Smallest defining sets of directed triple systems

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Abstract

A directed triple system of order v, DTS(v), is a pair $(V, \mathcal{B})$ where V is a set of v elements and $\mathcal{B}$ is a collection of ordered triples of distinct elements of V with the property that every ordered pair of distinct elements of V occurs in exactly one triple as a subsequence. A set of triples in a DTS(v) $\mathcal{D}$ is a defining set for $\mathcal{D}$ if it occurs in no other DTS(v) on the same set of points. A defining set for $\mathcal{D}$ is a smallest defining set for $\mathcal{D}$ if $\mathcal{D}$ has no defining set of smaller cardinality. In this paper we are interested in the quantity

$$f = \frac{\text{number of triples in a smallest defining set for } \mathcal{D}}{\text{number of triples in } \mathcal{D}}.$$

We show that for all $v \equiv 0, 1 \pmod 3$, $v \geq 3$ there exists a DTS with $f \geq \frac{1}{2}$, and improve this result for certain residue classes. In particular we show that for all $v \equiv 1 \pmod{18}$, $v \geq 19$ there exists a DTS with $f \geq \frac{2}{3}$. We also prove that, for all $\epsilon > 0$ and all sufficiently large admissible v, there exists a DTS(v) with $f \geq \frac{2}{3} - \epsilon$.

Results are also obtained for pure, regular and Mendelsohn directed triple systems.

Key words: directed triple system, smallest defining set
AMS classification: 05B07

1 Introduction

A *directed triple system* (DTS) with parameters v and λ, denoted by DTS(v, λ), is a pair $(V, \mathcal{B})$ where V is a set of v elements, called *points*, and $\mathcal{B}$ is a collection of ordered triples, more succinctly just called *triples*, of distinct elements

Preprint submitted to Elsevier

of V, with the property that every ordered pair of distinct elements of V occurs in exactly λ triples, as a subsequence. In this paper we are concerned with DTSs with $\lambda = 1$. A DTS$(v, 1)$ is denoted by DTS(v).

We usually specify a DTS by listing its triples. For example, the following triples form a DTS(4):

$$(0, 2, 1),\ (2, 0, 3),\ (1, 3, 0),\ (3, 1, 2).$$

Here, for example, the triple $(0, 2, 1)$ contains the ordered pairs $(0, 2)$, $(0, 1)$ and $(2, 1)$.

A set of triples in a DTS(v) $\mathcal{D}$ is a *defining set* for $\mathcal{D}$ if it occurs in no other DTS(v) on the same set of points. A defining set of a DTS $\mathcal{D}$ is a *smallest* defining set for $\mathcal{D}$ if $\mathcal{D}$ has no defining set of smaller cardinality.

A set of triples in a DTS(v) $\mathcal{D}$ is a *trade* in $\mathcal{D}$ if it can be replaced by a different set of triples, called a *replacement trade*, to give another DTS(v). For example, the set $\{(0, 1, 2), (2, 1, 3)\}$ is a trade in any DTS that contains it, since it covers the same set of ordered pairs as the set $\{(0, 2, 1), (1, 2, 3)\}$.

Each defining set of a DTS $\mathcal{D}$ contains at least one triple in every trade in $\mathcal{D}$. In particular, if $\mathcal{D}$ contains m mutually disjoint trades then the smallest defining set of D contains at least m triples.

In this paper we are interested in the quantity

$$f = \frac{\text{number of triples in a smallest defining set for } \mathcal{D}}{\text{number of triples in } \mathcal{D}},$$

where $\mathcal{D}$ is a DTS(v). We are also interested in the analogues of this quantity for several special types of directed triple system, which we now define.

A DTS is *pure* if no two triples contain the same three points. A DTS is *regular* if there is a constant r such that each point appears exactly r times in each of the three possible positions in a triple. For example, the DTS(4) at the beginning of this section is regular with $r = 1$.

A DTS is *Mendelsohn*, and we write that it is an MDTS, if each of the two non-identity cyclic shifts of all its triples results in a DTS. The DTS(4) at the beginning of this section is an MDTS: the two non-identity cyclic shifts of its triples give

$$(1, 0, 2),\ (3, 2, 0),\ (0, 1, 3),\ (2, 3, 1)$$

$$\text{and } (2, 1, 0),\ (0, 3, 2),\ (3, 0, 1),\ (1, 2, 3),$$

respectively, and each of these lists of triples is a DTS(4).

MDTSs are related to *Mendelsohn triple systems* (MTSs). The definition of an MTS is similar to that of a DTS. The difference is that the containment of ordered pairs in triples is cyclic instead of transitive: that is, a triple (x, y, z) contains the ordered pairs (x, y), (y, z) and (z, x) instead of (x, y), (y, z) and (x, z). Every MDTS is both an MTS and a DTS, and remains so under any of the six permutations of the positions of the entries in all the triples. This follows from the following property, called the *order conditions*, which is proved in [9]:

> For any MDTS, let $S_{a,b}$ denote the multiset of ordered pairs of points in positions a and b of the triples. Then $S_{1,2} = S_{2,1}$, $S_{3,1} = S_{1,3}$ and $S_{2,3} = S_{3,2}$.

It follows from the order conditions that every MDTS is regular. Also, every MDTS with $\lambda = 1$ is pure, because any DTS(v) which is not pure contains a pair of triples of the form (a, b, c), (c, b, a), and hence the Mendelsohn property is not satisfied.

