then*

$$(V_i, \{\{a,b,c\} : \{a,b,c,d\} \in \mathcal{B}, \{a,b,c\} \subseteq V_i\}) \tag{1}$$

is a Steiner triple system of order $|V_i|$.

Proof. If $|V_i| > 1$ and $\{a,b\} \subseteq V_i$, then a and b must both occur in a block of S together with precisely one other point of the same colour. On the other hand, if $|V_i| = 1$, then $(V_i, \emptyset)$ forms an STS(1). □

The next two lemmas come directly from [7].

Lemma 2.2 *Let $S = (V, \mathcal{B})$ be an $S(2,4,v)$ with $V = \{q_1, q_2, \ldots, q_v\}$, and let $(K, \mathcal{C})$, $K \cap V = \emptyset$, be a KTS$(2v+1)$ with resolution classes $\mathcal{C}_1$, $\mathcal{C}_2$, ... , $\mathcal{C}_v$. Let*

$$\mathcal{Q} = \bigcup_{i=1}^{v} \{\{x,y,z,q_i\} : \{x,y,z\} \in \mathcal{C}_i\}.$$

Then $S' = (V \cup K, \mathcal{B} \cup \mathcal{Q})$ is an $S(2,4,3v+1)$. Furthermore, if S is type B χ-colourable, then S' is type B $(\chi+1)$-colourable.

Proof. This is the well-known $3v+1$ construction. Also it is plain that assigning a $(\chi+1)$-th colour to the points of the Kirkman triple system results in a valid type B $(\chi+1)$-colouring of S'. □

Remark. We will show in Theorem 2.1 that, by careful choice of the KTS$(2v+1)$, it is generally possible to obtain an alternative type B $(\chi+1)$-colouring pattern for the $S(2,4,3v+1)$.

Lemma 2.3 *For* $v = (3^\chi - 1)/2$, $\chi = 2, 3, \ldots$, *there exists a type B* χ*-colourable* $S(2, 4, v)$.

Proof. Clearly, the $S(2, 4, 4)$ system has a type B 2-colouring. Apply Lemma 2.2 recursively to obtain systems with orders given as follows.

χ	2	3	4	5	6	7	$\ldots$	χ
v	4	13	40	121	364	1093	$\ldots$	$\frac{1}{2}(3^\chi - 1)$

For the Kirkman triple system, one can use the affine STS(3^χ) and resolution classes described in [1, pp 149–150]. □

Until recently, it seemed that Lemma 2.3 accounted for the only known examples of type B χ-colourable $S(2, 4, v)$ systems. All such systems, however, must satisfy the conditions in the next lemma.

Lemma 2.4 *Let* $(V, \mathcal{B})$ *be a type B* χ*-colourable* $S(2, 4, v)$ *with colour class sizes* $\gamma_1, \gamma_2, \ldots, \gamma_\chi$. *Then*

(i) for $i = 1, 2, \ldots, \chi$, $\gamma_i \equiv 1$ *or* $3 \pmod 6$, *with precisely one* $\gamma_i \equiv 1 \pmod 6$*;*

(ii) $\sum_{i=1}^{\chi} \binom{\gamma_i}{2} = \sum_{1 \le i < j \le \chi} \gamma_i \gamma_j = \frac{1}{4} v(v - 1)$*;*

(iii) for $i = 1, 2, \ldots, \chi$, $\gamma_i \le \frac{1}{3}(2v + 1)$*;*

(iv) $\gamma_i = \frac{1}{3}(2v + 1)$ *for some* i *if and only if the* $S(2, 4, v)$ *can be obtained from an* $S(2, 4, v - \gamma_i)$ *via Lemma 2.2;*

(v) for $0 \le i < j \le \chi$, $(\gamma_i - \gamma_j)^2 \ge \gamma_i + \gamma_j$.

Proof. For items (i)–(iii), see Lemma 3.7 of [7]; and (iv) follows easily from the proof of Lemma 2.2, above. For (v), denote the i-th colour class by Γ_i and observe that the $\gamma_i \gamma_j$ $\{\Gamma_i, \Gamma_j\}$ pairs , $i \ne j$, must come from blocks of the form $\{\Gamma_i, \Gamma_i, \Gamma_i, \Gamma_j\}$ or $\{\Gamma_j, \Gamma_j, \Gamma_j, \Gamma_i\}$. Hence

$$\frac{1}{2}\gamma_i(\gamma_i - 1) + \frac{1}{2}\gamma_j(\gamma_j - 1) \ge \gamma_i \gamma_j.$$

Note in particular that one cannot have two equal colour class sizes. □

Table 1: Parameters of possible type B χ-colourable $S(2,4,v)$s

v	χ	$\gamma_1, \gamma_2, \ldots, \gamma_\chi$	v	χ	$\gamma_1, \gamma_2, \ldots, \gamma_\chi$
61	3	3, 19, 39	313	5	1, 3, 9, 105, 195
100	2	45, 55	328	4	1, 3, 135, 189
109	3	1, 45, 63	328	4	1, 45, 63, 219
184	4	1, 9, 57, 117	328	4	9, 15, 91, 213
184	4	3, 19, 39, 123	361	5	1, 9, 21, 93, 237
196	2	91, 105	361	5	3, 9, 15, 99, 235
232	4	3, 9, 73, 147	397	3	19, 129, 249
232	4	3, 19, 57, 153	424	4	9, 19, 123, 273
301	3	1, 135, 165	457	5	3, 15, 27, 109, 303
301	3	9, 109, 183	484	2	231, 253
301	3	33, 69, 199	505	5	3, 9, 21, 147, 325
301	3	45, 55, 201	505	5	9, 15, 21, 127, 333

For $v \leq 505$, the only possible parameter sets satisfying Lemma 2.4, other than those arising from Lemma 2.3, are given in Table 1.

