Switching cycles in Steiner triple systems

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Abstract

We investigate equivalence classes under the operation of n-cycle switching on isomorphism classes of STS(v)s for $v \in \{7, 9, 13, 15\}$ and $n \in \{4, 6, 8, 10, 12\}$. We also investigate equivalence classes under the operation of n-cycle switching on realizations of STS(v)s on a fixed base set V for $v \in \{7, 9, 13\}$ and $n \in \{4, 6, 10\}$, and for $(v, n) = (15, 12)$.

1 Introduction

A *Steiner triple system* of order v, briefly STS(v), is a pair $(V, \mathcal{B})$ where V is a set of v elements and $\mathcal{B}$ is a collection of 3-element subsets of V, with the property that every 2-element subset of V appears in exactly one member of $\mathcal{B}$. The members of $\mathcal{B}$ are called *blocks*. For each pair $\{a, b\} \subset V$, a graph G_{ab} can be defined as follows. The vertices of G_{ab} are $V \backslash \{a, b, c\}$ where $\{a, b, c\}$ is a block; $\{x, y\}$ is an edge if either $\{a, x, y\}$ or $\{b, x, y\} \in \mathcal{B}$. Clearly G_{ab} is a union of disjoint cycles; it is called the *cycle graph* through $\{a, b\}$. We can associate with this a *cycle list* $(c_1, c_2, c_3, ..., c_n)$ where the c_is are the lengths of the individual cycles in G_{ab} and $c_i \geq c_{i+1}, i = 1, 2, 3, ..., n-1$. Clearly $\sum_{i=1}^{n} c_i = v - 3$. The *cycle structure* of an STS(v) is the collection of all $\binom{v}{2}$ cycle lists for that STS(v). An STS(v) is said to be *uniform* if all its cycle lists are identical. A uniform STS(v) is said to be *perfect* when all of the cycles are of length $v - 3$. In this case each cycle list is just $(v - 3)$.

An operation which can be carried out on an STS(v) is that of cycle switching. For a pair $\{a, b\} \subset V$, select one of the components of the cycle graph G_{ab}. For each edge $\{x, y\}$ of this component resulting from a block $\{a, x, y\}$ replace this block by $\{b, x, y\}$ and conversely. If the component is a cycle of length n, this operation is called an *n-cycle switch*. Applying an n-cycle switch to an STS(v) will result in another STS(v). STS(v)s for $v = 7$ and $v = 9$ are unique and perfect. It is well-known that the two non-isomorphic STS(13)s can be obtained from one another by switching appropriate 4-cycles. Theorem 2 below shows that this result continues to hold by switching appropriate 6-cycles.

There exist 80 pairwise non-isomorphic STS(15)s, all but one of which give rise to 4-cycles [5] (page 10). Gibbons [1] analysed the 79 STS(15)s containing 4-cycles and showed that all of them can be obtained from one another by successive operations of 4-cycle switching. This can be described in general graph-theoretic terms which is more appropriate for our purposes. For every $v \equiv 1$ or 3 (mod 6) and every even n such that $4 \leq n \leq v - 3$ (except for $n = v - 5$ which can not occur) define a graph $M(v, n)$, which we will call the *isomorphism classes graph*, as follows.

The vertices of $M(v, n)$ are the set of pairwise non-isomorphic STS(v). Two vertices are joined by an edge if and only if there is an n-cycle switch which will transform either system into the other. (Note that it is possible and permissible for a graph $M(v, n)$ to contain loops). Gibbons' result is that the graph M(15,4) consists of a connected component having 79 vertices together with an isolated point. Isolated points correspond to those systems whose cycle structure contains no n-cycles. Apart from such points, a fundamental question is whether the graphs $M(v, n)$ are connected. In [2], the present authors showed that for $v = 19, 21, 25, 27$ and 31 the graphs

$M(v,4)$ are not connected. In this paper we consider the switching of cycles of length greater than 4 paying particular attention to the 80 STS(15)s. However we first note some elementary results about switching cycles.

2 General Results

Theorem 1 *Suppose S is an STS(v) with a cycle graph G_{ab} consisting of a single cycle. Let T be the STS(v) obtained by performing the $(v-3)$-cycle switch on this cycle. Then S and T are isomorphic.*
Proof The permutation $(a\ b)$ is the required isomorphism. □

Corollary 1 *Any perfect STS(v) under the operation of $(v-3)$-cycle switching transforms to an isomorphic copy of itself.* □

Theorem 2 *Suppose S is an STS(v) with a cycle graph G_{ab} consisting of an x-cycle and a y-cycle with $x+y=v-3$. Let T_x be the STS(v) obtained by performing the x-cycle switch and T_y the STS(v) obtained by performing the y-cycle switch. Then T_x and T_y are isomorphic.*
Proof Again the permutation $(a\ b)$ is the required isomorphism. □

Theorem 3
(i) For $v \equiv 1$ or $3 \pmod 6$, $v \geq 13$, the graph $M(v, v-7)$ is a subgraph of the graph $M(v,4)$.
(ii) For $v \equiv 1$ or $3 \pmod 6$, $v \geq 13$, the graph $M(v, v-9)$ is a subgraph of the graph $M(v,6)$.
Proof Observe that if a cycle graph G_{ab} contains a $v-7$-cycle (respectively a $v-9$-cycle) then it must also contain a 4-cycle (respectively a 6-cycle). □

It is almost trivial now to describe the graphs $M(v,n)$ for $v \leq 13$ and we do so only for completeness. $M(7,4)$ and $M(9,6)$ each consist of a single vertex together with a loop. $M(13,4)$ has two vertices, corresponding to the cyclic and non-cyclic STS(13)s respectively, which are joined by an edge. In addition the vertex corresponding to the non-cyclic STS(13) has a loop. By Theorem 2, $M(13,6)$ is isomorphic to $M(13,4)$. By Theorem 1, $M(13,10)$ comprises two disconnected vertices with a loop on each.

3 Analysis of the STS(15)s

This section is concerned with the operation of cycle switching on the 80 STS(15)s. Throughout this paper we follow the system of identification as given in [4], from which the cycle structure and all other relevant information concerning these systems may be obtained. Full details of the cycle switching is given in the Appendix.

There is only one STS(15) which is 4-cycle (quadrilateral) free, this being system #80. We have verified Gibbons' result that the remaining 79 STS(15)s form a connected graph under 4-cycle switching. However, we find that, in addition to the switching recorded in [1], the systems #15 and #58 each switch to themselves.

There are 16 STS(15)s which are 6-cycle free, these being systems #1 to #10 and #13 to #18 inclusive. These form isolated points in the graph M(15,6). In addition we find that there are two disconnected components. Systems #11, #12, #19, #20, #21, #22 and #61 form the vertices of one component and the other 57 STS(15)s which contain 6-cycles form the vertices of the other.

From Theorem 3, the graph M(15,8) is a subgraph of the graph M(15,4). M(15,8) consists of a single connected component together with three isolated vertices which correspond to the three systems #1, #16 and #80 which are 8-cycle free.