The concepts of trade, defining set and smallest defining set, and the quantity that we have denoted by f, all have analogues for pure, regular and Mendelsohn DTSs. For example, a set of triples of a pure DTS $\mathcal{D}$ is called a *(pure) trade* for $\mathcal{D}$ if it can be replaced by a different set of triples to give another pure DTS(v). Similarly, a set of triples of a pure DTS $\mathcal{D}$ is called a *(pure) defining set* for $\mathcal{D}$ if it occurs in no other pure DTS on the same set of points. Where it is necessary to distinguish trades and defining sets of ordinary DTSs from their analogues for special types of DTS, we call them ordinary trades and ordinary defining sets. Thus, for example, any pure trade for a pure DTS $\mathcal{D}$ is also an ordinary trade for $\mathcal{D}$, and any ordinary defining set for D is also a pure defining set for $\mathcal{D}$, but the converses of these statements are not necessarily true. A pure defining set for $\mathcal{D}$ (known to be pure) could be smaller than a smallest ordinary defining set for $\mathcal{D}$. In this paper any mention of trades, defining sets or f refers to the version for the type of DTS that is being considered at that point, unless otherwise stated.

The concepts of trade, defining set and smallest defining set, and the quantity f, can also be defined for Steiner triple systems (STSs), in the obvious way. In [10] it is shown that for all admissible values of v (that is, all values of v satisfying the necessary conditions) there is an STS(v) with $f > \frac{1}{4}$. In this paper we show that for ordinary DTSs, pure DTSs and regular DTSs, for all admissible values, there is a system with $f \geq \frac{1}{2}$. We also obtain a result for MDTSs. In [10], an asymptotic result, $f \geq \frac{16}{35}$, is obtained for Steiner triple systems. Using a similar argument we show that $f \geq \frac{2}{3}$ can be obtained asymptotically for ordinary DTSs.

The proofs in this paper use various types of combinatorial objects. The definitions of these objects are either given in the paper or can be found in the references.

Several proofs depend on the following result, which involves pairwise balanced designs (PBDs) and is a special case of a result (the *Replacement Lemma* [13]) that is used in several earlier papers on directed designs.

Lemma 1 *If there exist a* 2-$(v, K, 1)$ *design and a* DTS(k) *for each* $k \in K$*, then there exists a* DTS(v)*.*

PROOF. Replacing each block of the 2-$(v, K, 1)$ design with a copy of a DTS(k) with point set the points of that block gives a DTS(v). $\square$

A lower bound for f for the DTS(v) constructed in Lemma 1 can be calculated from lower bounds for f for the various DTS(k)s. In particular, if there is a constant c such that each of the DTS(k)s has $f \geq c$, then the resulting DTS(v) also has $f \geq c$.

Clearly, analogues of Lemma 1 hold for pure, regular and Mendelsohn DTSs, and the above comment about f applies to these analogues also.

2 Directed triple systems and pure directed triple systems

A necessary and sufficient condition for the existence of a DTS(v) is $v \equiv 0, 1 \pmod 3$, $v \geq 3$ [12].

There is only one DTS(3) up to isomorphism, namely the system given by the triples $(0, 1, 2), (2, 1, 0)$. Clearly this system has $f = \frac{1}{2}$. Results for DTS(4)s and DTS(6)s are given in [14]. In summary, these are as follows. Up to isomorphism there are three DTS(4)s, and each of these has $f = \frac{1}{2}$. Up to isomorphism there are 32 DTS(6)s; of these 28 have $f = \frac{1}{2}$ and four have $f = \frac{2}{5}$. We can use these results to prove the following theorem.

Theorem 2 *For all* $v \equiv 0, 1 \pmod 3$*,* $v \geq 3$*, there exists a* DTS(v) *with* $f \geq \frac{1}{2}$*.*

PROOF. For all $v \equiv 0, 1 \pmod 3$, $v \geq 3$, except $v = 6$, there exists a 2-$(v, \{3, 4\}, 1)$ design [1]. Replacing the blocks of size 3 and 4 in this design with DTS(3)s and DTS(4)s, respectively, gives a DTS(v) with $f \geq \frac{1}{2}$. Since there is also a DTS(6) with $f = \frac{1}{2}$, this proves the result. $\square$

A necessary and sufficient condition for the existence of a pure DTS(v) is $v \equiv 0, 1 \pmod 3$, $v \geq 4$ (see [6], Subsection 24.4). A result similar to Theorem 2 holds for pure DTSs, as we show next.

The proof involves pure trades of three types, as below.

Type	Trade	Replacement trade
1	$\{(a,b,c),(b,a,d)\}$	$\{(b,a,c),(a,b,d)\}$
2	$\{(c,a,b),(d,b,a)\}$	$\{(c,b,a),(d,a,b)\}$
3	$\{(c,a,b),(b,a,d)\}$	$\{(c,b,a),(a,b,d)\}$

Each of the pairs of triples on the left is a trade in any pure DTS that contains it, since it covers the same ordered pairs of points as the pair of triples on the right, and each of the triples of the trade contains the same points as a triple in the replacement trade.

The proof also involves group divisible designs (GDDs). The existence of all the GDDs used in the proof is confirmed in [7].

The following theorem is used both here and in Section 5.

Theorem 3 (Chu [4]) *A* DTS(w) *can be embedded in a* DTS($w+v$) *if and only if* $w + v \equiv 0, 1 \pmod 3$ *(the admissibility condition) and* $v \geq w + 1$.

Since every DTS(v) that is not pure contains a pair of triples of the form (a,b,c), (c,b,a), that is, an embedded DTS(3), it follows from Theorem 3 that every DTS(4) and every DTS(6) is pure.