The existence of a type B 2-colourable $S(2,4,100)$ is a long-standing problem. It was already raised by de Resmini [8] in a paper published in 1981. It is the first non-trivial system of a potential infinite sequence of type B 2-colourable Steiner systems $S(2,4,v)$ where $v = (12s+2)^2$ or $v = (12s+10)^2$, $s \geq 0$, and the two colour class sizes are $(v \pm \sqrt{v})/2$. In [7], the explicit issue of finding a type B 3-colourable $S(2,4,v)$ for each of $v = 61$ and 109 was formulated. All of these problems were restated in [9]. The closely related problem of embedding Steiner triple systems into $S(2,4,v)$ systems is discussed at length in [6].

A type B 2-colourable $S(2,4,100)$ was recently constructed by the authors and appears in [3]. In the current paper we give constructions for other systems listed in Table 1; specifically for $v = 61$ and 109. Also we briefly describe the system for $v = 100$ from [3] for completeness. We make no claim that any of the systems are unique up to isomorphism for their types. Indeed, for $v = 109$, which we found to be the easiest to construct, we have several systems and we list two examples. Further systems arise from repeated use of Theorem 2.1.

Theorem 2.1 may be regarded as an extension of Lemma 2.2, where the KTS$(2v+1)$ is obtained from the $S(2,4,v)$ and the resulting $S(2,4,3v+1)$ then generally has two alternative type B $(\chi+1)$-colouring patterns.

Theorem 2.1 *Let S be a type B χ-colourable $S(2,4,v)$ system with colour class sizes $\{\gamma_1,\ \gamma_2,\ \ldots\ ,\ \gamma_\chi\}$. Then there exists a type B $(\chi+1)$-colourable $S(2,4,3v+1)$ which may be coloured either with colour class sizes $\{\gamma_1,\ \gamma_2,\ \ldots\ ,\ \gamma_\chi,\ 2v+1\}$ or with colour class sizes $\{1, 3\gamma_1,\ 3\gamma_2,\ \ldots\ ,\ 3\gamma_\chi\}$.*

Proof. The former colouring pattern is generated by Lemma 2.2 using any KTS$(2v+1)$ in that construction.

To deal with the latter colouring pattern, let $S=(V,\mathcal{B})$, where $V=\{i_0 : i=1,2,\ldots,v\}$. For each block $\{x_0,y_0,z_0,w_0\}\in\mathcal{B}$, take a fixed ordering of the block, (x_0,y_0,z_0,w_0). From these ordered blocks we create a KTS$(2v+1)$ on the point set $V'=\{i_1,i_2 : i=1,2,\ldots,v\}\cup\{\infty\}$. We list the blocks of this design in v parallel classes each of which is associated with a single point of V. The ordered block (x_0,y_0,z_0,w_0) obtained from $\mathcal{B}$ contributes the following triples to these classes.

(i) $\{y_1,z_2,w_1\}$ and $\{y_2,z_1,w_2\}$ associated with x_0,

(ii) $\{x_2,z_1,w_1\}$ and $\{x_1,z_2,w_2\}$ associated with y_0,

(iii) $\{x_1,y_2,w_1\}$ and $\{x_2,y_1,w_2\}$ associated with z_0,

(iv) $\{x_1,y_1,z_1\}$ and $\{x_2,y_2,z_2\}$ associated with w_0.

In addition, the class associated with i_0 contains the triple $\{\infty,i_1,i_2\}$. Thus each class contains $2(v-1)/3+1=(2v+1)/3$ disjoint blocks and so forms a parallel class of triples on V'. It is also easy to see that the complete set of triples forms an STS$(2v+1)$ and hence, with the specified resolution, a KTS$(2v+1)$. From the KTS$(2v+1)$ and the original $S(2,4,v)$ we form an $S(2,4,3v+1)$ using the method of Lemma 2.2 and taking care to adjoin to each parallel class the point of V with which it is associated. We now colour the points i_1,i_2 with the same colour as i_0 for each $i=1,2,\ldots,v$ and we assign a new colour to the point ∞.

