On directed designs with block size five

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Abstract

In a t-(v,k,λ) *directed design* the blocks are ordered k-tuples and every ordered t-tuple of distinct points occurs in exactly λ blocks (as a subsequence). We study t-$(v,5,1)$ directed designs with $t=3$ and $t=4$. In particular, we construct the first known examples, and an infinite class, of 3-$(v,5,1)$ directed designs, and the first known infinite class of 4-$(v,5,1)$ directed designs.

1 Introduction

A t-(v,k,λ) *directed design* is a pair $(\mathcal{P},\mathcal{B})$ where $\mathcal{P}$ is a set of v elements, called *points*, and $\mathcal{B}$ is a collection of ordered k-tuples of distinct elements of $\mathcal{P}$, called *blocks*, with the property that every ordered t-tuple of distinct elements of $\mathcal{P}$ occurs in exactly λ blocks (as a subsequence). Background information on directed designs can be found in [2] and [5].

We usually specify a directed design by listing its blocks. For example, the following blocks form a 3-$(4,4,1)$ directed design:

$$(1,2,3,4),\ (2,1,4,3),\ (3,1,4,2),\ (4,2,3,1),\ (3,2,4,1),\ (4,1,3,2).$$

Here, for example, the block $(1,2,3,4)$ contains the ordered triples $(1,2,3)$, $(1,2,4)$, $(1,3,4)$ and $(2,3,4)$.

A t-(v,k,λ) directed design is *cyclic* if it has an automorphism which permutes its points in a cycle of length v. The base blocks below, developed modulo 6, form a cyclic 3-$(6,4,1)$ directed design. This design is given by Soltankhah [14].

$$(0,1,3,5),\ (0,4,2,1),\ (0,3,1,2),\ (0,5,1,4),\ (0,5,2,3).$$

The following result (which is straightforward to prove) gives necessary conditions for the existence of a t-(v,k,λ) directed design.

Result 1.1 *Let $\mathcal{D}$ be a t-(v,k,λ) directed design. Then $\mathcal{D}$ is an s-(v,k,λ_s) directed design for $0 \le s < t$ where*

$$\lambda_s = \lambda \frac{\binom{v-s}{t-s} t!}{\binom{k-s}{t-s} s!}.$$

Hence λ_s must be an integer for $s = 0, 1, 2, \ldots, t-1$.

2-(v,k,λ) directed designs have been studied quite extensively. For such designs, the necessary conditions of Result 1.1 reduce to $2\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ and $2\lambda(v-1) \equiv 0 \pmod{k-1}$. It has been shown [1, 9, 13, 15, 16] that for $k \in \{3,4,5,6\}$ these necessary conditions are sufficient, with two exceptions, namely that no directed designs with parameters 2-$(15,5,1)$ or 2-$(21,6,1)$ exist.

Existence results for t-(v,k,λ) directed designs with $t \ge 3$ seem to be more elusive. However, the question of the existence of 3-$(v,4,\lambda)$ directed designs has been settled recently. The necessary conditions for these designs reduce to the single condition $\lambda v \equiv 0 \pmod 2$. In [11], Levenshtein showed that a 3-$(v,4,1)$ directed design exists for all even v. Soltankhah [14] further proved that the necessary condition is sufficient except possibly when $v \equiv 3$ or $11 \pmod{12}$. In [8], some of the present authors built on Levenshtein's and Soltankhah's results to obtain a proof that the necessary condition is sufficient in general.

The proof for 3-$(v,4,\lambda)$ directed designs relies on the following result involving *t-wise balanced designs*, which we shall also need in the present paper. A t-(v,K,λ) (t-wise balanced) design is a pair $(\mathcal{P},\mathcal{B})$ where $\mathcal{P}$ is a set of v elements, called *points*, and $\mathcal{B}$ is a collection of subsets of $\mathcal{P}$, called *blocks*, with the property that the size of every block is in the set K and every t-element subset of $\mathcal{P}$ is contained in exactly λ blocks. In particular, $K = \{k\}$ gives an 'ordinary' t-(v,k,λ) design. When $\lambda = 1$, t-wise balanced designs are also known as generalized Steiner systems and denoted by $\mathrm{S}(t,K,v)$.