Theorem 4 *For all* $v \equiv 0, 1 \pmod 3$*,* $v \geq 4$*, there exists a pure* DTS(v) *with* $f \geq \frac{1}{2}$.

PROOF. For all $v \equiv 0, 1 \pmod 3$, $v \geq 4$, except $v = 10, 12, 15, 18, 19, 24, 27$, there exists a 2-($v, \{4, 6, 7, 9\}, 1$) design [1]. Hence the result follows from the existence of a pure DTS(v) with $f \geq \frac{1}{2}$ for $v = 4, 6, 7, 9, 10, 12, 15, 18, 19, 24, 27$. We now demonstrate the existence of these designs.

Since each of the three DTS(4)s is pure, and each has $f = \frac{1}{2}$ as an ordinary DTS, it follows that each has $f = \frac{1}{2}$ as a pure DTS.

Similarly, since each of the 32 DTS(6)s is pure, and some of them have $f = \frac{1}{2}$ as ordinary DTSs, these designs have $f = \frac{1}{2}$ as pure DTSs.

The pair of triples $\{(0,1,3),(1,0,5)\}$ generates a pure DTS(7) under the mapping $i \mapsto i+1 \pmod 7$. Further, each pair of triples generated by this pair is a type 1 trade, and these seven trades are disjoint. Hence this pure DTS has $f \geq \frac{7}{14} = \frac{1}{2}$.

The following triples form a pure DTS(9).

$$\begin{array}{ccccc}
(0,5,1) & (0,6,4) & (0,7,2) & (0,8,3) & (5,6,7) \\
(1,5,2) & (4,6,3) & (2,7,4) & (3,8,1) & (6,5,8) \\
(2,5,3) & (3,6,2) & (4,7,1) & (1,8,4) & (7,8,5) \\
(3,5,4) & (2,6,1) & (1,7,3) & (4,8,2) & (8,7,6) \\
(4,5,0) & (1,6,0) & (3,7,0) & (2,8,0) &
\end{array}$$

Each pair of triples appearing consecutively (cyclically) in any of the first four columns above is a type 3 trade. Hence any defining set for this DTS(9) must contain at least three triples from each of the first four columns. The final column of triples forms a DTS(4) and so any defining set for the DTS(9) must also contain at least two triples from the final column. Hence any defining set must contain at least $4 \times 3 + 2 = 14$ triples, so for this DTS(9) we have $f \geq \frac{14}{24} = \frac{7}{12} > \frac{1}{2}$.

The existence of a pure DTS(10) and a pure DTS(19) with $f \geq \frac{1}{2}$ follows from the existence of an MDTS(10) and an MDTS(19), as follows. Every MDTS(v) is pure, and, by the order conditions, its set of triples can be partitioned into pairs of the form $\{(a,b,c),(b,a,d)\}$. Each such pair is a type 1 trade. Hence every MDTS(v), when considered just as a pure DTS(v), has $f \geq \frac{1}{2}$. It is shown in [9] that an MDTS(v) exists if and only if $v \equiv 1 \pmod 3$, $v \geq 4$, so it follows that there exist a pure DTS(10) and a pure DTS(19) with $f \geq \frac{1}{2}$.

A pure DTS(12) with $f \geq \frac{1}{2}$ can be constructed as follows. Begin with a 3-GDD(2^3): for example, such a design, with groups $\{\mathrm{A},\mathrm{B}\}, \{\mathrm{C},\mathrm{D}\}, \{\mathrm{E},\mathrm{F}\}$, is given by the following blocks:

$$\{\mathrm{A},\mathrm{C},\mathrm{E}\}, \{\mathrm{A},\mathrm{D},\mathrm{F}\}, \{\mathrm{B},\mathrm{C},\mathrm{F}\}, \{\mathrm{B},\mathrm{D},\mathrm{E}\}.$$

Replace each point of the GDD with two points, to give 12 points altogether. These will be the points of the DTS and we refer to them as DTS points. For each group of the GDD, take the triples of a DTS(4) on the four DTS points in that group. Each block of the GDD contains six DTS points, say a,b,c,x,y,z, where $\{a,x\}$, $\{b,y\}$, $\{c,z\}$ are the pairs of DTS points corresponding to the same GDD points. For each such block take the following eight triples (listed in pairs):

$$\{(a,b,c),(b,a,z)\},\ \{(z,a,y),(c,y,a)\},$$
$$\{(z,b,x),(c,x,b)\},\ \{(y,x,c),(x,y,z)\}.$$

The set of triples constructed in this way forms a DTS(12). Further, this DTS has $f \geq \frac{1}{2}$, since each of the DTS(4)s has $f \geq \frac{1}{2}$, and each pair of triples in the list above is a type 1 or type 2 trade.