We note that the two colour patterns presented are identical if and only if $\{\gamma_1,\ \gamma_2,\ \ldots\ ,\ \gamma_\chi,\ 2v+1\}=\{1,\ 3\gamma_1,\ 3\gamma_2,\ \ldots\ ,\ 3\gamma_\chi\}=\{1,\ 3,\ 3^2,\ \ldots\ ,\ 3^\chi\}$, which is the case covered by Lemma 2.3. □

3 A type B 3-colourable $S(2,4,61)$

Here we consider the first entry in Table 1. We construct an $S(2,4,61)$ together with a type B 3-colouring having colour class sizes 39, 19 and 3. Denote the corresponding colours by A, B and C, respectively. For each of the three STS(v)s identified by Lemma 2.1, let the points be the integers $0,1,\ldots,v-1$ indexed by the system's colour. We denote a point by any of the descriptions X_n, Xn and n, where X is the colour and n is the integer; the second option appears only in Tables 2, 4, 5 and 6, and the third option is used only if the colour is clear from the context. Arithmetic on points is performed on the integer parts in an appropriate ring.

Let the A system be an STS(39) with the automorphism α defined by

$$\alpha : A_i \mapsto A_{i+13 \pmod{39}}.$$

For the B system, we choose the cyclic STS(19) with starter blocks $\{0,1,4\}$, $\{0,7,9\}$ and $\{0,11,6\}$, and automorphism $\beta : B_j \mapsto B_{7j \pmod{19}}$. Note that β leaves B_0 fixed and partitions the other B points into six orbits of size 3.

We begin with the block

$$\{C_0, C_1, C_2, B_0\}$$

and we assign blocks of the B system to A and C points as in Table 2. Observe that if $\{B_x, B_y, B_z\}$ is assigned to point A_i, then $\beta(\{B_x, B_y, B_z\})$ is assigned to point $\alpha(A_i)$ while if $\{B_x, B_y, B_z\}$ is assigned to point C_i then $\beta(\{B_x, B_y, B_z\})$ is also assigned to the point C_i. This latter assignment is also done in such a way that each C_i is paired with each $B_j, j \neq 0$. Thus we have dealt with the 171 BB pairs, the 57 BC pairs, the three CC pairs, and, so far, also 117 AB pairs.

For the STS(39), we first create the set of ten A blocks

$$\mathcal{U}_0 = \{\{A_i, A_{13+i}, A_{26+i}\} : i = 1,2,3,4,5,6,7,8,9,11\}.$$

and assign them to B_0. These blocks are fixed under the action of α and they account for the remaining AB_0 pairs. For $j = 1,2,\ldots 18$, let Ω_j denote the set of A points that have so far been paired with B_j. Then from Table 2

Table 2: STS(19) – the B system for the $S(2,4,61)$

0	1	4	$A0$	0	7	9	$A13$	0	11	6	$A26$
1	2	5	$A1$	1	8	10	$A20$	1	12	7	$A30$
2	3	6	$A2$	2	9	11	$C0$	2	13	8	$A35$
3	4	7	$C0$	3	10	12	$A22$	3	14	9	$A28$
4	5	8	$C1$	4	11	13	$C2$	4	15	10	$A32$
5	6	9	$A3$	5	12	14	$C2$	5	16	11	$A37$
6	7	10	$C2$	6	13	15	$A19$	6	17	12	$C1$
7	8	11	$A4$	7	14	16	$A14$	7	18	13	$A33$
8	9	12	$A5$	8	15	17	$C0$	8	0	14	$A38$
9	10	13	$A6$	9	16	18	$C1$	9	1	15	$C2$
10	11	14	$C1$	10	17	0	$A23$	10	2	16	$A34$
11	12	15	$A7$	11	18	1	$A17$	11	3	17	$A27$
12	13	16	$C0$	12	0	2	$A25$	12	4	18	$A31$
13	14	17	$A8$	13	1	3	$C1$	13	5	0	$A36$
14	15	18	$A9$	14	2	4	$A15$	14	6	1	$C0$
15	16	0	$A10$	15	3	5	$A21$	15	7	2	$C1$
16	17	1	$A11$	16	4	6	$A16$	16	8	3	$C2$
17	18	2	$C2$	17	5	7	$A24$	17	9	4	$A29$
18	0	3	$A12$	18	6	8	$A18$	18	10	5	$C0$

we have

$$\begin{aligned}
\Omega_1 &= \{A_0, A_1, A_{11}, A_{17}, A_{20}, A_{30}\}, \\
\Omega_2 &= \{A_1, A_2, A_{15}, A_{25}, A_{34}, A_{35}\}, \\
\Omega_4 &= \{A_0, A_{15}, A_{16}, A_{29}, A_{31}, A_{32}\}, \\
\Omega_8 &= \{A_4, A_5, A_{18}, A_{20}, A_{35}, A_{38}\}, \\
\Omega_{16} &= \{A_{10}, A_{11}, A_{14}, A_{16}, A_{34}, A_{37}\}, \\
\Omega_{13} &= \{A_6, A_8, A_{19}, A_{33}, A_{35}, A_{36}\},
\end{aligned}$$

and $\Omega_{\beta(j)} = \alpha(\Omega_j)$.