Result 1.2 **(Replacement Lemma)** *If there exist a t-(v,K,λ_1) design and a t-(k',k,λ_2) directed design for each $k' \in K$, then there exists a t-$(v,k,\lambda_1\lambda_2)$ directed design.*

Proof Replacing each block of the t-(v,K,λ_1) design with a copy of a t-(k',k,λ_2) directed design with point set the points of that block gives a t-$(v,k,\lambda_1\lambda_2)$ directed design. □

In this paper, we are concerned with t-(v,k,λ) directed designs in the cases $t = 3$, $k = 5$ and $t = 4$, $k = 5$. We restrict our attention to $\lambda = 1$.

For $t = 3$, $k = 5$ and $\lambda = 1$, no designs were previously known. The only published result is a computer search by Mahmoodi [12], showing that there is no such design for $v = 5$, 6 or 7. We give an analytic proof of this result. More significantly, we have been able to construct the first known 3-$(v,5,1)$ directed designs, namely for $v = 26$ and $v = 37$. The first of these allows us to obtain an infinite class of 3-$(v,5,1)$ directed designs.

For $t = 4$, $k = 5$ and $\lambda = 1$, the existence of some 4-$(v,5,1)$ directed designs follows immediately from the known existence of corresponding Steiner systems $\mathrm{S}(4,5,v)$, as follows.

Result 1.3 *There exists a* 4-$(v, 5, 1)$ *directed design for* $v = 11$, *23, 35, 47, 71, 83, 107, 131 and* 167.

Proof There exists a 4-$(5, 5, 1)$ directed design [6, 11]. Also, there exists a 4-$(v, 5, 1)$ design for the above values of v [3, 4, 7]. The result follows by the replacement lemma. □

We have been able to construct new 4-$(v, 5, 1)$ directed designs not obtained in this way. Our results extend the set of values of v for which a 4-$(v, 5, 1)$ directed design is known to exist; the new values are $v = 7, 8, 13, 18$ and 48. The first of these allows us to obtain an infinite class of 4-$(v, 5, 1)$ directed designs. We have also found, by means of an exhaustive computer search, that there is no 4-$(6, 5, 1)$ directed design.

2 Results on 3-$(v, 5, 1)$ directed designs

The necessary conditions given by Result 1.1 for 3-$(v, 5, 1)$ directed designs are $v \equiv 0, 1, 2 \pmod 5$. We prove below that these designs do not exist for $v = 5, 6, 7$. The proof for $v = 5$ is trivial and already known, but we include it for completeness.

Theorem 2.1 *There exists no* 3-$(5, 5, 1)$ *directed design.*

Proof Suppose that there exists a 3-$(5, 5, 1)$ directed design; let its points be $0, 1, 2, 3, 4$. The design contains six blocks, and therefore there are two blocks starting with the same point, say $x_1 = (0, 1, 2, 3, 4)$ and $x_2 = (0, *, *, *, *)$. In the final four positions of x_2, no two points can appear in the order in which they appear in x_1, and hence we must have $x_2 = (0, 4, 3, 2, 1)$. Now consider the directed triple $(1, 3, 2)$. This cannot appear in any block without repeating a directed triple already contained in x_1 or x_2. □

Theorem 2.2 *There exists no* 3-$(6, 5, 1)$ *directed design.*

Proof Suppose that there exists a 3-$(6, 5, 1)$ directed design; let its points be $0, 1, 2, 3, 4, 5$. Let the number of blocks in which 0 occupies the ith position be n_i, for $i = 1, 2, 3, 4, 5$.

We now count the number of triples of the form $(0, *, *)$. There are $6n_1 + 3n_2 + n_3$ occurrences of such triples. Since there are 20 such triples altogether, we have $6n_1 + 3n_2 + n_3 = 20$. Similarly, counting triples of the forms $(*, 0, *)$ and $(*, *, 0)$, we obtain $3n_2 + 4n_3 + 3n_4 = 20$ and $n_3 + 3n_4 + 6n_5 = 20$.

Looking at the second of these three equations, we see that $20 - 4n_3$ is a non-negative multiple of 3; it follows that $n_3 = 2$. Hence, by this same equation, $n_2 + n_4 = 4$, and by the other two equations, n_2 and n_4 are even. We can rule out the possibility $n_2 = 4$, $n_4 = 0$ as follows. Suppose that 0 appears in the second position in some block, say $(P, 0, *, *, *)$. This block contains exactly three of the four triples of the form $(P, 0, *)$. Hence there is a block containing exactly one of these triples; this block therefore has 0 occurring in the fourth position, which implies that $n_4 > 0$. Similarly, by assuming that 0 appears in the fourth position in some block, we may rule out the possibility $n_2 = 4$, $n_4 = 0$. Hence $n_2 = n_4 = 2$.

