On independent sets

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Abstract

In a general set-theoretic context, an *independent set* is defined as a set which avoids certain specified structures called *blocks*. A formula is given for the number of independent sets of cardinality k in terms of the numbers of *configurations* (i.e. non-empty collections) of blocks.

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This paper is concerned with a formula for counting sets which avoid certain structures. The formula was originally produced in the context of Steiner triple systems [1] but is capable of considerable generalization. The authors believe that the result may be of some wider interest. Accordingly, we here present a version of the result in a more general setting. The proof relies on an application of the inclusion-exclusion principle.

Consider a finite set of points S with $|S| = s$. Define $S_1 = \mathcal{P}(S)$ and, for $i = 1, 2, 3, \ldots$, put $S_{i+1} = S_i \cup \mathcal{P}(S_i)$, where $\mathcal{P}(X)$ denotes the set of all subsets of the set X. Then put $\mathbf{S} = \cup_{i=1}^{\infty} S_i$. We will say that a set $X \in \mathbf{S}$ *covers* the point $a \in S$ if $a \in X_0 \in X_1 \in \ldots \in X_n = X$ for some sequence of sets $X_0, X_1, \ldots, X_n \in \mathbf{S}$. If $X \in \mathbf{S}$ then the set $\{a \in S : X \text{ covers } a\}$ will be called the *foundation* of X and denoted by $\underline{X}$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be a fixed finite set of elements of $\mathbf{S}$, each having a non-empty foundation; we will call the elements of $\mathcal{B}$ *blocks*.

As an example of these definitions, we might take $\mathcal{B}$ to be a set of 4-cycles on the set S. Note that a 4-cycle (a, b, c, d) may be equated with the set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\} \in \mathbf{S}$. The block $B = (a, b, c, d)$ covers the points $a, b, c, d \in S$ and $\underline{B} = \{a, b, c, d\}$. By a similar method, we may describe any undirected graph by listing its edges, and any directed graph by representing a directed edge (a, b) by $\{\{a\}, \{a, b\}\}$.

An *independent set* in S is a subset of S which does not contain the foundation of any block $B \in \mathcal{B}$. A *configuration* X is any non-empty set of blocks. Two configurations X_1 and X_2 are called *isomorphic* if one may be obtained from the other by means of a bijective mapping from the points of $\underline{X_1}$ to the points of $\underline{X_2}$. We denote by $b(X)$ the the number of blocks in X and by $p(X)$ the number of points in $\underline{X}$.

Continuing our earlier example, if $a, b, \ldots, g$ are distinct points of S then the configuration of two 4-cycles $X_1 = \{(a, b, c, d), (a, e, f, g)\}$ is isomorphic to $X_2 = \{(a, b, c, d), (b, e, f, g)\}$, but not to $X_3 = \{(a, b, c, d), (a, e, c, g)\}$. We also have $b(X_1) = b(X_2) = b(X_3) = 2$, and $p(X_1) = p(X_2) = 7, p(X_3) = 6$.

Next consider any k-element set $W \subseteq S$ with $k \geq 1$. Denote by $n(X, W)$ the number of isomorphic copies of the configuration X with foundation in the set W. If there are exactly l blocks with foundation in W then

$$\begin{aligned}
\text{if } l = 0,\ \sum_X (-1)^{b(X)} n(X, W) &= 0, \quad \text{and} \\
\text{if } l \geq 1,\ \sum_X (-1)^{b(X)} n(X, W) &= -\left[l - \binom{l}{2} + \binom{l}{3} + \ldots + (-1)^{l-1} \binom{l}{l} \right] \\
&= -1,
\end{aligned}$$

where the sums extend over all isomorphism classes of configurations X with $p(X) \leq k$.

It follows that the number of independent sets of cardinality k in S, denoted by $I_k(S)$, is given by

$$\begin{aligned} I_k(S) &= \binom{s}{k} + \sum_{|W|=k} \sum_X (-1)^{b(X)} n(X,W) \\ &= \binom{s}{k} + \sum_X (-1)^{b(X)} \sum_{|W|=k} n(X,W). \end{aligned}$$

However, $\sum_{|W|=k} n(X,W)$ is evaluated by listing the k-element sets $W \subseteq S$ and scoring $+1$ for every copy of X with foundation in each such W. This is the same number as that found by taking each configuration X and extending its foundation in all possible ways to form a k-element subset of S, i.e. $n(X,S)\binom{s-p(X)}{k-p(X)}$. In consequence, we arrive at the following formula which we state formally as a theorem.

Theorem 1 *The number of independent sets of cardinality $k \geq 1$ in the set S is given by*

$$I_k(S) = \binom{s}{k} + \sum_X (-1)^{b(X)} n(X,S) \binom{s-p(X)}{k-p(X)},$$

where $s = |S|$ and the summation extends over all isomorphism classes of configurations X with $p(X) \leq k$.

In the case of a Steiner triple system on a point set S, the blocks are those triples of points which form the system. In this case the formula is efficacious for low values of k ($k \leq 8$) because it is possible to determine or to estimate the values of $n(X,S)$ for configurations covering at most k points. By this method we were able to determine the spectrum of maximum independent set sizes and of chromatic numbers for Steiner triple systems of order 21. Both of these results are presented in [1]. We offer the more general version of the formula here in the hope that it will prove useful in other contexts.

We conclude by remarking that if $\mathcal{B}$ and $\mathcal{B}'$ are collections of blocks whose elements have identical foundations (i.e. if for each $B \in \mathcal{B}$ there is a $B' \in \mathcal{B}'$ such that $\underline{B} = \underline{B}'$, and vice-versa) then the independent sets corresponding to $\mathcal{B}$ are identical to those corresponding to $\mathcal{B}'$. In other words, the number $I_k(S)$ depends on the foundations of the blocks rather than on the blocks themselves. A consequence of this observation is that there is no need to consider the possibility of repeated blocks either in $\mathcal{B}$ or in the configurations $X \subseteq \mathcal{B}$.

References

[1] A. D. Forbes, M. J. Grannell and T. S. Griggs, Independent sets in Steiner triple systems, *Ars Combin.* **72** (2004), 161-169.