A pure DTS(15) with $f \geq \frac{1}{2}$ can be constructed as follows. Some of its triples are listed in pairs on the left below. Each of these pairs is a type 1 trade. The remaining 56 triples are given by the table on the right. Each pair of numbers nm in the body of the table is used to give two triples, namely (r, n, m) and (m, n, c), where r and c are the letters in the row and column headings, respectively. For example, the pair 02 gives the triples $(\mathrm{A}, 0, 2)$ and $(2, 0, \mathrm{B})$. Each such pair of triples is a type 3 trade. This gives a DTS(15) whose set of triples is partitioned into disjoint trades of size 2. Hence this design has $f \geq \frac{1}{2}$.

$$\{(\mathrm{A},\mathrm{B},\mathrm{D}),(\mathrm{B},\mathrm{A},\mathrm{F})\}$$
$$\{(\mathrm{B},\mathrm{C},\mathrm{E}),(\mathrm{C},\mathrm{B},\mathrm{G})\}$$
$$\{(\mathrm{C},\mathrm{D},\mathrm{F}),(\mathrm{D},\mathrm{C},\mathrm{A})\}$$
$$\{(\mathrm{D},\mathrm{E},\mathrm{G}),(\mathrm{E},\mathrm{D},\mathrm{B})\}$$
$$\{(\mathrm{E},\mathrm{F},\mathrm{A}),(\mathrm{F},\mathrm{E},\mathrm{C})\}$$
$$\{(\mathrm{F},\mathrm{G},\mathrm{B}),(\mathrm{G},\mathrm{F},\mathrm{D})\}$$
$$\{(\mathrm{G},\mathrm{A},\mathrm{C}),(\mathrm{A},\mathrm{G},\mathrm{E})\}$$

	A	B	C	D	E	F	G
A		02	56	13		47	
B			03	67	24		15
C	26			04	17	35	
D		37			05	12	46
E	57		14			06	23
F	34	16		25			07
G	01	45	27		36		

A pure DTS(18) with $f \geq \frac{1}{2}$ can be constructed as follows. Begin with a 3-GDD(2^3). Replace each point of the GDD with three points, to give 18 DTS points. For each group of the GDD, take the triples of a DTS(6) with $f \geq \frac{1}{2}$ on the six DTS points in that group. Each block of the GDD contains nine DTS points, say $a, b, c, d, e, f, g, h, i$, where $\{a, b, c\}$, $\{d, e, f\}$, $\{g, h, i\}$ are the triples of DTS points corresponding to the same GDD points. For each such block take the following 18 triples:

$$\{(a,d,g),(d,a,h)\},\ \{(i,a,e),(a,i,f)\},\ \{(f,h,a),(h,f,b)\},$$
$$\{(b,e,h),(e,b,i)\},\ \{(g,b,f),(b,g,d)\},\ \{(d,i,b),(i,d,c)\},$$
$$\{(c,f,i),(f,c,g)\},\ \{(h,c,d),(c,h,e)\},\ \{(e,g,c),(g,e,a)\}.$$

The set of triples constructed in this way forms a DTS(18). Further, this DTS has $f \geq \frac{1}{2}$, since each of the DTS(6)s has $f \geq \frac{1}{2}$, and each pair of triples in the list above is a type 1 trade.

A pure DTS(24) with $f \geq \frac{1}{2}$ can be constructed in a similar way to the DTS(18). In this case begin with a 3-GDD(2^4) and replace each point by three points. For each group use a DTS(6) with $f \geq \frac{1}{2}$, and for each block use the 18 triples in the DTS(18) construction.

A pure DTS(27) with $f \geq \frac{1}{2}$ can also be constructed in a similar way to the DTS(18). In this case begin with a 3-GDD(3^3) and replace each point by three points. For each group use a DTS(9) with $f \geq \frac{1}{2}$, and for each block use the 18 triples in the DTS(18) construction. □

3 Regular and Mendelsohn directed triple systems

A necessary and sufficient condition for the existence of a regular DTS(v) is $v \equiv 1 \pmod 3$, $v \geq 4$ [5]. A result similar to those for ordinary and pure DTSs holds for regular DTSs, as we show next. The proof uses the fact that any type 1 pure trade is also a regular trade in any regular DTS that contains it, since the numbers of times that the points appear in the positions in the trade are the same as in the replacement trade. The same is true of type 2 trades, but not type 3.

Theorem 5 *For all $v \equiv 1 \pmod 3$, $v \geq 4$, there exists a regular DTS(v) with $f \geq \frac{1}{2}$.*

PROOF. For all $v \equiv 1 \pmod 3$, $v \geq 4$, except $v = 10, 19$, there exists a 2-$(v, \{4, 7\}, 1)$ design [1]. Hence the result follows from the existence of a regular DTS(v) with $f \geq \frac{1}{2}$ for $v = 4, 7, 10, 19$.

Each of the three DTS(4)s is regular (see [14]), and hence since each has $f = \frac{1}{2}$ as an ordinary DTS, each has $f = \frac{1}{2}$ as a regular DTS.

The DTS(7) in the proof of Theorem 4 is regular and its set of triples is a union of disjoint trades of type 1, so it has $f \geq \frac{1}{2}$ as a regular DTS. The same is true of the DTS(10) and DTS(19) in the proof of Theorem 4. □

By a result in [9], a necessary and sufficient condition for the existence of an MDTS(v) is $v \equiv 1 \pmod 3$, $v \geq 4$.

There is just one MDTS(4) up to isomorphism (namely the DTS(4) given at the beginning of Section 1), and it is easy to check, using the order conditions, that it has $f = \frac{1}{4}$. It is shown in [9] that there are precisely two non-isomorphic MDTS(7)s.

Lemma 6 *Each of the two non-isomorphic* MDTS(7)*s has* $f = \frac{3}{14}$.

PROOF. It is shown in [9] that the two non-isomorphic MDTS(7)s are generated by the pairs of triples $\{(0,1,3),(0,6,4)\}$ and $\{(0,3,1),(0,4,6)\}$, respectively, under the mapping $i \mapsto i+1 \pmod 7$. Here we denote these designs by $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. We show that $f = \frac{3}{14}$ for $\mathcal{D}_1$; analogous arguments prove the same result for $\mathcal{D}_2$.