Now consider the six blocks of $\mathcal{D}$ containing 0 in the second, third or fourth positions:

$$x_1 = (*, 0, *, *, *), \quad x_3 = (*, *, 0, *, *), \quad x_5 = (*, *, *, 0, *),$$
$$x_2 = (*, 0, *, *, *), \quad x_4 = (*, *, 0, *, *), \quad x_6 = (*, *, *, 0, *).$$

Blocks x_1 and x_2 cannot have the same point, P say, in the first position, because this would give a repeated triple of the form $(P, 0, *)$. So without loss of generality we may write $x_1 = (1, 0, *, *, *)$ and $x_2 = (2, 0, *, *, *)$. Now x_3 and x_4 cannot contain either 1 or 2 in their first two positions, since this would give a repeated triple of the form $(1, 0, *)$ or $(2, 0, *)$. Also, x_3 and x_4 cannot be of the form $x_3 = (\{P, Q\}, 0, *, *)$, $x_4 = (\{P, Q\}, 0, *, *)$, where the inclusion of the set brackets around P, Q indicates that they occur in those positions in some order, because this would give a repeated triple of the form $(P, 0, *)$. So without loss of generality we may write $x_3 = (\{3, 4\}, 0, *, *)$ and $x_4 = (\{3, 5\}, 0, *, *)$. To ensure the correct number of triples of the form $(P, 0, *)$ for $P = 1, 2, 3, 4, 5$, we must now have $x_5 = (\{1, 4, 5\}, 0, *)$, $x_6 = (\{2, 4, 5\}, 0, *)$. Further, to ensure the correct number of triples of the form $(*, 0, 4)$ and $(*, 0, 5)$, the six blocks we are considering must have the form:

$$x_1 = (1, 0, \{4, 5, *\}), \quad x_3 = (\{3, 4\}, 0, \{5, *\}), \quad x_5 = (\{1, 4, 5\}, 0, *),$$
$$x_2 = (2, 0, \{4, 5, *\}), \quad x_4 = (\{3, 5\}, 0, \{4, *\}), \quad x_6 = (\{2, 4, 5\}, 0, *).$$

We now see that there is no way to replace some of the stars by 1s to give the correct number of directed triples of the form $(*, 0, 1)$ with no repeats. Hence no such design exists. $\square$

Theorem 2.3 *There exists no* 3-$(7, 5, 1)$ *directed design.*

Proof Suppose that there exists a 3-$(7, 5, 1)$ directed design; let its points be 0, 1, 2, 3, 4, 5, 6. Let the number of blocks in which 0 occupies the ith position be n_i, for $i = 1, 2, 3, 4, 5, 6$.

As in Theorem 2.2, we now count the numbers of triples of the forms $(0, *, *)$, $(*, 0, *)$ and $(*, *, 0)$ to obtain the equations $6n_1 + 3n_2 + n_3 = 30$, $3n_2 + 4n_3 + 3n_4 = 30$ and $n_3 + 3n_4 + 6n_5 = 30$. Further, we can deduce that $n_2 \leq 6$, since otherwise for some point $P \neq 0$ there would exist two blocks of the form $(P, 0, *, *, *)$, which would imply a repeated triple of the form $(P, 0, *)$. Similarly $n_4 \leq 6$. We can also deduce that $n_3 \leq 6$, since otherwise for some P there would be three blocks of the form $(\{P, *\}, 0, *, *)$, which would imply a repeated triple of the form $(P, 0, *)$. Moreover, if $n_3 = 6$, then for every point $P \neq 0$ there are precisely two blocks of the form $(\{P, *\}, 0, *, *)$ and hence also a block $(\{P, *, *\}, 0, *)$. Thus in this case $n_4 > 0$ and similarly $n_2 > 0$. Solving the above equations subject to these further conditions gives $(n_1, n_2, n_3, n_4, n_5) = (4, 1, 3, 5, 2)$, $(2, 5, 3, 1, 4)$, $(3, 4, 0, 6, 2)$, $(2, 6, 0, 4, 3)$ or $(3, 3, 3, 3, 3)$. The above argument applies to any point. Let the number of points having each of the above distributions for $(n_1, n_2, n_3, n_4, n_5)$ be x_1, x_2, y_1, y_2 and z respectively. Then, counting the number of points, we obtain $x_1 + x_2 + y_1 + y_2 + z = 7$, and counting the number of occurences of points in the third position, we obtain $3x_1 + 3x_2 + 3z = 21$. Hence $y_1 = y_2 = 0$. Now, counting the number of occurences of points in the first and second positions, we obtain $4x_1 + 2x_2 + 3z = 21$ and $x_1 + 5x_2 + 3z = 21$. Hence $x_1 = x_2 = x$ (say) giving $2x + z = 7$, from which it follows that $z \geq 1$.