Next we search for a set $\mathcal{T}$ of A blocks such that (i) the blocks of $\mathcal{T}$ form a parallel class, (ii) no two blocks of $\mathcal{T}$ lie in the same orbit under the action of α, and (iii) blocks which are fixed by α are not used in $\mathcal{T}$. A computer search produces 1038 sets, of which we select one,

$$\begin{aligned}\mathcal{T}_0 &= \{\{0,1,17\},\{11,20,30\},\{14,15,9\},\{2,12,21\},\{13,28,29\},\\ &\quad \{3,5,6\},\{4,18,35\},\{31,7,25\},\{10,16,37\},\{24,27,8\},\\ &\quad \{32,34,22\},\{19,33,36\},\{23,26,38\}\},\end{aligned}$$

and we assign the blocks of $\mathcal{T}_0$ to C_0. Together with $\mathcal{T}_1 = \alpha(\mathcal{T}_0)$ (assigned to C_1), $\mathcal{T}_2 = \alpha^2(\mathcal{T}_0)$ (assigned to C_2) and $\mathcal{U}_0$, we now have 49 A blocks and have dealt with 117 AC pairs.

The A system is extended to an STS(39) by hill climbing [10] and these A blocks are assigned to points B_j, $j = 1, 2, 4, 8, 16, 13$, under the following conditions.

(i) During the hill-climbing process, blocks are added to the system or removed from the system in triples: X, $\alpha(X)$ and $\alpha^2(X)$. Blocks in $\mathcal{U}_0 \cup \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ are never removed. Blocks fixed by α are never added.

(ii) The blocks assigned to B_j form a partial parallel class $\mathcal{U}_j$ of size 11 that does not contain any of the points in Ω_j.

(iii) At most one block in each orbit under the action of α is assigned to B_j.

A solution is given by assigning the blocks of $\mathcal{U}_j$ to B_j, $j = 1, 2, 4, 8, 16, 13$, where the $\mathcal{U}_j$ are defined by

$$\begin{aligned}\mathcal{U}_1 &= \{\{2,10,33\},\{3,25,32\},\{4,22,37\},\{5,8,23\},\\ &\quad \{6,28,31\},\{7,12,16\},\{9,26,36\},\{13,21,24\},\\ &\quad \{14,18,38\},\{15,29,35\},\{19,27,34\}\},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_2 &= \{\{0,19,31\},\{3,7,27\},\{4,14,36\},\{5,11,33\},\\ &\quad \{6,10,30\},\{8,9,13\},\{12,20,22\},\{16,21,28\},\\ &\quad \{17,18,23\},\{24,26,32\},\{29,37,38\}\},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_4 &= \{\{1,4,6\},\{2,36,38\},\{3,19,20\},\{5,7,14\},\\ &\quad \{8,18,22\},\{9,11,25\},\{10,26,35\},\{12,30,34\},\\ &\quad \{13,27,37\},\{17,24,28\},\{21,23,33\}\},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_8 &= \{\{0,2,32\},\{1,24,29\},\{3,10,21\},\{6,9,27\},\\ &\quad \{7,11,22\},\{8,12,14\},\{13,16,31\},\{15,17,26\},\\ &\quad \{19,25,30\},\{23,36,37\},\{28,33,34\}\},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_{16} &= \{\{0,25,38\},\{1,7,28\},\{2,9,18\},\{3,30,35\},\\ &\quad \{4,21,29\},\{5,27,36\},\{6,22,33\},\{8,26,31\},\\ &\quad \{12,23,32\},\{13,17,20\},\{15,19,24\}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}_{13} &= \{\{0,27,29\},\{1,9,12\},\{2,24,31\},\{3,22,23\},\\ &\quad \{4,20,34\},\{5,16,17\},\{7,13,26\},\{10,14,28\},\\ &\quad \{11,21,32\},\{15,30,38\},\{18,25,37\}\}.\end{aligned}$$

To complete the construction we assign the blocks of $\alpha(\mathcal{U}_j)$ to $\beta(B_j)$ and the blocks of $\alpha^2(\mathcal{U}_j)$ to $\beta^2(B_j)$, $j = 1, 2, 4, 8, 16, 13$. The entire $S(2,4,61)$ is listed in Table 4 at the end of the paper.

4 A type B 2-colourable $S(2,4,100)$

Using the same conventions as in Section 3, let the colours corresponding to colour class sizes $(55,45)$ be (A,B) and let the $S(2,4,100)$ have the automorphism σ defined by

$$\sigma : A_i \mapsto A_{i+5 \pmod{55}}, B_j \mapsto B_{j+4 \pmod{44}}, j = 0,1,\ldots,43, B_{44} \mapsto B_{44}.$$

The A system is an STS(55) and the B system is an STS(45). Both systems have automorphism σ. As an aside, we remark that the B system is an example of a *4-rotational* STS(v) and that such systems exist for all $v \equiv 1, 9, 13$ or $21 \pmod{24}$ [2].

Blocks in the B system are assigned to points A_i, $i = 0,1,\ldots,4$, subject to the conditions: (i) the B blocks assigned to A_i form a partial parallel class of size 6, and (ii) no more than one block in an orbit under the action of σ is assigned to A_i. Then we apply σ to assign the rest of the blocks: if $\{B_x, B_y, B_z\}$ is assigned to A_k, then $\sigma(\{B_x, B_y, B_z\})$ is assigned to $\sigma(A_k)$.