First we show that $\mathcal{D}_1$ has no defining set of size 2. Let S be a set of two triples of $\mathcal{D}_1$; then the triples in S have 0, 1 or 2 points in common.

First suppose that the triples in S are disjoint. Since each triple in $\mathcal{D}_1$ has precisely one triple disjoint from it, and one of these two triples is generated by $(0,1,3)$ and the other by $(0,6,4)$, we may take $S = \{(0,1,3),(5,4,2)\}$ without loss of generality. Then the MDTS(7) obtained by applying the permutation $(1\ 3)(2\ 4)$ to the points of $\mathcal{D}_2$ includes the triples in S but is different from $\mathcal{D}_1$ since it is isomorphic to $\mathcal{D}_2$. Hence S is not a defining set.

Now suppose that the triples in S have one or two points in common. Let a and b be two points of $\mathcal{D}_1$ that do not appear in either triple in S. Applying the permutation $q = (a\ b)$ to $\mathcal{D}_1$ gives a MDTS(7) that includes the triples in S. Now, by the order conditions, D_1 includes a pair of triples of one of the following forms: $\{(a,b,*),(b,a,*)\}$, or $\{(a,*,b),(b,*,a)\}$, or $\{(*,a,b),(*,b,a)\}$. In each case the two other points appearing in these triples are different, since $\mathcal{D}_1$ is pure. Thus the images of these triples under the point permutation $(a\ b)$ are not triples of D_1. Hence $(a\ b)(\mathcal{D}_1)$ is different from $\mathcal{D}_1$. Thus S is not a defining set in this case either.

We complete the proof by showing that $\mathcal{D}_1$ has a defining set of size 3. We do this by showing that the only MDTS(7)s that include the triples (0,1,3) and (5,4,2) are $\mathcal{D}_1$ and $p(\mathcal{D}_2)$, where $p = (1\ 3)(2\ 4)$. Since $\mathcal{D}_1$ and $p(\mathcal{D}_2)$ have no other triples in common, this shows that (0,1,3) and (5,4,2) together with any other triple in $\mathcal{D}_1$ form a defining set for $\mathcal{D}_1$.

Let $\mathcal{D}_3$ be an MDTS(7) that includes the triples $(0,1,3)$ and $(5,4,2)$. Then $\mathcal{D}_3 \cong \mathcal{D}_1$ or $\mathcal{D}_3 \cong \mathcal{D}_2$. First consider the case $\mathcal{D}_3 \cong \mathcal{D}_1$. Since $\mathcal{D}_1$ is generated by $\{(0,1,3),(5,4,2)\}$, and since there is an automorphism of $\mathcal{D}_1$ which maps $(0,1,3)$ to $(5,4,2)$ (namely $(0\ 5)(1\ 4)(2\ 3)$), there is an isomorphism $\phi : \mathcal{D}_1 \to \mathcal{D}_3$ such that $\phi((0,1,3)) = (0,1,3)$. Since $(5,4,2)$ is the only triple in $\mathcal{D}_1$ disjoint from $(0,1,3)$ (and hence also the only triple in $\mathcal{D}_3$ disjoint from

$(0,1,3)$), it follows that $\phi((5,4,2)) = (5,4,2)$. Hence ϕ is the identity and therefore $\mathcal{D}_3 = \mathcal{D}_1$.

Now consider the case $\mathcal{D}_3 \cong \mathcal{D}_2$. Since $\mathcal{D}_2$ is generated by $\{(0,3,1),(5,2,4)\}$, and since there is an automorphism of $\mathcal{D}_2$ which maps $(0,3,1)$ to $(5,2,4)$ (namely $(0\ 5)(1\ 4)(2\ 3)$), there is an isomorphism $\phi : \mathcal{D}_2 \to \mathcal{D}_3$ such that $\phi((0,3,1)) = (0,1,3)$. Since $(5,2,4)$ is the only triple in $\mathcal{D}_2$ disjoint from $(0,3,1)$ (and hence $(5,4,2)$ is the only triple in $\mathcal{D}_3$ disjoint from $(0,1,3)$), it follows that $\phi((5,2,4)) = (5,4,2)$. Hence $\phi = (1\ 3)(2\ 4) = p$ and therefore $\mathcal{D}_3 = p(\mathcal{D}_2)$. □

We can use the results for MDTSs for $v = 4$ and $v = 7$ to prove a result for MDTSs for general v.

Theorem 7 *For all* $v \equiv 1 \pmod 3$, $v \geq 4$, *except possibly* $v = 10, 19$, *there exists an* MDTS(v) *with*

$$f \geq \begin{cases} \frac{1}{4} & \text{if } v \equiv 1,4 \pmod{12}, \\ \frac{1}{4} - \frac{3}{2v(v-1)} & \text{if } v \equiv 7,10 \pmod{12}. \end{cases}$$

PROOF. For all $v \equiv 1,4 \pmod{12}$, $v \geq 4$, there exists a 2-$(v,4,1)$ design [11]. Replacing each block of this design with an MDTS(4) whose points are the points of that block gives an MDTS(v). Since each of the MDTS(4)s has $f = \frac{1}{4}$, the MDTS(v) has $f \geq \frac{1}{4}$.

The case $v = 7$ is dealt with in Lemma 6.