For 12-cycle switching it has been shown in Theorem 1 that any STS(15) will switch to an isomorphic copy of itself. Of the 80 STS(15)s only #1 and #2 are 12-cycle free. Thus the graph M(15,12) consists of 80 disconnected vertices of which 78 have a loop and 2 do not.

4 Realizations of STS(v)s

We now consider the operation of cycle switching applied to realizations, rather than isomorphism classes, of Steiner triple systems. For every $v \equiv 1$ or 3 (mod 6) and every even n such that $4 \leq n \leq v-3, n \neq v-5$, a further graph, the realizations graph $R(v,n)$, can be defined whose vertices are the realizations of STS(v) on a fixed base set V; two vertices are joined by an edge if and only if there is an n-cycle switch which will transform either system into the other. Again our focus is on the connectedness of these graphs. Note that unlike the graphs $M(v,n)$, the graphs $R(v,n)$ cannot contain loops. The situations where $v = 7$ and $v = 9$ are straightforward and are given in the following theorem, again mainly for completeness.

Theorem 4 *The graphs $R(7,4)$ and $R(9,6)$ are connected.*
Proof We first present the proof for $R(7,4)$. It is well known that STS(7) is unique to within isomorphism and perfect. By Theorem 1, any 4-cycle

switch is equivalent to a transposition. Conversely any transposition and hence any sequence of transpositions can be represented by a corresponding sequence of 4-cycle switches. Since the set of all transpositions generates the symmetric group, the result follows. The proof for R(9,6) is analogous.

□

We observe that both of these graphs have undoubtedly been studied further and possess many other interesting properties. Indeed the recent paper by Lloyd [3], together with the references therein, present more information on R(7,4). We note here that the number of vertices in the graph $R(7,4)$ is $|S_7|/|Aut(\text{STS}(7))| = 7!/168 = 30$ and in the graph $R(9,6)$ the number of vertices is $|S_9|/|Aut(\text{STS}(9))| = 9!/432 = 840$.

We next consider the graph $R(13,4)$ and show that it too is connected. Denote the cyclic STS(13) by $C(13)$ and the non-cyclic STS(13) by $N(13)$. There are $|S_{13}|/|Aut(C(13))| = 13!/39$ realizations of $C(13)$ and $|S_{13}|/|Aut(N(13))| = 13!/6$ realizations of $N(13)$. There are thirteen 4-cycles in a $C(13)$ and switching any one of these gives an $N(13)$. There are eight 4-cycles in an $N(13)$, two of which switch the system to $C(13)$s and the remaining six switch the system to other $N(13)$s. Thus, starting with a given $C(13)$, we may switch firstly to an $N(13)$ and then to a *different* $C(13)$. There will be 39 permutations that map the given $C(13)$ to the new $C(13)$. Therefore a consecutive pair of 4-cycle switches of the form described may be represented by any one of these permutations. To prove that all the vertices of $R(13,4)$ which correspond to $C(13)$s are in fact connected, it suffices to prove that suitable combinations of these permutations generate the symmetric group S_{13}. Since every vertex corresponding to an $N(13)$ is connected to a vertex corresponding to a $C(13)$, this will establish that $R(13,4)$ is connected.

In order to investigate the permutations we choose as the given $C(13)$ the system with the following blocks: $\{i, i+1, i+4\}$, $\{i, i+2, i+7\}$, $i = 0, 1, \ldots, 12$ with addition modulo 13. This has a 4-cycle with blocks $\{0,1,4\}$, $\{0,2,7\}$, $\{2,4,9\}$ and $\{7,9,1\}$. If we switch this 4-cycle we obtain blocks $\{0,2,4\}$, $\{0,1,7\}$, $\{1,4,9\}$ and $\{7,9,2\}$, and thereby create an $N(13)$. This contains a 4-cycle with blocks $\{7,8,11\}, \{8,9,12\}, \{10,12,2\}$ and $\{7,9,2\}$. Switching this 4-cycle results in a new $C(13)$ containing the blocks of the given $C(13)$ apart from $\{0,1,4\}$, $\{0,2,7\}$, $\{2,4,9\}$, $\{7,9,1\}$, $\{7,8,11\}$, $\{8,9,12\}$ and $\{11,12,2\}$ which are replaced by $\{0,2,4\}$, $\{0,1,7\}$, $\{1,4,9\}$, $\{7,9,8\}$, $\{7,2,11\}$, $\{2,9,12\}$ and $\{11,12,8\}$.

Amongst the permutations mapping the given $C(13)$ to the new $C(13)$ are the following:

A: (12)(4 10)(5 6 9)(0 8 7 1 2 3 11)
B: (10)(0 1 7 2 5 12 9 11 6 8 3 4)

We will show that suitable combinations of these permutations generate the symmetric group S_{13} by obtaining all transpositions through a single element. It is well-known that these generate the symmetric group.

There is a subtle but important point concerning the combination of these permutations. The two permutations given above represent mappings from a *single* $C(13)$, namely the *given* one. In applying one of these permutations to a derived realization, the expression of this permutation in terms of the elements of this derived realization will be different from the original expression of the permutation in terms of the given system. The difficulty may be overcome by regarding the given system as a template and interpreting the permutations as mappings of the positions of the template. Thus, if P is a permutation containing $(\ldots\ a\ b\ \ldots)$, then P maps the element in template position a to the element in template position b. If X denotes a derived realization of $C(13)$ (including the given system S) then we will use the notation $P[X]$ to denote the result of applying P to X in the sense just described. As an example, using the mappings A and B given above, and denoting the given system by S we may obtain the following table.

Template	0	1	2	3	4	5	6	7	8	9	10	11	12
S	0	1	2	3	4	5	6	7	8	9	10	11	12
$A[S]$	8	2	3	11	10	6	9	1	7	5	4	0	12
$B[A[S]]$	2	1	6	10	8	12	7	3	11	0	4	9	5

Table 1. Composition of Mappings.

The result of applying firstly the mapping A followed by B to the given system S may be obtained by reading lines two and four of the Table. Using this method of combining A and B, we have $B[A[S]] = (AB)(S)$ where the expression AB is interpreted in the usual sense of composition of permutations. In general, if $\{P_i : i = 1, 2, \ldots, n\}$ is a set of such permutations then $P_n[P_{n-1}[\ldots[P_1[S]]\ldots]] = (P_1P_2\ldots P_n)(S)$. It will therefore suffice to prove that suitable compositions, in the usual sense, of the permutations A and B generate all transpositions through a single element. To do this, firstly observe that $A^{21} = (4\ 10)$. Then note that $\{B^nA^{21}B^{12-n} : n = 1, 2, \ldots, 11\}$ gives all the remaining transpositions through the element 10.