So there exists at least one point, say 0, for which $n_1 = n_2 = n_3 = n_4 = n_5 = 3$. Now consider the nine blocks of $\mathcal{D}$ containing 0 in the second, third or fourth positions. Those in which 0 occupies the second position must have different points in the first position.

$$\begin{array}{lll} x_1 = (1,0,*,*,*), & x_4 = (*,*,0,*,*), & x_7 = (*,*,*,0,*) \\ x_2 = (2,0,*,*,*), & x_5 = (*,*,0,*,*), & x_8 = (*,*,*,0,*) \\ x_3 = (3,0,*,*,*), & x_6 = (*,*,0,*,*), & x_9 = (*,*,*,0,*). \end{array}$$

The proof proceeds by considering all the possibilities for the distribution of the points 1, 2 and 3 before the point 0 in blocks x_4, x_5, x_6, x_7, x_8 and x_9. There are nine of these, all of which we eliminate. The arguments given are to be understood to be without loss of generality.

(I) $x_4 = (1,2,0,*,*)$, $x_5 = (\{3,*\},0,*,*)$. Then $x_7 = (4,5,6,0,*)$ and it follows that $x_8 = (6,5,4,0,*)$. Now it is impossible to complete x_9.

(II) $x_4 = (\{1,*\},0,*,*)$, $x_5 = (\{2,*\},0,*,*)$, $x_6 = (\{3,*\},0,*,*)$. The same argument as in (I) applies.

(III) $x_4 = (1,2,0,*,*)$, $x_7 = (\{3,*,*\},0,*)$, $x_8 = (\{3,*,*\},0,*)$. Then $x_9 = (\{4,5,6\},0,*)$. Now the points 4, 5 and 6 occur in the first, second or third positions of blocks x_7, x_8 and x_9 either once or three times. Hence $x_7 = (\{3,4,5\},0,*)$ and $x_8 = (\{3,4,5\},0,*)$ and it is impossible to order blocks x_7, x_8 and x_9 without repeating one of the triples $(4,5,0)$ or $(5,4,0)$.

(IV) $x_4 = (\{1,*\},0,*,*)$, $x_5 = (\{2,*\},0,*,*)$, $x_7 = (\{3,*,*\},0,*)$, $x_8 = (\{3,*,*\},0,*)$. The same argument as in (III) applies.

(V) $x_4 = (\{1,*\},0,*,*)$, $x_7 = (\{2,3,*\},0,*)$, $x_8 = (\{2,3,*\},0,*)$. The 3-$(7,5,1)$ directed design contains five triples of the form $(2,*,0)$ and five triples of the form $(*,2,0)$. Four of these triples occur in blocks x_7 and x_8. Now consider the blocks $x_{10} = (*,*,*,*,0)$, $x_{11} = (*,*,*,*,0)$ and $x_{12} = (*,*,*,*,0)$. The point 2 occurs in precisely two of these. The same argument applies to the point 3 and hence there is a block $x_{10} = (\{2,3,*,*\},0)$. Now it is impossible to order blocks x_7, x_8 and x_{10} without repeating one of the triples $(2,3,0)$ and $(3,2,0)$.