Eleven A blocks in a single orbit under the action of σ are assigned to B_{44} such that the points in these A blocks together with the A points to which the B blocks containing B_{44} are already assigned cover a complete set of residues modulo 55. Thus all AB_{44} pairs are accounted for.

Then blocks in the A system are assigned to B_x, $x = 0, 1, 2, 3$, subject to the conditions: (i) the A blocks assigned to B_x form a partial parallel class of size 11 which contains none of the 22 A points to which the B blocks containing B_x have already been assigned, and (ii) no more than one block in an orbit under the action of σ is assigned to B_x. We complete the assignment by applying σ: if $\{A_i, A_j, A_k\}$ is assigned to B_z, then $\sigma(\{A_i, A_j, A_k\})$ is assigned to $\sigma(B_z)$.

Orbit representatives under σ of the blocks of the $S(2, 4, 100)$ are listed in Table 5.

5 Type B 3-colourable $S(2, 4, 109)$s

Let the colours corresponding to colour class sizes $(63, 45, 1)$ be (A, B, C) and let the $S(2, 4, 109)$ have the automorphism τ defined by

$$\tau : \begin{cases} A_i \mapsto A_{i+3 \pmod{63}}, & i = 0, 1, \ldots, 62, \\ B_j \mapsto B_{j+2 \pmod{42}}, & j = 0, 1, \ldots, 41, \\ B_{42} \mapsto B_{43} \mapsto B_{44} \mapsto B_{42}, & \\ C_0 \mapsto C_0. & \end{cases}$$

The B system is an STS(45) containing the blocks

$$\{\{B_n, B_{n+14}, B_{n+28}\} : n = 0, 1, \ldots, 13\} \text{ and } \{B_{42}, B_{43}, B_{44}\},$$

which we assign to the point C_0. The remaining blocks are partitioned into fifteen 21-block orbits by τ. We deal with these 315 blocks by assigning B blocks to A_i, $i = 0, 1, 2$, such that (i) the B blocks assigned to A_i form a partial parallel class of size 5, and (ii) no more than one block in an orbit under the action of τ is assigned to A_i. The assignment is then completed by applying the automorphism τ.

The A system is an STS(63) whose blocks are partitioned by τ into 31 orbits of size 21, at least one of which is a parallel class. We assign one of these parallel classes to C_0.

In the STS(45), the fifteen 21-block orbits under the action of τ collectively contain each B point 21 times. The points B_{42}, B_{43}, B_{44} must each occur precisely seven times in three of these orbits. Hence there are precisely three distinct A points, x_1, x_2, x_3, paired with B_{42} such that $0 \leq x_1, x_2, x_3 < 9$. In the A system we choose two blocks from different orbits, $\{x_4, x_5, x_6\}$ and $\{x_7, x_8, x_9\}$, such that $\{x_1, x_2, \ldots, x_9\}$ covers a complete set of residues modulo 9. We assign these blocks to B_{42} and extend the assignment in the usual manner by applying τ.

For the remaining 588 blocks in the STS(63), we assign A blocks to B_x, $x = 0, 1$, such that (i) the A blocks assigned to B_x form a partial parallel class of size 14 which contains none of the 21 A points to which the B blocks containing B_x have already been assigned, and (ii) no more than one block in an orbit under the action of τ is assigned to B_x. Finally, the assignment is completed by applying τ. Orbit representatives of two $S(2, 4, 109)$s are presented as Table 6.

6 Summary

In the previous sections we have constructed three systems $S(2, 4, v)$ with type B χ-colourings, namely those for $(v, \chi) = (61, 3)$, $(100, 2)$ and $(109, 3)$. By applying the constructions of Section 2 these generate infinite families of systems with type B χ-colourings, which are given by $v = 3^n v_0 + \frac{1}{2}(3^n - 1)$ for $v_0 = 61$, 100 and 109, and $n = 0, 1, 2, \ldots$. We note that in these new families at each stage there is a choice of two colour patterns. With reference to Table 1, we have now constructed systems with the parameters as shown in Table 3.

Table 3: Parameters of some known type B χ-colourable $S(2, 4, v)$s

v	χ	$\gamma_1, \gamma_2, \ldots, \gamma_\chi$	v	χ	$\gamma_1, \gamma_2, \ldots, \gamma_\chi$
61	3	3, 19, 39	301	3	1, 135, 165
100	2	45, 55	301	3	45, 55, 201
109	3	1, 45, 63	328	4	1, 3, 135, 189
184	4	1, 9, 57, 117	328	4	1, 45, 63, 219
184	4	3, 19, 39, 123			

The results of this paper yield some improvements on bounds for minimum embeddings of Steiner triple systems into $S(2, 4, v)$ systems. The upper bound on the quantity $m(19)$ (see [6], Theorem 22(iii)) is improved from 85 to 61, and the upper bound on $q(19)$ (see Corollary 23 of the same paper) is improved from 256 to 184. More generally, some improvements can be made to the results of [6, Theorem 9]. By consistently selecting the second colour class pattern from our Theorem 2.1 for the new families of $S(2,4,v)$ systems, the following result is easily established.