The argument concerning the graph $R(13, 6)$ is similar. We take the same given $C(13)$. This has a 6-cycle with blocks $\{1, 3, 8\}, \{1, 10, 11\}$, $\{1, 6, 12\}$, $\{2, 8, 10\}$, $\{2, 11, 12\}$ and $\{2, 3, 6\}$. If we switch this 6-cycle we obtain blocks $\{2, 3, 8\}, \{2, 10, 11\}, \{2, 6, 12\}, \{1, 8, 10\}, \{1, 11, 12\}$ and $\{1, 3, 6\}$, and thereby create an $N(13)$. This contains a 6-cycle with blocks $\{1, 0, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 6\}, \{8, 4, 5\}, \{8, 2, 3\}$ and $\{8, 0, 6\}$. Switching this 6-cycle results in a new $C(13)$ containing the blocks of the given $C(13)$ apart from $\{0, 1, 4\}, \{1, 2, 5\}, \{2, 3, 6\}, \{4, 5, 8\}, \{10, 11, 1\}, \{11, 12, 2\}, \{1, 3, 8\}, \{6, 8, 0\}$,

$\{8,10,2\}$ and $\{12,1,6\}$ which are replaced by $\{0,8,4\},\{8,2,5\},\{8,3,6\}$, $\{4,5,1\},\{10,11,2\},\{11,12,1\},\{2,3,1\},\{6,1,0\},\{8,10,1\}$ and $\{12,2,6\}$. Amongst the permutations mapping the given $C(13)$ to the new $C(13)$ are the following:

C: (12)(4 10)(5 6 9)(0 1 8 7 2 3 11)

D: (10)(0 6 2 4 3 9 7 11 1 8 12 5)

We observe that $C^{21} = (4\ 10)$ and that $\{D^n C^{21} D^{12-n} : n = 1, 2, \ldots, 11\}$ gives all the remaining transpositions through the element 10.

Finally in this section we consider the graph $R(13,10)$. By Theorem 1, this must comprise at least two disconnected components corresponding to the cyclic and the non-cyclic STS(13)s respectively. We show that each of these components is connected using an argument similar to that given above for $R(13,4)$ and $R(13,6)$. Each 10-cycle switch is equivalent to the transposition of two elements and we show that there are enough of these to generate the symmetric group S_{13}.

Consider the cyclic STS(13) given previously. This has 10-cycle switches giving rise to the following transpositions: $(n\ n+2)$, $(n\ n+5)$, $(n\ n+6)$, $(n\ n+7)$, $(n\ n+8)$ and $(n\ n+11)$, $n = 0, 1, \ldots, 12$, with addition modulo 13. The remaining transpositions through the element 0 may be obtained as follows: (0 1) = (1 6)(0 6)(1 6), (0 3) = (3 8)(0 8)(3 8), (0 4) = (4 6)(0 6)(4 6), (0 9) = (9 11)(0 11)(9 11), (0 10) = (10 5)(0 5)(10 5) and (0 12) = (12 5)(0 5)(12 5).

Take the non-cyclic STS(13) to be the one obtained from the cyclic STS(13) given previously by switching the 4-cycle with blocks $\{0,1,4\}$, $\{0,2,7\}$, $\{2,4,9\}$ and $\{7,9,1\}$. These blocks are replaced by $\{0,2,4\}$, $\{0,1,7\}$, $\{1,4,9\}$ and $\{7,9,2\}$. Amongst others, this has 10-cycle switches giving rise to the following transpositions: (0 2), (0 5), (0 6), (0 7), (0 8), (0 10), (0 11), (0 12), (1 5), (3 5), (4 6) and (6 9). Now, (0 1) = (1 5)(0 5)(1 5), (0 3) = (3 5)(0 5)(3 5), (0 4) = (4 6)(0 6)(4 6) and (0 9) = (9 6)(0 6)(9 6), thus giving all transpositions through the element 0.

We summarize the results of this Section concerning realizations of the STS(13)s in the following Theorem.

Theorem 5
(i) The graphs $R(13,4)$ and $R(13,6)$ are connected.
(ii) The graph $R(13,10)$ comprises precisely two disconnected components corresponding to the cyclic and to the non-cyclic STS(13)s. □

5 The graph $R(15, 12)$

The only graph $R(15, n)$ which it appears tractable to study is that for $n = 12$. By Theorem 1, if S is an STS(15) every 12-cycle switch will give an STS(15) isomorphic to S. Thus the graph $R(15, 12)$ must comprise at least 80 mutually disconnected components, each of these being a subgraph obtained by consideration of one of the 80 pairwise non-isomorphic STS(15)s. Denote by $R(i, 15, 12)$, $1 \leq i \leq 80$, the component corresponding to STS(15) $\#i$ as in [4]. We explore the connectedness of each of the components $R(i, 15, 12)$.

Firstly, as observed earlier, systems #1 and #2 are 12-cycle free. It follows therefore that the graphs $R(i, 15, 12)$, $i = 1, 2$, consist of $|S_{15}|/|Aut(\#i)|$ isolated vertices. The actual numbers are $15!/20160$ for $R(1, 15, 12)$ and $15!/192$ for $R(2, 15, 12)$.

We then find that precisely 72 of the remaining 78 graphs $R(i, 15, 12)$ are connected, namely those for $i \notin \{1, 2, 3, 4, 5, 6, 7, 13\}$. To prove this, recall from Theorem 1 that a 12-cycle switch is equivalent to a transposition of a pair of elements. As before, it then suffices to show that for a given value of i, compositions of these transpositions generate all transpositions through a single element, and hence that they generate the symmetric group S_{15}. Having established this, it follows that each graph $R(i, 15, 12)$, for $i \geq 8$ and $i \neq 13$, forms a connected graph on $15!/|Aut(\#i)|$ vertices. For reasons of space we do not give the details for all 72 cases but, as an example, take the case $i = 51$. Here there are 12-cycles giving rise to transpositions $(1\ b)$ for $b \in \{3, 4, 5, 6, 8, 9, 12, 13, 14, 15\}$, as well as $(2\ 6)$, $(3\ 7)$, $(3\ 10)$ and $(3\ 11)$. We obtain all remaining transpositions through the element 1 as follows: $(1\ 2) = (1\ 6)(2\ 6)(1\ 6)$, $(1\ 7) = (1\ 3)(3\ 7)(1\ 3)$, $(1\ 10) = (1\ 3)(3\ 10)(1\ 3)$ and $(1\ 11) = (1\ 3)(3\ 11)(1\ 3)$.

Each of the remaining six cases is now discussed individually using the representation of the system and its automorphism group given in [4].

#3 There exist 12-cycles corresponding to the transpositions $(a\ b)$ for $a \in \{4, 5, 6, 7\}$ and $b \in \{8, 9, 10, 11, 12, 13, 14, 15\}$. These generate the symmetric group S_{12} on $\{4, 5, \ldots, 15\}$. There are no 12-cycles giving rise to transpositions involving the elements 1, 2 or 3, and the automorphism group of #3 partitions $\{1, 2, 3\}$ from the remaining elements. The restriction of $Aut(\#3)$ to the set $\{1, 2, 3\}$ is the cyclic group C_3. It follows that the 15! permutations of $\{1, 2, \ldots, 15\}$ partition into sets, each of cardinality $3 \times 12!$, such that the permutations in each set give rise either to identical realizations of #3 or to realizations which are equivalent under a sequence of 12-cycle switches. Therefore the graph $R(3, 15, 12)$ has $15!/3.12! = 910$ components, each of which is a connected graph on $3.12!/|Aut(\#3)| = 3.12!/96$

vertices.