(VI) $x_4 = (\{1,*\},0,*,*)$, $x_7 = (\{2,3,*\},0,*)$, $x_8 = (\{2,*,*\},0,*)$, $x_9 = (\{3,*,*\},0,*)$. As in (III), the points 4, 5 and 6 occur in the first, second or third positions of blocks x_7, x_8 and x_9 either once or three times. Therefore $x_7 = (\{2,3,4\},0,*)$, $x_8 = (\{2,4,5\},0,*)$ and $x_9 = (\{3,4,6\},0,*)$. Further, the point 4 occurs precisely once in the first or second positions of blocks x_4, x_5 and x_6. Now consider the blocks $x_{10} = (*,*,*,*,0)$, $x_{11} = (*,*,*,*,0)$ and $x_{12} = (*,*,*,*,0)$. By the same reasoning as in (V), the points 2 and 3 occur in precisely two of these and the point 4 in precisely one. It is easily verified that no matter how these are distributed it is impossible to order blocks x_7, x_8, x_9, x_{10}, x_{11} and x_{12} without repeating a triple $(2,3,0)$, $(3,2,0)$, $(2,4,0)$, $(4,2,0)$, $(3,4,0)$ or $(4,3,0)$.

(VII) $x_7 = (1,2,3,0,*)$, $x_8 = (3,2,1,0,*)$. Then $x_9 = (4,5,6,0,A)$ and it follows that $x_4 = (6,5,0,B,C)$, $x_5 = (6,4,0,D,E)$ and $x_6 = (5,4,0,F,G)$, where the points A,B,C,D,E,F,G are all different. But none of these can be the point 0.

(VIII) $x_7 = (1,2,3,0,*)$, $x_8 = (\{1,2,*\},0,*)$, $x_9 = (\{3,*,*\},0,*)$. The argument is similar to that in (VII). We have $x_9 = (\{3,5,6\},0,A)$, which implies that $x_8 = (\{1,2,4\},0,*)$, which in turn implies that $x_4 = (\{5,6\},0,B,C)$, $x_5 = (\{4,6\},0,D,E)$ and $x_6 = (\{4,5\},0,F,G)$ where the points A,B,C,D,E,F,G are all different. Again none of these can be the point 0.

(IX) $x_7 = (\{1,2,*\},0,*)$, $x_8 = (\{1,3,*\},0,*)$, $x_9 = (\{2,3,*\},0,*)$. Then it follows that $x_7 = (\{1,2,6\},0,C)$, $x_8 = (\{1,3,5\},0,B)$ and $x_9 = (\{2,3,4\},0,A)$, where $A \neq B \neq C \neq A$. This implies that $x_4 = (\{5,6\},0,\{A,*\})$, $x_5 = (\{4,6\},0,\{B,*\})$ and $x_6 = (\{4,5\},0,\{C,*\})$. Now $A = 1$, $B = 2$ and $C = 3$. Finally, $x_1 = (1,0,\{4,5,6\})$, $x_2 = (2,0,\{4,5,6\})$ and $x_3 = (3,0,\{4,5,6\})$, but it is impossible to order these without repeating a triple.

Hence no such design exists. □

Mahmoodi [12] has shown that 3-$(v,5,\lambda)$ directed designs exist for $v = 5,6,7$ and $\lambda = 2,3$. Since a t-(v,k,λ_1) directed design and a t-(v,k,λ_2) directed design on the same point set together give a t-$(v,k,\lambda_1+\lambda_2)$ directed design, this leads to the following theorem.

Theorem 2.4 *For $v = 5,6,7$, there exists a 3-$(v,5,\lambda)$ directed design if and only if $\lambda > 1$.*

We now consider 3-$(v,5,1)$ directed designs for $v = 10$, 11, 12. The authors have tried, without success, to construct examples of such designs. However, we can rule out the existence of such designs with certain automorphisms. First we state and prove an easy lemma.

Lemma 2.5 *Let D be a t-(v,k,λ) directed design and let b be the number of blocks. For $0 \leq m < k$, if $v-m$ does not divide b then D does not have an automorphism containing a $(v-m)$-cycle.*

Proof Suppose that D has an automorphism α containing a $(v-m)$-cycle, where $0 \leq m < k$. Then each block of D contains at least one point of the $(v-m)$-cycle, and hence generates an orbit under $\langle\alpha\rangle$ whose size is a multiple of $v-m$. It follows that b is also a multiple of $v-m$. □

By using the above lemma and exhaustive computer searches we are able to prove the following result.

Theorem 2.6 *For $m = 0,1,2$ and $v = 10,11,12$ there exists no 3-$(v,5,1)$ directed design with an automorphism containing a $(v-m)$-cycle.*

Proof The values of b, the number of blocks in the directed design, corresponding to $v = 10, 11, 12$ are $b = 72, 99, 132$. Lemma 2.5 rules out the cases $(m, v) = (0, 10)$, $(1, 11)$, $(2, 12)$. We have eliminated the other cases by exhaustive computer search. □

For 3-$(v, 5, 1)$ directed designs and $m = 3, 4$, a stronger result holds.