Theorem 6.1 *For* $n = 0, 1, \dots,$

$$\begin{aligned}
m(v) &\leq \frac{41v-13}{26} && \text{if } v = 3^n \cdot 39, \\
m(v) &\leq \frac{67v-15}{30} && \text{if } v = 3^n \cdot 45, \\
m(v) &\leq \frac{201v-55}{110} && \text{if } v = 3^n \cdot 55, \\
m(v) &\leq \frac{73v-21}{42} && \text{if } v = 3^n \cdot 63.
\end{aligned}$$

Since $q(v) \leq 3m(v) + 1$, this also leads to corresponding improvements to Corollary 10 of [6].

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Table 4: $S(2, 4, 61)$

A1 A14 A27 B0	A2 A15 A28 B0	A3 A16 A29 B0	A4 A17 A30 B0	A5 A18 A31 B0	A6 A19 A32 B0
A7 A20 A33 B0	A8 A21 A34 B0	A9 A22 A35 B0	A11 A24 A37 B0		
A0 A2 A32 B8	A13 A15 A6 B18	A26 A28 A19 B12	A0 A3 A18 B12	A13 A16 A31 B8	A26 A29 A5 B18
A0 A4 A7 B5	A13 A17 A20 B16	A26 A30 A33 B17	A0 A5 A21 B17	A13 A18 A34 B5	A26 A31 A8 B16
A0 A6 A37 B14	A13 A19 A11 B3	A26 A32 A24 B2	A0 A8 A11 B11	A13 A21 A24 B1	A26 A34 A37 B7
A0 A9 A23 B9	A13 A22 A36 B6	A26 A35 A10 B4	A0 A10 A22 B7	A13 A23 A35 B11	A26 A36 A9 B1
A0 A13 A33 B10	A13 A26 A7 B13	A26 A0 A20 B15	A0 A14 A24 B6	A13 A27 A37 B4	A26 A1 A11 B9
A0 A19 A31 B2	A13 A32 A5 B14	A26 A6 A18 B3	A0 A25 A38 B16	A13 A38 A12 B17	A26 A12 A25 B5
A0 A27 A29 B13	A13 A1 A3 B15	A26 A14 A16 B10	A0 A28 A30 B18	A13 A2 A4 B12	A26 A15 A17 B8
A0 A34 A35 B3	A13 A8 A9 B2	A26 A21 A22 B14	A1 A4 A6 B4	A14 A17 A19 B9	A27 A30 A32 B6
A1 A5 A25 B11	A14 A18 A38 B1	A27 A31 A12 B7	A1 A7 A28 B16	A14 A20 A2 B17	A27 A33 A15 B5
A1 A8 A32 B7	A14 A21 A6 B11	A27 A34 A19 B1	A1 A9 A12 B13	A14 A22 A25 B15	A27 A35 A38 B10
A1 A10 A18 B17	A14 A23 A31 B5	A27 A36 A5 B16	A1 A15 A36 B10	A14 A28 A10 B13	A27 A2 A23 B15
A1 A16 A20 B14	A14 A29 A33 B3	A27 A3 A7 B2	A1 A19 A22 B18	A14 A32 A35 B12	A27 A6 A9 B8
A1 A23 A30 B3	A14 A36 A4 B2	A27 A10 A17 B14	A1 A24 A29 B8	A14 A37 A3 B18	A27 A11 A16 B12
A1 A31 A33 B6	A14 A5 A7 B4	A27 A18 A20 B9	A1 A34 A38 B12	A14 A8 A12 B8	A27 A21 A25 B18
A2 A5 A19 B7	A15 A18 A32 B11	A28 A31 A6 B1	A2 A6 A11 B5	A15 A19 A24 B16	A28 A32 A37 B17
A2 A7 A8 B18	A15 A20 A21 B12	A28 A33 A34 B8	A2 A9 A18 B16	A15 A22 A31 B17	A28 A35 A5 B5
A2 A10 A33 B1	A15 A23 A7 B7	A28 A36 A20 B11	A2 A16 A22 B11	A15 A29 A35 B1	A28 A3 A9 B7
A2 A17 A25 B10	A15 A30 A38 B13	A28 A4 A12 B15	A2 A24 A31 B13	A15 A37 A5 B15	A28 A11 A18 B10
A2 A29 A34 B14	A15 A3 A8 B3	A28 A16 A21 B2	A2 A30 A37 B9	A15 A4 A11 B6	A28 A17 A24 B4
A2 A36 A38 B4	A15 A10 A12 B9	A28 A23 A25 B6	A3 A4 A31 B10	A16 A17 A5 B13	A29 A30 A18 B15
A3 A10 A21 B8	A16 A23 A34 B18	A29 A36 A8 B12	A3 A11 A12 B14	A16 A24 A25 B3	A29 A37 A38 B2