#4 There exist 12-cycles corresponding to the transpositions $(a\ b)$ for $a \in \{4, 6\}$ and $b \in \{8, 11, 13, 14\}$. These generate the symmetric group S_6 on $\{4, 6, 8, 11, 13, 14\}$. There also exist 12-cycles corresponding to the transpositions $(a\ b)$ for $a \in \{5, 7\}$ and $b \in \{9, 10, 12, 15\}$. These generate the symmetric group S_6 on $\{5, 7, 9, 10, 12, 15\}$. There are no 12-cycles giving rise to transpositions involving the elements 1, 2 or 3, or to transpositions $(a\ b)$ with $a \in \{4, 6, 8, 11, 13, 14\}$ and $b \in \{5, 7, 9, 10, 12, 15\}$. The automorphism partition of #4 is $\{1\}, \{2\}, \{3\}$, $\{4, 6\}, \{5, 7\}, \{8, 11, 13, 14\}, \{9, 10, 12, 15\}$ and the restriction of $Aut(\#4)$ to $\{1, 2, 3\}$ is the identity group. It follows that the 15! permutations of $\{1, 2, \ldots, 15\}$ partition into sets, each of cardinality $(6!)^2$ within which the corresponding realizations of #4 are equivalent under a sequence of 12-cycle switches. Therefore the graph $R(4, 15, 12)$ has $15!/(6!)^2$ components, each of which is a connected graph on $(6!)^2/|Aut(\#4)| = (6!)^2/8$ vertices.

#5 There exist 12-cycles corresponding to the transpositions $(a\ b)$ for $a \in \{4, 5, 6, 7\}$ and $b \in \{8, 9, 10, 11\}$. These generate the symmetric group S_8 on $\{4, 5, \ldots, 11\}$. There are no 12-cycles giving rise to transpositions involving the elements 1, 2, 3, 12, 13, 14 or 15, and the automorphism partition of #5 is $\{1, 3\}, \{2\}, \{4, 5, 6, 7, 8, 9, 10, 11\}, \{12, 13, 14, 15\}$. The restriction of $Aut(\#5)$ to the set $\{1, 2, 3, 12, 13, 14, 15\}$ is the dihedral group D_4 of order 8. It follows that the 15! permutations of $\{1, 2, \ldots, 15\}$ partition into sets, each of cardinality $8 \times 8!$, such that the permutations in each set give rise either to identical realizations of #5 or to realizations which are equivalent under a sequence of 12-cycle switches. Therefore the graph $R(5, 15, 12)$ has $15!/8.8!$ components, each of which is a connected graph on $8.8!/|Aut(\#5)| = 8.8!/32$ vertices.

#6 There exist 12-cycles corresponding to the transpositions $(a\ b)$ for $a \in \{4, 5\}$ and $b \in \{8, 11, 12, 14\}$. These generate the symmetric group S_6 on $\{4, 5, 8, 11, 12, 14\}$. There also exist 12-cycles corresponding to the transpositions $(a\ b)$ for $a \in \{6, 7\}$ and $b \in \{9, 10, 13, 15\}$. These generate the symmetric group S_6 on $\{6, 7, 9, 10, 13, 15\}$. There are no 12-cycles giving rise to transpositions involving the elements 1, 2 or 3, or to transpositions $(a\ b)$ with $a \in \{4, 5, 8, 11, 12, 14\}$ and $b \in \{6, 7, 9, 10, 13, 15\}$. The automorphism partition of #6 is $\{1, 2, 3\}$, $\{4, 5, 8, 11, 12, 14\}, \{6, 7, 9, 10, 13, 15\}$ and the restriction of $Aut(\#6)$ to $\{1, 2, 3\}$ is the symmetric group S_3. It follows that the 15! permutations of $\{1, 2, \ldots, 15\}$ partition into sets, each of cardinality $3!(6!)^2$, such that the permutations in each set give rise either to identical realizations of #6 or to realizations which are equivalent under a sequence of 12-cycle switches. Therefore the graph $R(6,15,12)$ has $15!/3!(6!)^2$ components, each of which is a connected graph on $3!(6!)^2/|Aut(\#6)| = 3!(6!)^2/24$ vertices.

#7 There exist 12-cycles corresponding to the transpositions $(a\ b)$ for $a \in \{4,5,6,7\}$ and $b \in \{8,9,10,11,12,13,14,15\}$. These generate the symmetric group S_{12} on $\{4,5,\ldots,15\}$. There are no 12-cycles giving rise to transpositions involving the elements 1, 2 or 3, and the automorphism group of #7 partitions $\{1,2,3\}$ from the remaining elements. The restriction of $Aut(\#7)$ to the set $\{1,2,3\}$ is the symmetric group S_3. It follows that the 15! permutations of $\{1,2,\ldots,15\}$ partition into sets, each of cardinality 3!12!, such that the permutations in each set give rise either to identical realizations of #7 or to realizations which are equivalent under a sequence of 12-cycle switches. Therefore the graph $R(7,15,12)$ has $15!/3!12! = 455$ components, each of which is a connected graph on $3!12!/|Aut(\#7)| = 3!12!/288$ vertices.

#13 There exist 12-cycles corresponding to the transpositions $(a\ b)$ for $a \in \{6,7\}$ and $b \in \{8,9,10,11,12,13,14,15\}$, as well as (2 9), (3 8), (4 9) and (5 8). These generate the symmetric group S_{14} on $\{2,3,\ldots,15\}$. There are no 12-cycles giving rise to transpositions involving the element 1, and the automorphism group of #13 partitions $\{1\}$ from the remaining elements. It follows that the 15! permutations of $\{1,2,\ldots,15\}$ partition into sets, each of cardinality 14! within which the corresponding realizations of #13 are equivalent under a sequence of 12-cycle switches. Therefore the graph $R(13,15,12)$ has $15!/14! = 15$ components, each of which is a connected graph on $14!/|Aut(\#13)| = 14!/8$ vertices.

We summarize the results of this Section concerning realizations of the STS(15)s in the following Theorem.

Theorem 6 *The graph $R(15,12)$ comprises 80 mutually disconnected components, $R(i,15,12), 1 \le i \le 80$, corresponding to the 80 pairwise non-isomorphic STS(15)s. Of these components, eight are themselves disconnected and these are described in Table 2 below. The remaining 72 form connected graphs with $R(i,15,12)$ having $15!/|Aut(\#i)|$ vertices.*

i	*Number of mutually disconnected components in* $R(i, 15, 12)$	*Number of vertices in each connected component of* $R(i, 15, 12)$
1	15!/20160	1
2	15!/192	1
3	15!/3.12! = 910	3.12!/96
4	$15!/(6!)^2$	$(6!)^2/8$
5	15!/8.8!	8.8!/32
6	$15!/3!(6!)^2$	$3!(6!)^2/24$
7	15!/3!12! = 455	3!12!/288
13	15!/14! = 15	14!/8

Table 2. *The Disconnected Components of* $R(15, 12)$.