Theorem 2.7 *For $m = 3, 4$ and $v \geq 2m + 1$, there exists no 3-$(v, 5, 1)$ directed design with an automorphism containing a $(v - m)$-cycle.*

Proof Suppose that such a design exists, with an automorphism α of the above type. Let P, Q, R be points not in the $(v - m)$-cycle. Under $\langle\alpha\rangle$, the triple (P, Q, R) generates a sub-orbit of size at most m. However, the block containing (P, Q, R) contains a point of the $(v - m)$-cycle, and hence generates an orbit of size at least $v - m$, that is, at least $2m + 1 - m = m + 1$. Hence the orbit contains a repeated triple, which is a contradiction. □

Finally in this section, we present our calculations relating to 3-$(v, 5, 1)$ directed designs for $v = 17, 26, 37$. We adopt a similar approach in all three cases. Let the set of points of the design be $\mathbf{Z}_v = \{0, 1, 2, \ldots, v - 1\}$. For $v = 17$ and 37 we seek a directed design invariant under the group of mappings $\{z \mapsto a^2 z + b \,:\, a, b \in \mathbf{Z}_v, a \neq 0\}$, that is, the Frobenius group $F_{v,(v-1)/2}$. For $v = 26$ we use the group of mappings $\{z \mapsto az + b \,:\, a, b \in \mathbf{Z}_{26}, (a, 26) = 1\}$.

An exhaustive computer search shows that there exists no 3-$(17, 5, 1)$ directed design under the action of the stated group.

For $v = 26$, we find that there are precisely two non-isomorphic solutions. These are generated respectively by the following blocks:

(a) $(0, 1, 13, 24, 19)$, $(0, 1, 4, 12, 17)$, $(0, 1, 16, 5, 6)$, $(0, 1, 11, 15, 7)$, $(0, 2, 14, 18, 21)$;

(b) $(0, 1, 17, 12, 4)$, $(0, 2, 13, 9, 12)$, $(0, 1, 23, 7, 25)$, $(0, 1, 24, 21, 22)$, $(0, 2, 6, 20, 23)$.

To prove that the two designs are non-isomorphic we calculate an *invariant*. A 3-$(v, 5, 1)$ directed design is also a 2-$(v, 5, v - 2)$ directed design, by Result 1.1. With each ordered pair $(x, y) \in V \times V$ with $x \neq y$, we can associate a 5×5 upper triangular matrix $T(x, y)$ whose entries $t_{i,j}$, $1 \leq i \leq 5$, $1 \leq j \leq 5$ are defined as follows. For $1 \leq i < j \leq 5$, $t_{i,j}$ is the number of occurences of the pair (x, y) in positions i and j respectively of a block of the design; otherwise $t_{i,j} = 0$. The invariant is easy to calculate in this case because, by the action of the automorphism group, every ordered pair can be mapped onto $(0, 1)$, $(0, 2)$ or $(0, 13)$.

For design (a), the matrices $T(0, 1)$, $T(0, 2)$, $T(0, 13)$ are respectively

$$\begin{pmatrix} 0 & 4 & 1 & 2 & 4 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For design (b), the matrices $T(0,1)$, $T(0,2)$, $T(0,13)$ are respectively

$$\begin{pmatrix} 0 & 3 & 2 & 3 & 2 \\ 0 & 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 2 & 2 & 3 \\ 0 & 0 & 3 & 3 & 2 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that both designs are *self-converse*, that is, isomorphic to the directed design obtained by reversing the blocks.

An exhaustive computer search was not possible for $v = 37$. However two such designs are generated by the following blocks under the action of the stated group:

(a) $(0,1,2,5,32)$, $(0,2,7,19,25)$, $(0,1,12,34,16)$, $(0,2,1,28,33)$, $(0,2,10,8,11)$, $(0,2,14,6,17)$, $(0,1,18,36,22)$;

(b) $(0,1,2,5,32)$, $(0,2,28,11,25)$, $(0,1,28,34,3)$, $(0,2,24,30,1)$, $(0,1,29,14,13)$, $(0,1,24,20,6)$, $(0,2,22,13,3)$.