A3 A17 A34 B17	A16 A30 A8 B5	A29 A4 A21 B16	A3 A19 A20 B4	A16 A32 A33 B9	A29 A6 A7 B6
A3 A22 A23 B13	A16 A35 A36 B15	A29 A9 A10 B10	A3 A25 A32 B1	A16 A38 A6 B7	A29 A12 A19 B11
A3 A30 A35 B16	A16 A4 A9 B17	A29 A17 A22 B5	A3 A33 A38 B11	A16 A7 A12 B1	A29 A20 A25 B7
A4 A5 A10 B3	A17 A18 A23 B2	A30 A31 A36 B14	A4 A8 A25 B9	A17 A21 A38 B6	A30 A34 A12 B4
A4 A19 A23 B14	A17 A32 A36 B3	A30 A6 A10 B2	A4 A20 A34 B13	A17 A33 A8 B15	A30 A7 A21 B10
A4 A22 A37 B1	A17 A35 A11 B7	A30 A9 A24 B11	A4 A32 A38 B18	A17 A6 A12 B12	A30 A19 A25 B8
A5 A8 A23 B1	A18 A21 A36 B7	A31 A34 A10 B11	A5 A9 A34 B6	A18 A22 A8 B4	A31 A35 A21 B9
A5 A11 A33 B2	A18 A24 A7 B14	A31 A37 A20 B3	A5 A12 A24 B10	A18 A25 A37 B13	A31 A38 A11 B15
A6 A22 A33 B16	A19 A35 A7 B17	A32 A9 A20 B5	A6 A24 A34 B15	A19 A37 A8 B10	A32 A11 A21 B13
A6 A25 A36 B17	A19 A38 A10 B5	A32 A12 A23 B16	A7 A9 A38 B3	A20 A22 A12 B2	A33 A35 A25 B14
A7 A11 A22 B8	A20 A24 A35 B18	A33 A37 A9 B12	A7 A34 A36 B9	A20 A8 A10 B6	A33 A21 A23 B4
A9 A11 A25 B4	A22 A24 A38 B9	A35 A37 A12 B6	A10 A11 A36 B18	A23 A24 A10 B12	A36 A37 A23 B8
A0 A1 A17 C0	A11 A20 A30 C0	A14 A15 A9 C0	A2 A12 A21 C0	A13 A28 A29 C0	A3 A5 A6 C0
A4 A18 A35 C0	A31 A7 A25 C0	A10 A16 A37 C0	A24 A27 A8 C0	A32 A34 A22 C0	A19 A33 A36 C0
A23 A26 A38 C0					
A13 A14 A30 C1	A24 A33 A4 C1	A27 A28 A22 C1	A15 A25 A34 C1	A26 A2 A3 C1	A16 A18 A19 C1
A17 A31 A9 C1	A5 A20 A38 C1	A23 A29 A11 C1	A37 A1 A21 C1	A6 A8 A35 C1	A32 A7 A10 C1
A0 A12 A36 C1					
A26 A27 A4 C2	A37 A7 A17 C2	A1 A2 A35 C2	A28 A38 A8 C2	A0 A15 A16 C2	A29 A31 A32 C2
A30 A5 A22 C2	A18 A33 A12 C2	A36 A3 A24 C2	A11 A14 A34 C2	A19 A21 A9 C2	A6 A20 A23 C2
A10 A13 A25 C2					C0 C1 C2 B0
B0 B1 B4 A0	B0 B7 B9 A13	B0 B11 B6 A26	B1 B2 B5 A1	B1 B8 B10 A20	B1 B12 B7 A30
B2 B3 B6 A2	B2 B13 B8 A35	B3 B10 B12 A22	B3 B14 B9 A28	B4 B15 B10 A32	B5 B6 B9 A3
B5 B16 B11 A37	B6 B13 B15 A19	B7 B8 B11 A4	B7 B14 B16 A14	B7 B18 B13 A33	B8 B9 B12 A5
B8 B0 B14 A38	B9 B10 B13 A6	B10 B17 B0 A23	B10 B2 B16 A34	B11 B12 B15 A7	B11 B18 B1 A17
B11 B3 B17 A27	B12 B0 B2 A25	B12 B4 B18 A31	B13 B14 B17 A8	B13 B5 B0 A36	B14 B15 B18 A9
B14 B2 B4 A15	B15 B16 B0 A10	B15 B3 B5 A21	B16 B17 B1 A11	B16 B4 B6 A16	B17 B5 B7 A24
B17 B9 B4 A29	B18 B0 B3 A12	B18 B6 B8 A18			
B2 B9 B11 C0	B3 B4 B7 C0	B8 B15 B17 C0	B12 B13 B16 C0	B14 B6 B1 C0	B18 B10 B5 C0
B4 B5 B8 C1	B6 B17 B12 C1	B9 B16 B18 C1	B10 B11 B14 C1	B13 B1 B3 C1	B15 B7 B2 C1
B4 B11 B13 C2	B5 B12 B14 C2	B6 B7 B10 C2	B9 B1 B15 C2	B16 B8 B3 C2	B17 B18 B2 C2