For all $v \equiv 7,10 \pmod{12}$, $v \geq 22$, there exists a 2-$(v,\{4,7\},1)$ design with a single block of size 7 [7]. Replacing each block of size 4 with an MDTS(4), and the block of size 7 with an MDTS(7), gives an MDTS(v). Since any defining set of the MDTS(v) contains at least one triple of each of the MDTS(4) and, by Lemma 6, at least 3 triples of the MDTS(7), the MDTS(v) has

$$f \geq \left(\frac{1}{4} \left(\frac{v(v-1)}{3} - 14 \right) + 3 \right) \Big/ \left(\frac{v(v-1)}{3} \right).$$

Simplifying the right-hand side of this inequality gives the stated result. □

4 Some particular classes of DTSs

In Section 2 we showed that for all admissible values of v there is a DTS(v) with $f \geq \frac{1}{2}$ and a pure DTS(v) with $f \geq \frac{1}{2}$. In this section we show that these results can be improved for some infinite classes of DTSs.

Theorem 8 *For all $v \equiv 0, 1 \pmod 9$, $v \geq 19$, except possibly $v = 64$ and $v = 27, 36, 54, 72, 81, 90, 135, 144, 162, 216, 234$, there exists a pure* DTS(v) *with $f \geq \frac{2}{3}$ both as a pure DTS and as an ordinary DTS.*

PROOF. Let v be as defined in the statement of the theorem. We construct a pure DTS(v) as follows. The complete graph K_v can be decomposed into copies of K_5 with one edge removed [8]. Take the v vertices of such a decomposition to be points. For each copy of K_5 with one edge removed in the decomposition, with points a, b, c, d, e and missing edge de, take the following as triples of the DTS:

$$(a, d, b),\ (b, e, a),$$
$$(b, d, c),\ (c, e, b),$$
$$(c, d, a),\ (a, e, c).$$

Then the resulting set of triples forms a pure DTS(v).

Each pair of triples appearing in the same column above is a type 3 pure trade. Hence any pure or ordinary defining set for the pure DTS(v) contains at least four triples (two from each column) from each such set of six triples. Hence the DTS(v) has $f \geq \frac{2}{3}$ as a pure DTS and as an ordinary DTS. □

Theorem 9 *For all $v \equiv 0 \pmod{15}$, $v \geq 15$, except possibly $v = 30$, there exists a pure* DTS(v) *with $f > \frac{31}{50}$ both as a pure DTS and as an ordinary DTS.*

PROOF. Let $s \geq 1$, $s \neq 2$. We construct a pure DTS($15s$) as follows. Begin with a 4-GDD($(3s)^4$); this is equivalent to a pair of mutually orthogonal Latin squares of side $3s$. Replace each point in one of the groups by two points, and leave the points in the other three groups unchanged, to give a total of $15s$ DTS points.

For each of the three groups of the GDD that contain $3s$ DTS points, take as triples of the DTS the triples of a pure DTS($3s$) with $f \geq \frac{1}{2}$ on the DTS points in that group. Similarly, for the group that contains $6s$ DTS points, take as triples of the DTS the triples of a pure DTS($6s$) with $f \geq \frac{1}{2}$ on the DTS points in that group. Each block of the GDD contains five DTS points, say a, b, c, d, e, where d and e are the points corresponding to the same GDD point. For each such block take as triples of the DTS the six triples listed in the proof of Theorem 8. The resulting set of triples forms a pure DTS($15s$).

There are $(3s)^2$ sets of six triples corresponding to the blocks of the GDD, and any pure or ordinary defining set for the DTS($15s$) contains at least two-thirds of these triples. It also contains at least half of the other triples, since the DTS($3s$)s and the DTS($6s$) all have $f \geq \frac{1}{2}$. Hence for the pure DTS($15s$), considered either as a pure DTS or as an ordinary DTS, we have

$$f \geq \left(\tfrac{2}{3} \cdot 6(3s)^2 + \tfrac{1}{2} \cdot 3 \cdot \frac{3s(3s-1)}{3} + \tfrac{1}{2} \cdot \frac{6s(6s-1)}{3}\right) \Big/ \left(\frac{15s(15s-1)}{3}\right).$$

Simplifying gives

$$f \geq \frac{31}{50} + \frac{3}{25(15s-1)}. \qquad \square$$

The proof of the next result involves Kirkman triple systems (KTSs). A KTS(v) exists for all $v \equiv 3 \pmod 6$ [15].

Theorem 10 *For all* $v \equiv 3 \pmod{12}$, $v \geq 15$, *there exists a* DTS(v) *with* $f > \frac{5}{8}$.

PROOF. Let $s \geq 1$, and construct a DTS($12s+3$) as follows. Begin with a KTS($6s+3$); this has $3s+1$ resolution classes, each containing $2s+1$ triples. For the DTS points, take the $6s+3$ points of the KTS and a further $6s$ points, two corresponding to each resolution class of the KTS, except for one resolution class which is to have no such points.

For each triple $\{a, b, c\}$ of the KTS that lies in a resolution class corresponding to extra points, take as triples of the DTS the six triples in the proof of Theorem 8, where d and e are the points corresponding to the resolution class in which $\{a, b, c\}$ lies. For each triple $\{a, b, c\}$ of the KTS in the resolution class not corresponding to extra points, take as triples of the DTS the two triples

$$(a, b, c), \quad (c, b, a).$$

Finally, take as triples of the DTS the triples of a DTS($6s$) with $f \geq \frac{1}{2}$ on the points corresponding to the resolution classes. Then the resulting set of triples forms a DTS($12s+3$) (which is not pure).