Checking the invariants of these two designs and their converses, as described above, shows that all four designs are pairwise non-isomorphic.

We can use the replacement lemma (Result 1.2), together with one of the 3-$(26,5,1)$ directed designs constructed above, to give an infinite class of 3-$(v,5,1)$ directed designs, as follows.

Theorem 2.8 *There exists a* 3-$(25^n+1,5,1)$ *directed design for all* $n \geq 1$.

Proof There exist Steiner systems $S(3, q+1, q^n+1)$ for all prime powers q and all $n \geq 1$; these are the spherical geometries. Let $q = 25$ and use the replacement lemma with a 3-$(26,5,1)$ directed design constructed above. □

3 Results on 4-$(v,5,1)$ directed designs

The necessary conditions given by Result 1.1 for 4-$(v,5,1)$ directed designs are $v \equiv 0,1,2,3 \pmod 5$. As in the previous section, the strategy for constructing these designs will be to find suitable unions of orbits under an assumed automorphism group. Promising candidates are the sharply triply transitive groups $\mathrm{PGL}_2(p^\alpha)$, and we first present the results of our investigations on these.

Let $v \equiv 8 \pmod{10}$ and $v-1 = p^\alpha$ be a prime or prime power. Further, let $V = \mathrm{GF}(p^\alpha) \cup \{\infty\}$ be the base set of a 4-$(v,5,1)$ directed design, D, and let $\mathrm{PGL}_2(p^\alpha)$ act on the ordered 5-tuples of V in the usual way. The number of blocks of D is $v(v-1)(v-2)(v-3)/5$. The order of $\mathrm{PGL}_2(p^\alpha)$ is $(p^\alpha+1)p^\alpha(p^\alpha-1)$. Hence, the possibility exists that D may be constructed as the union of $(v-3)/5$, which is integral, full orbits of ordered 5-tuples under $\mathrm{PGL}_2(p^\alpha)$.

The first case to consider is $v = 8$, in which case a single orbit is required. Since there are in total only 20 orbits of $\mathrm{PGL}_2(7)$ acting on ordered 5-tuples of $\mathrm{GF}(7) \cup \{\infty\}$, all calculations can be done by hand. It is easily verified that precisely four of these orbits, namely those generated by the ordered 5-tuples (a) $(\infty, 0, 1, 2, 5)$, (b) $(\infty, 0, 1, 2, 6)$, (c) $(\infty, 0, 1, 3, 4)$ and (d) $(\infty, 0, 1, 4, 5)$ form a 4-$(8, 5, 1)$ directed design. It is equally easy to verify, by seeking possible isomorphisms directly, that the four systems are pairwise non-isomorphic and that systems (a) and (c) are converses, as are systems (b) and (d).

For $v = 18$, three orbits are required for the design and it is more appropriate to perform the calculations by computer. There are precisely 40 pairwise non-isomorphic 4-$(18, 5, 1)$ directed designs invariant under the group $\mathrm{PGL}_2(17)$. Twenty of these are listed in tabular form below; the other twenty are the converses of those given.

System	Orbits $(\infty, 0, 1, x_n, y_n)$					
no.	x_1	y_1	x_2	y_2	x_3	y_3
1	4	3	7	6	2	11
2	4	3	7	6	8	10
3	4	3	9	15	2	11
4	4	3	9	15	8	10
5	4	9	2	6	7	16
6	4	9	2	6	8	11
7	4	9	13	5	7	16
8	4	9	13	5	8	11
9	5	3	12	16	2	11
10	5	3	12	16	8	10

System	Orbits $(\infty, 0, 1, x_n, y_n)$					
no.	x_1	y_1	x_2	y_2	x_3	y_3
11	5	3	13	6	2	11
12	5	3	13	6	8	10
13	8	4	5	12	3	6
14	8	4	5	12	14	11
15	8	4	10	7	3	6
16	8	4	10	7	14	11
17	9	4	2	6	15	14
18	9	4	2	6	16	7
19	9	4	13	5	15	14
20	9	4	13	5	16	7

We have confirmed that the forty designs given above are non-isomorphic by the following means. With each design we associate a vector $v = (v_2, v_3, \ldots, v_{16})$ defined in the following way. Given a sub-orbit starter x, we define $n(x)$ as follows. The sub-orbit generated by x occurs in one of the three orbits of the design; if it is obtained by deleting the ith element from each block in that orbit, let $n(x) = i - 1$. Then for $j = 2, 3, \ldots, 16$, let $v_j = n((0, 1, \infty, j)) + 5n((0, 1, j, \infty)) + 25n((0, j, 1, \infty)) + 125n((j, 0, 1, \infty))$. The vector v, considered as a multiset, is invariant under isomorphism. These invariants distinguish all the designs except that they do not distinguish systems 13, 14, 15 and 16 from their respective converses. In each of these four cases, taking account of the correspondences between the points of the designs implied by their associated vectors v reduces the number of possible isomorphisms to one, and this one is easily ruled out.