□

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Appendix

Below are given the tables for switching 4-cycles, 6-cycles and 8-cycles in the 80 STS(15)s. The systems are identified numerically in the first column, following the enumeration given in [4]. Within each table the second column gives the number of n-cycles contained in that STS(15). The subsequent entries in each row indicate which STS(15)s are obtained by switching each *particular* n-cycle. Thus the Table for switching 4-cycles gives more information than that given by Gibbons [1] (Table 7.5, page 153). The n-cycles are ordered as described in the next paragraph. We use the realization of each isomorphism class on the base set $\{1, 2, \ldots, 15\}$ as given in [4].

The blocks forming each n-cycle span $n + 2$ distinct elements. We will identify an ordered triple $\langle\alpha, \beta, \gamma\rangle$ of these elements by the process described below. These ordered triples are then used to order the n-cycles. If the n-cycles C_i and C_j correspond to the ordered triples $\langle\alpha_i, \beta_i, \gamma_i\rangle$ and $\langle\alpha_j, \beta_j, \gamma_j\rangle$ respectively, then we place C_i before C_j if either

(i) $\alpha_i < \alpha_j$, or

(ii) $\alpha_i = \alpha_j$ and $\beta_i < \beta_j$, or

(iii) $\alpha_i = \alpha_j$, $\beta_i = \beta_j$ and $\gamma_i < \gamma_j$.

It will be easily seen that ordered triples corresponding to distinct n-cycles of a particular STS(15) are themselves distinct.

For a 4-cycle $\{x, a, b\}$, $\{x, c, d\}$, $\{y, b, c\}$, $\{y, d, a\}$, put $\alpha = \min\{x, y, a, b, c, d\}$. Take β to be the unique element of $\{x, y, a, b, c, d\}$ such that the pair $\{\alpha, \beta\}$ is not contained in a block of the 4-cycle, and then put $\gamma = \min(\{x, y, a, b, c, d\} \setminus \{\alpha, \beta\})$.

For a 6-cycle $\{x, a, b\}$, $\{x, c, d\}$, $\{x, e, f\}$, $\{y, b, c\}$, $\{y, d, e\}$, $\{y, f, a\}$, put $\alpha = \min\{x, y\}$, $\beta = \max\{x, y\}$ and $\gamma = \min\{a, b, c, d, e, f\}$.

For an 8-cycle $\{x, a, b\}$, $\{x, c, d\}$, $\{x, e, f\}$, $\{x, g, h\}$, $\{y, b, c\}$, $\{y, d, e\}$, $\{y, f, g\}$, $\{y, h, a\}$, put $\alpha = \min\{x, y\}$, $\beta = \max\{x, y\}$ and $\gamma = \min\{a, b, c, d, e, f, g, h\}$.

4 - Cycle Switching.

1	105	ALL 4 - CYCLES SWITCH TO SYSTEM # 2
2	73	3 5 5 3 5 5 3 5 5 3 5 5 3 5 5 3 5 5 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 1 2 2 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 2 2 4 4 4 4 4 4 4 4 2 2
3	57	2 5 5 2 5 5 16 8 8 16 8 8 16 8 8 16 8 8 14 14 14 14 14 14 14 14 2 5 5 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 4 4 4 4 4 4 4 4 4 4 4 4
4	49	4 4 4 4 6 6 8 10 10 8 8 10 10 8 9 10 12 12 10 12 12 9 10 12 12 9 9 10 12 12 2 4 4 9 9 9 9 9 9 14 13 9 9 14 13 5 5 2 3
5	49	2 5 3 5 5 7 8 10 10 8 10 10 8 10 10 8 10 10 8 8 8 8 13 13 13 13 5 2 3 10 10 10 10 10 10 10 10 6 6 4 4 4 4 4 4 4 4 6 6
6	37	4 6 4 4 4 6 15 11 11 15 11 11 18 18 18 15 15 18 18 15 18 15 6 4 4 11 11 11 11 11 11 11 11 7 5 5 5
7	33	5 5 5 5 5 5 17 17 17 17 17 17 17 17 17 17 17 17 5 5 5 6 6 6 6 6 6 6 6 6 6 6 6
8	37	4 10 10 4 8 8 8 8 8 15 15 8 18 18 26 26 26 26 5 15 15 18 18 8 8 16 3 17 13 14 13 9 9 9 9 10 10
9	31	9 11 11 4 9 15 10 18 9 10 10 18 20 20 23 24 24 23 10 13 14 29 24 25 29 24 25 8 4 4 8
10	31	4 12 12 9 18 10 13 14 10 15 23 31 31 23 10 11 11 9 9 9 15 8 4 26 23 25 25 26 23 5 8
11	31	11 9 9 12 11 20 20 19 27 28 32 27 30 32 28 12 32 32 30 15 6 10 10
12	32	12 10 4 12 20 12 11 11 23 24 24 26 25 29 26 11 24 23 24 26 25 26 10 4 24 26 24 26 25 23 10 4
13	33	14 9 9 14 14 14 9 9 17 17 15 15 25 25 25 25 25 25 25 25 5 18 18 10 10 8 8 8 4 4 8 10 10
14	37	13 9 9 3 13 13 13 10 10 13 10 10 26 26 26 26 13 9 9 9 9 26 26 26 26 26 26 26 26 8 4 8 4 8 4 10 10
15	25	15 13 17 15 19 19 9 10 9 10 27 27 27 27 6 11 11 18 10 10 18 8 8 8 8
16	49	3 8 8 3 8 8 3 8 8 3 8 8 3 8 8 3 8 8 3 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8
17	25	13 18 18 13 18 18 13 15 15 7 13 13 13 18 18 15 15 15 15 8 8 8 8 8 8
18	25	9 20 20 10 9 15 6 18 9 20 20 10 15 9 18 17 13 28 28 8 8 28 28 8 8
19	17	19 15 15 11 11 11 11 11 63 63 11 63 63 15 15 15 15
20	20	20 9 18 11 20 11 12 11 34 33 20 34 33 59 9 18 33 34 9 18
21	20	21 22 21 22 21 22 41 41 41 39 39 39 21 47 35 40 40 47 40 47
22	17	21 21 22 22 21 61 54 58 62 53 22 58 53 54 54 58 53
23	18	24 25 9 27 26 27 28 26 23 24 27 32 31 10 34 23 12 10
24	19	25 9 23 32 33 33 27 76 28 28 26 28 33 12 34 31 23 12 9
25	20	24 23 9 30 34 29 26 27 28 26 29 25 25 26 10 34 26 26 12 13
26	23	14 26 26 25 27 28 25 25 29 25 27 24 23 28 25 26 12 26 23 12 31 10 8
27	14	11 30 32 25 24 23 63 23 28 23 36 15 26 26
28	15	24 34 33 24 11 59 23 24 27 25 70 70 26 18 26
29	19	25 26 25 33 25 25 32 9 32 33 32 9 33 25 26 25 9 12 26
30	14	32 27 11 33 25 33 36 32 27 44 43 11 31 25
31	18	23 26 10 24 23 10 24 30 23 24 26 26 23 26 10 24 30 10
32	13	30 11 27 34 24 59 70 63 75 44 11 23 29
33	12	34 24 28 30 24 70 64 64 52 29 20 24
34	12	33 28 24 32 25 60 34 70 34 25 20 23
35	13	40 40 40 38 38 38 53 53 53 41 41 41 21
36	10	43 30 44 44 30 43 27 27 27 27
37	6	42 42 42 42 42 42
38	9	38 38 35 45 69 56 68 50 39
39	12	39 39 21 50 58 40 45 48 46 49 38 39
40	13	40 40 35 45 40 39 58 48 74 21 41 41 41
41	12	41 41 21 50 54 47 45 40 35 40 40 41
42	8	42 42 37 73 67 67 66 66
43	10	30 44 36 79 44 44 36 36 30 30
44	8	30 43 36 78 52 52 32 32
45	9	39 50 38 66 41 55 65 47 40
46	7	49 48 47 66 48 73 39
47	10	48 49 46 45 41 54 47 74 48 21
48	8	47 46 49 47 65 40 46 39
49	7	47 46 48 65 57 72 39
50	6	38 45 39 50 66 41
51	9	55 54 57 65 56 78 52 55 53
52	9	51 53 56 59 71 44 33 70 53
53	10	54 22 58 52 56 64 62 35 52 51
54	11	53 58 22 55 57 55 47 54 41 51 55
55	9	51 57 54 51 54 69 62 45 54
56	8	53 51 52 38 72 68 71 58
57	5	54 55 51 49 58
58	8	22 54 53 58 39 40 56 57
59	13	60 60 60 52 32 32 52 52 32 28 28 28 20
60	7	59 60 60 70 71 75 34
61	14	22 22 22 22 22 22 64 64 64 22 64 64 64 64
62	7	53 53 53 55 55 55 22
63	7	32 32 27 32 27 27 19
64	10	53 53 53 33 33 33 33 33 61 33
65	7	67 49 72 51 69 48 45
66	6	50 46 68 67 42 45
67	5	65 72 68 42 66
68	6	69 66 71 56 67 38
69	5	68 55 38 65 72
70	9	71 33 28 32 76 60 28 34 52
71	5	70 68 52 60 56
72	5	69 67 65 49 56
73	6	42 46 46 46 46 42
74	8	47 47 47 47 40 40 40 40
75	7	32 60 32 77 60 60 32
76	10	70 24 70 70 24 70 70 24 24 24
77	2	75 77
78	6	44 51 51 51 44 51
79	6	43 43 43 43 43 43
80	0	