There are $3s(2s+1)$ sets of six triples associated with blocks lying in resolution classes that correspond to extra points, and each of these is of the form given in the proof of Theorem 8. Any two triples from the same column form a type 3 pure trade and hence an ordinary trade, and so any defining set for the DTS$(12s+3)$ contains at least two-thirds of such triples. There are $2s+1$ pairs of triples associated with the resolution class that does not correspond to extra points, and each such pair forms a DTS(3), which has $f \geq \frac{1}{2}$. Since also the DTS$(6s)$ has $f \geq \frac{1}{2}$, the DTS$(12s+3)$ has

$$f \geq \left(\tfrac{2}{3} \cdot 18s(2s+1) + \tfrac{1}{2} \cdot 2(2s+1) + \tfrac{1}{2} \cdot \frac{6s(6s-1)}{3}\right) \Big/ \left(\frac{(12s+3)(12s+2)}{3}\right).$$

Simplifying gives

$$f \geq \frac{5}{8} + \frac{2s-1}{8(4s+1)(6s+1)} > \frac{5}{8}. \qquad \square$$

The proof of the next result is similar, but involves resolvable Mendelsohn triple systems. A resolvable MTS(v) exists for all $v \equiv 0, 1 \pmod 3$, $v \neq 6$ [3]. A resolvable MTS(v) with $v \equiv 1 \pmod 3$ has v resolution classes each of which is missing a single point, and each point of the MTS is missing from exactly one class.

Theorem 11 *For all $v \equiv 3 \pmod 6$, $v \geq 9$, there exists a* DTS(v) *with*

$$f \geq \frac{5}{8} - \frac{3}{8v}.$$

In particular, for all $v \equiv 9 \pmod{12}$, $v \geq 21$, there exists a DTS(v) *with* $f \geq \frac{17}{28}$.

PROOF. Let $s \geq 1$. We construct a DTS$(6s+3)$ as follows. Begin with a resolvable MTS$(3s+1)$. For the DTS points, take the $3s+1$ points of the MTS, a further $3s+1$ points, one corresponding to each resolution class of the MTS, and one final point ∞.

For each triple (a,b,c) of the MTS, take as triples of the DTS the three triples

$$(a,d,b), \quad (b,d,c), \quad (c,d,a)$$

(that is, the triples in the first column of triples in the proof of Theorem 8) where d is the point corresponding to the resolution class in which (a,b,c) lies.

Also, for each resolution class, take as triples of the DTS the two triples

$$(m, d, \infty), \quad (\infty, d, m),$$

where d is the point corresponding to the resolution class and m is the MTS point missing from the resolution class. Finally, take as triples of the DTS the triples of a DTS$(3s+1)$ with $f \geq \frac{1}{2}$ on the points corresponding to the resolution classes. Then the resulting set of triples forms a DTS$(6s+3)$ (which is not pure).

Any defining set for the DTS$(6s+3)$ contains at least two triples from each set of three triples of the form above. There are $s(3s+1)$ such sets. Each pair of triples of the form of the row of two triples above is a DTS(3), which has $f \geq \frac{1}{2}$. Since also the DTS$(3s+1)$ has $f \geq \frac{1}{2}$, the DTS$(6s+3)$ has

$$f \geq \left(\tfrac{2}{3} \cdot 3s(3s+1) + \tfrac{1}{2} \cdot 2(3s+1) + \tfrac{1}{2} \cdot \frac{(3s+1)3s}{3} \right) \Big/ \left(\frac{(6s+3)(6s+2)}{3} \right).$$

Simplifying gives

$$f \geq \frac{5}{8} - \frac{1}{8(2s+1)},$$

and putting $v = 6s+3$ gives the stated result. □

5 Asymptotic results

Theorem 8 establishes the existence of a DTS(v) with $f \geq \frac{2}{3}$ whenever $v \equiv 1 \pmod{18}$, $v \geq 19$. We now use this result to prove that, for all $\epsilon > 0$ and all sufficiently large admissible v, there exists a DTS(v) with $f \geq \frac{2}{3} - \epsilon$.

For each admissible value of v, we define $\overline{f_v}$ to be the maximum value of f for all DTS(v). We prove the main result of this section in two stages. First note that if l is admissible, then so is l^k for all $k \geq 1$. The proof involves transversal designs (see [2]).

Lemma 12 *Suppose that $u = l^k v + w$ where l, w and $v + w$ are admissible, $v \geq w + 1$ and there exists a transversal design* TD(l^k, v). *Then*

$$\overline{f_u} \geq \left(\frac{(u-w)(u-w-v)}{u(u-1)} \right) \overline{f_{l^k}}.$$

PROOF. Since $v \geq w+1$, by Theorem 3 there exists a DTS$(v+w)$ containing a DTS(w) as a subsystem. We now take l^k copies of this DTS$(v+w)$ intersecting in a common DTS(w) subsystem; we may take the points of the i^{th} copy to be

$$1, 2, \ldots, w, 1_i, 2_i, \ldots, v_i.$$

Altogether there are $l^k v + w = u$ points and we may form a DTS(u) on these points by taking as triples all the triples of all the DTS$(v+w)$s (the *horizontal* triples) together with certain other triples which we describe below (the *vertical* triples). The vertical triples must cover every pair of the form (c_i, d_j) for $c, d = 1, 2, \ldots, v, \quad i, j = 1, 2, \ldots, l^k$ and $i \neq j$. To form the vertical triples we take a TD(l^k, v) with groups $\{1_i, 2_i, \ldots, v_i\}$ for $i = 1, 2, \ldots, l^k$. We then replace each block of size l^k with a DTS(l^k) (on the same points) with the maximum value of f, namely $\overline{f_{l^k}}$. There are v^2 blocks in the TD and $l^k(l^k-1)/3$ triples in each DTS(l^k).