For $v = 38$, there are no 4-$(v, 5, 1)$ directed designs invariant under the group $\mathrm{PGL}_2(v - 1)$. This was a complete surprise to the authors, and, at the time of writing, we have been unable to determine a mathematical reason why there are no such designs. This requires further study. For $v = 48$ the computer produced over 2 000 directed designs in ten minutes. Based on an analysis of the fraction of the search space completed we estimate that there are approximately 2.5 million such solutions! An example of a 4-$(48, 5, 1)$ directed design invariant under the group $\mathrm{PGL}_2(47)$ is generated by the following nine ordered 5-tuples: $(\infty, 0, 1, x, y)$ for $(x, y) = (2, 6)$, $(7, 23)$, $(15, 22)$, $(12, 11)$, $(30, 26)$, $(13, 20)$, $(34, 37)$, $(9, 28)$, $(38, 17)$.

Three further values of v for which we have investigated the existence of a 4-$(v, 5, 1)$ directed design are $v = 6$, 7 and 13. An exhaustive computer search shows that there exists no 4-$(6, 5, 1)$ directed design.

A 4-$(7, 5, 1)$ directed design on the set $\mathbf{Z}_7$ is generated by the following 24 blocks under the action of the cyclic group of mappings $\{z \mapsto z + b \,:\, b \in \mathbf{Z}_7\}$:

$$\begin{array}{llllll}
(0,1,2,3,6), & (0,1,3,2,4), & (0,3,4,6,1), & (0,2,3,1,5), & 0,5,6,2,1), & (0,4,1,5,3), \\
(0,6,5,1,3), & (0,5,2,3,4), & (0,1,5,4,6), & (0,4,5,1,2), & (0,5,6,4,3), & (0,6,1,5,2), \\
(0,2,4,6,5), & (0,3,5,1,6), & (0,1,6,4,2), & (0,3,5,4,2), & (0,4,6,2,3), & (0,2,5,1,4), \\
(0,2,6,4,1), & (0,3,6,4,5), & (0,6,3,1,4), & (0,4,3,2,1), & (0,6,3,2,5), & (0,5,3,2,6).
\end{array}$$

A 4-$(13, 5, 1)$ directed design on the set $\mathbf{Z}_{13}$ is generated by the following 22 blocks under the action of the Frobenius group of mappings $\{z \mapsto az + b \,:\, a, b \in \mathbf{Z}_{13}, a \neq 0\}$:

$$\begin{array}{lllll}
(0,1,2,3,7), & (0,1,4,6,8), & (0,1,10,5,6), & (0,1,6,7,5), & (0,1,11,6,10), \\
(0,1,5,10,3), & (0,1,3,5,12), & (0,1,4,9,11), & (0,1,3,10,4), & (0,1,5,9,4), \\
(0,1,11,3,9), & (0,1,3,6,2), & (0,1,8,7,3), & (0,1,9,10,12), & (0,1,4,7,2), \\
(0,1,12,11,5), & (0,1,10,7,9), & (0,1,8,9,5), & (0,1,11,7,4), & (0,1,10,11,2), \\
(0,1,12,9,8), & (0,1,8,4,12). & & &
\end{array}$$

The existence of a 4-$(7, 5, 1)$ directed design is of particular importance because it permits the construction of the first infinite class of 4-$(v, 5, 1)$ directed designs.

Theorem 3.1 *There exists a* 4-$(2^n - 1, 5, 1)$ *directed design for all* $n \geq 3$.

Proof There exist generalized Steiner systems $S(4, \{5, 7\}, 2^n - 1)$ for all $n \geq 3$ (see [10]). Use the replacement lemma with the 4-$(5, 5, 1)$ directed design and the 4-$(7, 5, 1)$ directed design given above. □

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