Table 5: Orbit representatives of the $S(2,4,100)$

B0	B1	B9	A0	B4	B6	B23	A0	B8	B11	B13	A0
B12	B20	B10	A0	B36	B5	B7	A0	B2	B3	B38	A0
B0	B4	B33	A1	B40	B3	B6	A1	B8	B24	B5	A1
B16	B7	B11	A1	B1	B2	B21	A1	B9	B13	B42	A1
B0	B6	B24	A2	B8	B19	B40	A2	B4	B31	B3	A2
B9	B14	B43	A2	B1	B7	B29	A2	B5	B26	B2	A2
B0	B14	B21	A3	B4	B35	B41	A3	B1	B10	B31	A3
B5	B23	B44	A3	B2	B7	B18	A3	B22	B34	B3	A3
B28	B1	B14	A4	B4	B22	B26	A4	B0	B38	B44	A4
B29	B39	B2	A4	B37	B5	B19	A4	B3	B11	B35	A4
A25	A29	A19	B0	A20	A32	A5	B0	A35	A48	A18	B0
A15	A39	A11	B0	A41	A43	A8	B0	A21	A42	A13	B0
A31	A14	A28	B0	A16	A9	A12	B0	A17	A23	A49	B0
A37	A44	A33	B0	A22	A34	A38	B0				
A35	A36	A13	B1	A10	A17	A18	B1	A25	A44	A11	B1
A20	A43	A19	B1	A5	A37	A39	B1	A15	A6	A12	B1
A30	A23	A28	B1	A26	A29	A34	B1	A21	A38	A48	B1
A27	A42	A9	B1	A32	A49	A7	B1				
A5	A7	A21	B2	A20	A25	A46	B2	A35	A41	A11	B2
A40	A54	A15	B2	A30	A47	A19	B2	A36	A37	A23	B2
A26	A39	A24	B2	A16	A32	A53	B2	A31	A12	A22	B2
A17	A8	A9	B2	A13	A28	A49	B2				
A40	A43	A32	B3	A35	A44	A15	B3	A10	A20	A38	B3
A5	A27	A47	B3	A25	A6	A13	B3	A21	A26	A36	B3
A31	A42	A11	B3	A16	A28	A34	B3	A41	A9	A29	B3
A12	A17	A48	B3	A8	A24	A54	B3				
A0	A11	A37	B44								

Table 6: Orbit representatives of two $S(2,4,109)$s

B0 B1 B33 A0	B6 B8 B37 A0	B2 B5 B29 A0	B4 B20 B3 A0
B14 B7 B9 A0			
B0 B4 B13 A1	B2 B7 B42 A1	B8 B14 B31 A1	B30 B3 B6 A1
B1 B5 B35 A1			
B4 B11 B42 A2	B26 B34 B3 A2	B0 B10 B30 A2	B2 B23 B43 A2
B37 B1 B17 A2			
A6 A12 A44 B0	A24 A32 A35 B0	A30 A39 A23 B0	A9 A22 A27 B0
A15 A29 A11 B0	A21 A37 A41 B0	A48 A8 A47 B0	A18 A46 A10 B0
A45 A13 A31 B0	A36 A7 A16 B0	A3 A49 A56 B0	A4 A5 A38 B0
A28 A34 A53 B0	A25 A40 A17 B0		
A6 A7 A48 B1	A24 A26 A53 B1	A36 A39 A17 B1	A30 A40 A28 B1
A33 A45 A18 B1	A27 A44 A16 B1	A12 A31 A52 B1	A42 A3 A29 B1
A22 A25 A8 B1	A37 A47 A35 B1	A10 A23 A49 B1	A43 A59 A38 B1
A34 A56 A4 B1	A14 A20 A5 B1		
A0 A5 A25 B42	A6 A13 A39 B42		
B0 B14 B28 C0	B1 B15 B29 C0	B42 B43 B44 C0	A0 A4 A35 C0

B0 B1 B27 A0	B2 B4 B17 A0	B6 B9 B13 A0	B40 B7 B15 A0
B10 B3 B5 A0			
B0 B4 B25 A1	B2 B21 B1 A1	B6 B35 B3 A1	B14 B5 B44 A1
B7 B13 B37 A1			
B38 B1 B32 A2	B36 B2 B18 A2	B0 B10 B30 A2	B4 B27 B44 A2
B14 B3 B42 A2			
A24 A26 A23 B0	A21 A25 A47 B0	A27 A32 A42 B0	A39 A49 A18 B0
A6 A19 A30 B0	A51 A4 A46 B0	A36 A53 A22 B0	A9 A28 A31 B0
A15 A35 A52 B0	A45 A5 A16 B0	A12 A40 A56 B0	A33 A10 A29 B0
A13 A14 A37 B0	A34 A41 A7 B0		
A15 A21 A56 B1	A36 A44 A19 B1	A3 A12 A50 B1	A18 A30 A48 B1
A9 A23 A59 B1	A6 A31 A33 B1	A27 A7 A20 B1	A43 A47 A52 B1
A16 A22 A8 B1	A37 A49 A4 B1	A25 A53 A5 B1	A40 A14 A38 B1
A29 A35 A17 B1	A32 A41 A11 B1		
A3 A6 A35 B42	A0 A7 A55 B42		
B0 B14 B28 C0	B1 B15 B29 C0	B42 B43 B44 C0	A0 A1 A11 C0