6 - Cycle Switching.

1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	0	
9	0	
10	0	
11	8	22 22 21 21 22 22 21 21
12	8	22 22 21 21 21 21 21 21
13	0	
14	0	
15	0	
16	0	
17	0	
18	0	
19	8	21 21 21 21 21 21 21 21
20	8	61 61 22 22 22 22 22 22
21	14	19 19 12 12 11 11 12 12 11 11 12 12 11 11
22	14	11 11 20 20 11 11 20 20 20 20 11 11 12 12
23	6	45 45 47 47 58 58
24	10	47 47 39 39 53 53 47 47 39 39
25	8	55 55 41 41 35 35 40 40
26	8	41 41 54 54 40 40 40 40
27	10	74 74 41 41 58 58 50 50 57 57
28	14	48 48 39 39 51 51 45 45 54 54 38 38 48 48
29	8	41 41 41 41 41 41 62 62
30	12	40 40 41 41 53 53 40 40 53 53 41 41
31	2	74 74
32	18	47 47 41 41 54 54 66 66 53 53 49 49 39 39 47 47 62 62
33	16	49 49 45 45 52 52 50 50 46 46 54 54 64 64 58 58
34	20	65 65 48 48 56 56 46 46 51 51 69 69 38 38 54 54 53 53 58 58
35	20	63 63 44 44 25 25 25 25 44 44 44 44 25 25 38 38 38 38 38 38
36	8	40 40 40 40 40 40 40 40
37	24	47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47 47
38	26	70 70 51 51 52 52 34 34 78 78 28 28 50 50 55 55 45 45 35 35 39 39 69 69 53 53
39	20	70 70 76 76 28 28 24 24 32 32 24 24 65 65 56 56 51 51 38 38
40	14	36 36 25 25 26 26 43 43 30 30 26 26 68 68
41	16	27 27 26 26 30 30 25 25 32 32 29 29 71 71 56 56
42	40	55 55 55 55 54 54 54 54 45 45 45 45 48 48 48 48 46 46 46 46 49 49 49 49 72 72 72 72 75 75 75 75 60 60 60 60 65 65 65 65
43	12	40 40 40 40 40 40 40 40 40 40 40 40
44	16	45 45 45 45 35 35 35 35 48 48 48 48 52 52 52 52
45	24	33 33 51 51 49 49 23 23 52 52 44 44 75 75 28 28 38 38 66 66 42 42 71 71
46	28	70 70 60 60 33 33 59 59 34 34 42 42 72 72 66 66 72 72 57 57 71 71 55 55 68 68 67 67
47	18	63 63 24 24 32 32 24 24 32 32 23 23 37 37 68 68 51 51
48	20	28 28 44 44 28 28 34 34 70 70 42 42 67 67 56 56 69 69 55 55
49	24	75 75 33 33 32 32 52 52 70 70 42 42 54 54 45 45 68 68 66 66 72 72 71 71
50	24	70 70 78 78 72 72 27 27 51 51 63 63 77 77 33 33 38 38 67 67 58 58 57 57
51	30	69 69 50 50 39 39 53 53 66 66 45 45 28 28 60 60 60 60 34 34 67 67 38 38 47 47 56 56 52 52
52	24	69 69 71 71 38 38 45 45 33 33 56 56 44 44 53 53 49 49 55 55 51 51 72 72
53	18	38 38 52 52 24 24 34 34 30 30 32 32 51 51 68 68 56 56
54	26	59 59 60 60 26 26 42 42 66 66 34 34 33 33 28 28 32 32 49 49 65 65 56 56 68 68
55	30	38 38 52 52 46 46 75 75 68 68 67 67 25 25 59 59 42 42 60 60 70 70 70 70 48 48 72 72 71 71
56	24	69 69 70 70 39 39 34 34 51 51 68 68 41 41 48 48 52 52 65 65 54 54 53 53
57	22	73 73 58 58 64 64 67 67 65 65 27 27 60 60 66 66 60 60 50 50 46 46
58	14	33 33 23 23 67 67 50 50 34 34 27 27 57 57
59	32	46 46 46 46 46 46 64 64 68 68 68 68 68 68 55 55 55 55 55 55 54 54 65 65 65 65 65 65 54 54 54 54
60	36	67 67 42 42 69 69 72 72 73 73 65 65 66 66 54 54 51 51 57 57 57 57 69 69 46 46 73 73 72 72 51 51 69 69 55 55
61	14	20 20 20 20 20 20 20 20 20 20 20 20 20 20
62	14	71 71 71 71 71 71 29 29 32 32 32 32 32 32
63	14	50 50 50 50 50 50 47 47 35 35 47 47 47 47
64	20	66 66 57 57 57 57 57 57 66 66 66 66 59 59 33 33 33 33 33 33
65	28	60 60 34 34 70 70 54 54 66 66 57 57 39 39 71 71 59 59 68 68 56 56 66 66 42 42 66 66
66	28	73 73 51 51 46 46 65 65 57 57 45 45 65 65 64 64 60 60 32 32 49 49 69 69 54 54 65 65
67	32	70 70 67 67 77 77 57 57 72 72 60 60 50 50 55 55 58 58 51 51 69 69 48 48 77 77 46 46 73 73 67 67
68	22	71 71 55 55 53 53 59 59 54 54 65 65 47 47 40 40 49 49 46 46 56 56
69	28	56 56 77 77 72 72 60 60 60 60 34 34 51 51 71 71 60 60 52 52 67 67 48 48 66 66 38 38
70	24	78 78 48 48 49 49 55 55 55 55 56 56 67 67 46 46 50 50 39 39 38 38 65 65
71	22	55 55 62 62 46 46 72 72 41 41 69 69 52 52 65 65 45 45 68 68 49 49
72	32	77 77 75 75 46 46 49 49 67 67 50 50 46 46 55 55 52 52 72 72 71 71 72 72 69 69 42 42 60 60 60 60
73	42	66 66 67 67 60 60 60 60 60 60 57 57 57 57 60 60 60 60 60 60 57 57 60 60 66 66 80 80 57 57 67 67 67 67 66 66 67 67 66 66 60 60
74	14	27 27 27 27 27 27 27 27 79 79 31 31 78 78
75	30	72 72 45 45 45 45 42 42 42 42 55 55 55 55 72 72 49 49 49 49 42 42 49 49 55 55 72 72 45 45
76	10	39 39 39 39 39 39 39 39 39 39
77	30	67 67 67 67 67 67 69 69 69 69 67 67 67 67 69 69 67 67 72 72 72 72 50 50 50 50 72 72 50 50
78	26	38 38 70 70 70 70 70 70 38 38 74 74 50 50 50 50 50 50 50 50 70 70 38 38 38 38
79	18	74 74 74 74 74 74 74 74 74 74 74 74 74 74 74 74 74 74
80	30	73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73 73

8 - Cycle Switching.

1	0	
2	48	4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4
3	24	14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14
4	60	8 8 9 9 9 9 8 8 9 10 10 9 10 9 9 10 10 9 9 10 9 10 10 9 2 4 12 12 12 12 4 12 12 12 12 2 12 12 12 12 12 12 12 12 4 6 9 9 6 9 9 4 9 9 9 9 4 6 6 4
5	60	8 8 8 8 13 13 13 13 8 8 8 8 8 8 8 8 13 13 13 13 8 8 8 8 13 13 13 13 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10
6	66	18 18 18 15 15 18 18 15 18 15 18 18 15 15 18 15 15 18 18 18 18 18 15 15 18 18 18 15 18 15 4 4 11 11 11 11 4 4 11 11 11 11 11 11 11 11 11 11 11 11 4 4 11 11 4 11 11 4 11 11 11 11 4 4 4 4
7	36	17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17 17
8	36	4 8 26 26 26 26 5 8 26 26 26 26 8 8 26 26 26 26 5 4 9 10 18 15 10 18 15 9 15 18 15 18 9 10 10 9
9	54	4 9 23 24 24 23 10 18 9 15 24 29 29 24 18 9 15 24 23 23 24 10 4 9 24 24 4 29 25 29 25 10 24 24 29 25 29 25 4 11 20 10 11 20 10 8 18 20 18 20 8 11 11 4
10	54	9 18 10 15 23 31 31 23 10 15 31 23 23 31 15 18 10 23 25 23 25 10 9 4 26 25 25 26 4 9 23 23 23 23 25 26 26 25 5 12 9 11 12 8 11 15 9 11 11 15 5 12 12 8
11	54	12 11 20 20 19 27 28 32 28 12 19 20 20 11 28 32 28 27 20 19 20 28 27 28 27 12 12 30 32 32 11 12 30 32 30 32 12 32 32 32 30 32 10 9 15 9 10 6 6 15 10 9 9 10
12	57	12 20 12 11 11 23 24 24 26 11 11 11 12 20 24 23 24 26 11 11 20 24 24 23 26 11 12 24 23 24 26 12 12 24 24 24 11 23 26 24 24 23 24 26 24 10 10 10 10 4 4 4 10 10 4 4 4
13	36	14 14 17 25 25 25 25 25 25 25 25 5 17 14 25 25 25 25 17 14 25 25 25 25 5 14 25 25 25 25 14 25 25 25 25 5
14	36	3 13 13 26 26 26 26 13 13 13 13 13 26 26 26 26 3 13 26 26 26 26 13 3 26 26 26 26 26 26 26 26 26 26 26 26
15	42	9 10 9 10 27 27 27 27 10 9 10 9 10 10 6 15 27 27 15 27 27 6 27 27 27 27 8 19 11 10 19 8 10 11 11 10 8 19 10 11 19 8
16	0	
17	12	7 13 13 13 7 13 13 13 7 13 13 13
18	42	6 18 18 10 9 6 28 28 9 10 6 28 28 10 9 9 10 28 28 28 28 18 28 28 28 28 8 20 9 20 20 9 20 8 20 9 20 9 8 20 20 8
19	42	11 11 11 11 11 63 63 11 11 11 11 11 63 63 11 11 11 11 11 11 63 63 11 63 63 11 63 63 63 63 15 15 15 15 15 15 15 15 15 15 15 15
20	48	11 20 11 12 11 34 33 20 11 12 11 20 34 33 59 12 11 11 33 34 11 20 59 34 33 20 11 33 34 20 59 34 33 33 33 33 9 9 9 9 18 18 18 9 9 18 18 18
21	45	21 22 21 22 21 22 21 22 21 22 21 47 39 21 22 21 22 40 41 39 47 40 47 21 22 21 47 39 21 22 40 40 47 41 39 47 21 47 39 40 39 47 40 47 41
22	42	21 21 22 22 21 61 54 58 62 53 22 61 21 22 22 58 54 21 61 22 22 54 58 22 21 21 53 58 62 54 21 21 58 54 62 53 22 53 53 53 58 54
23	42	27 26 27 28 26 26 27 23 27 32 27 34 23 28 12 27 10 24 23 32 24 34 24 27 12 23 23 32 31 26 27 34 10 25 23 25 10 10 31 10 9 9
24	45	32 33 33 27 76 28 28 26 33 32 28 33 26 28 33 32 12 34 31 28 9 25 31 76 25 12 76 27 28 12 33 28 33 23 31 34 33 9 9 9 12 12 23 23 9
25	33	30 34 29 34 30 26 34 30 10 34 26 26 13 24 27 29 26 34 24 27 29 28 28 34 26 23 23 10 13 10 13 9 9
26	36	25 27 24 23 28 25 24 27 28 23 25 27 28 14 31 23 14 8 12 10 10 12 31 23 26 27 26 27 8 25 28 25 28 8 26 26
27	33	25 24 23 63 23 23 23 24 36 15 26 11 36 11 15 26 15 36 26 30 28 23 30 26 23 63 23 23 63 25 32 32 26
28	36	24 11 59 23 25 11 70 70 23 11 26 18 26 24 24 70 18 70 18 70 70 34 24 25 59 34 26 24 59 26 24 27 24 33 33 26
29	36	33 25 25 32 9 32 33 32 9 9 25 33 9 25 9 33 25 9 25 9 32 33 25 33 32 32 33 32 32 32 33 33 9 25 25 9
30	33	33 25 33 36 32 25 33 27 44 25 33 43 11 31 32 27 43 32 44 11 25 33 36 33 32 25 27 27 31 31 11 11 25
31	42	24 23 10 24 30 23 10 24 24 26 26 24 10 24 30 23 26 23 24 10 10 24 30 23 24 23 30 10 26 24 26 26 30 26 30 26 23 24 23 10 10 24
32	36	34 24 59 70 63 24 34 75 24 34 44 11 23 30 44 30 11 75 29 11 70 63 59 44 75 70 59 63 29 11 11 23 23 27 27 29
33	33	30 24 70 64 24 30 64 52 24 30 29 20 24 34 64 34 52 20 20 64 64 70 52 64 70 24 24 24 29 24 28 28 29
34	30	32 25 60 34 70 25 32 34 25 25 32 20 33 34 60 33 23 20 60 70 34 70 28 28 23 25 23 24 24 25
35	24	53 53 53 41 41 41 38 53 53 40 38 53 53 41 41 38 38 40 41 53 38 53 38 40
36	24	27 27 27 27 43 44 27 27 43 44 27 27 30 27 30 27 30 27 30 27 43 43 44 44
37	12	42 42 42 42 42 42 42 42 42 42 42 42
38	21	45 69 56 68 50 38 56 38 45 56 50 69 39 69 38 38 68 39 68 35 35
39	30	50 58 40 45 48 46 49 49 39 48 39 50 39 58 40 46 45 40 58 38 39 48 39 49 39 38 46 21 21 39
40	27	45 40 39 58 48 74 21 41 41 45 58 39 40 45 58 74 41 39 21 74 40 41 48 40 48 40 35
41	24	50 54 47 45 40 35 40 45 50 41 50 40 41 54 47 45 35 47 54 41 41 40 21 41
42	21	73 67 67 66 66 42 73 66 42 73 66 66 66 42 67 67 42 67 67 37 37
43	21	79 44 44 36 36 30 30 30 79 30 30 79 30 44 44 44 44 36 36 36 36
44	21	78 52 52 32 32 30 78 32 30 78 32 43 32 43 32 52 52 52 52 36 36
45	24	66 41 55 65 47 40 39 66 40 39 66 55 55 40 50 65 50 47 41 65 41 47 38 38
46	18	66 48 73 39 49 66 49 66 73 48 48 39 73 48 48 47 47 39
47	27	45 41 54 47 74 48 21 48 45 48 48 45 47 48 54 47 21 54 49 41 49 74 41 74 46 46 21
48	21	47 65 40 46 39 47 47 46 65 65 39 46 46 40 46 47 40 47 49 49 39
49	18	65 57 72 39 47 47 72 39 72 46 46 57 65 65 57 39 48 48
50	15	50 66 41 38 41 38 66 50 66 50 41 45 45 39 39
51	21	65 56 78 52 55 55 53 55 55 52 55 56 54 65 54 65 78 53 78 57 57
52	24	59 71 44 33 70 51 53 71 33 51 70 71 59 33 70 53 59 53 53 44 44 53 56 56
53	24	52 56 64 62 35 54 52 51 64 62 54 56 62 52 64 52 22 22 35 51 52 58 58 35
54	27	55 57 55 47 54 41 53 47 55 54 53 57 55 54 55 47 55 51 55 58 58 41 55 51 22 22 41
55	24	51 54 69 62 45 51 62 45 54 51 54 62 45 54 54 57 69 57 54 51 69 51 54 54
56	21	38 72 68 71 53 58 68 53 71 68 72 71 58 72 51 51 38 38 58 52 52
57	12	49 54 58 54 58 55 55 58 49 49 51 51
58	21	58 39 40 22 56 57 22 56 57 54 54 40 57 58 39 39 58 56 53 53 40
59	36	52 32 32 52 52 32 28 28 28 60 52 28 60 20 52 28 32 52 20 32 28 28 32 52 32 60 32 60 28 32 52 28 20 52 60 60
60	18	70 71 75 59 59 34 34 60 75 60 71 70 75 70 34 71 60 60
61	42	22 22 22 22 22 22 64 64 64 22 22 22 22 22 64 64 64 22 22 22 22 64 64 64 22 22 22 64 64 64 22 22 64 64 64 22 64 64 64 64 64 64
62	21	53 53 53 55 55 53 53 55 55 53 53 22 55 53 53 22 55 22 55 55 55
63	21	32 32 27 32 32 27 32 32 32 27 19 32 32 19 19 27 27 27 27 27 27
64	30	53 53 53 33 33 33 53 53 33 33 33 53 53 61 33 33 53 33 33 33 53 33 61 33 61 33 33 33 33 33
65	21	67 49 72 51 69 48 69 45 69 67 49 72 67 49 72 48 51 51 45 48 45
66	18	50 46 68 67 42 46 46 50 67 50 45 67 42 68 45 68 42 45
67	15	65 72 68 42 68 66 68 65 42 65 42 72 66 72 66
68	18	69 66 71 56 67 38 67 67 69 66 71 69 66 71 56 56 38 38
69	15	68 55 38 65 72 65 65 68 72 68 72 55 55 38 38
70	27	71 33 28 32 76 60 28 34 52 71 28 28 71 60 28 76 52 34 28 32 60 76 32 33 33 52 34
71	15	70 68 52 60 56 70 68 70 68 56 60 52 52 56 60
72	15	69 67 65 49 69 69 56 49 67 65 56 67 65 56 49
73	18	42 46 46 46 46 46 42 46 42 46 46 42 42 46 46 46 46 42
74	24	47 47 47 47 40 40 47 47 40 40 47 40 47 40 47 47 40 40 40 40 40 47 40 47
75	21	32 60 32 77 60 60 32 77 32 32 77 32 60 32 32 60 32 60 60 60 60
76	30	70 24 70 70 24 70 70 24 24 24 24 24 70 24 70 70 70 70 70 70 70 70 70 24 24 24 24 24 24 24
77	6	75 77 75 75 77 77
78	18	44 51 51 51 44 51 51 44 51 44 51 44 51 51 51 51 51 44
79	18	43 43 43 43 43 43 43 43 43 43 43 43 43 43 43 43 43 43
80	0	