Ignoring contributions from the horizontal triples, we have

$$\overline{f_u} \geq \left(\frac{v^2 l^k (l^k-1)}{3}\overline{f_{l^k}}\right) \Big/ \left(\frac{u(u-1)}{3}\right) = \frac{(u-w)(u-w-v)}{u(u-1)}\overline{f_{l^k}}. \qquad \square$$

Theorem 13 *Suppose that $l \geq 3$ is admissible and that $\overline{f_{l^k}} \geq f^*$ for all $k \geq 1$. Then, for any $\epsilon > 0$, there exists u_0 such that for all admissible $u > u_0$,*

$$\overline{f_u} > f^* - \epsilon.$$

PROOF. There exists v_0 such that for all $v > v_0$ the number of mutually orthogonal Latin squares of side v, denoted by $N(v)$, satisfies $N(v) \geq v^{\frac{1}{14.8}}$ [2]. Hence, for $v > v_0$ and $m \leq v^{\frac{1}{14.8}}$, there exists a transversal design TD(m, v) [2]. We will assume that v_0 is so large that

$$14.8\frac{\log(v_0+2)}{\log v_0} < 15.$$

Take $u \geq \max\{(v_0+2)^{\frac{16}{15}}, l^{16}\}$ and admissible. Define $k = \lfloor(\log_l u)/16\rfloor$ so that $1 \leq k \leq (\log_l u)/16 < k+1$, and hence $l^{16k} \leq u < l^{16(k+1)}$. We may write $u = \sum_{i=0}^{n} u_i l^i$, where $0 \leq u_i < l$ and $u_n \neq 0$. Since l is admissible, $l \equiv 0$ or 1 (mod 3).

Next we will choose $\alpha \in \{0, 1\}$ and define

$$v = u_n l^{n-k} + u_{n-1} l^{n-k-1} + \cdots + u_k - \alpha,$$

$$w = \alpha l^k + u_{k-1}l^{k-1} + u_{k-2}l^{k-2} + \cdots + u_0,$$

so that $u = l^k v + w$. We specify the choice of α as follows to ensure that both w and $v + w$ are admissible.

Suppose first that $l \equiv 1 \pmod 3$. Then $w \equiv \alpha + u_{k-1} + u_{k-2} + \cdots + u_0 \pmod 3$, so simply choose α to ensure that w is admissible. Then observe that

$$v + w \equiv (u_n + u_{n-1} + \cdots + u_k - \alpha) + (\alpha + u_{k-1} + u_{k-2} + \cdots + u_0) \equiv u \pmod 3,$$

so that $v + w$ is admissible.

Now suppose that $l \equiv 0 \pmod 3$. We then have $w \equiv u_0 \pmod 3$. But $u \equiv u_0 \pmod 3$, so that $u_0 \equiv 0$ or $1 \pmod 3$, and so w is admissible. Also, $v + w \equiv u_k - \alpha + u_0 \equiv u_k - \alpha + u \pmod 3$. If $u \equiv 0 \pmod 3$ then select α as per Table 1 below, whereas if $u \equiv 1 \pmod 3$ then select α as per Table 2 below.

$u_k \pmod 3$	0	1	2
α	0	0	1

Table 1

$u_k \pmod 3$	0	1	2
α	0	1	0

Table 2

For either residue class for u, w and $v + w$ are then both admissible.

By our choice of α, we have $0 \le w < 2l^k$. Hence $l^k(v + 2) > l^k v + w = u$ and so $v + 2 > ul^{-k} \ge u^{\frac{15}{16}} \ge v_0 + 2$, giving $v > v_0$. Also, $l^k \le u^{\frac{1}{16}} < (l^k(v + 2))^{\frac{1}{16}}$ and so $l^k < (v + 2)^{\frac{1}{15}}$. But $v > v_0$ and so $(v + 2)^{\frac{1}{15}} < v^{\frac{1}{14.8}}$, giving $l^k < v^{\frac{1}{14.8}}$. It follows that there is a $\mathrm{TD}(l^k, v)$.

Since $v + 2 > u^{\frac{15}{16}}$, $u \ge l^{16}$ and $w < 2l^k \le 2u^{\frac{1}{16}}$, we have

$$v > u^{\frac{15}{16}} - 2 > 2u^{\frac{1}{16}} + 1 > w + 1.$$

From Lemma 12 we now have

$$\begin{aligned}\overline{f_u} &\ge \left(\frac{(u - w)(u - w - v)}{u(u - 1)}\right)\overline{f_{l^k}}\\ &\ge \left(\frac{(u - w)(u - w - v)}{u(u - 1)}\right) f^*.\end{aligned}$$

But $0 \leq w < v$ and $0 < v \leq ul^{-k} < u(lu^{-\frac{1}{16}}) = lu^{\frac{15}{16}}$.

$$\text{Hence} \quad \frac{(u-w)(u-w-v)}{u(u-1)} \to 1 \text{ as } u \to \infty.$$

Consequently, for any $\epsilon > 0$ there exists u_0 such that for all admissible $u > u_0$, $\overline{f_u} > f^* - \epsilon$. □

Corollary 14 *For any $\epsilon > 0$ there exists u_0 such that, for all admissible $u > u_0$, there exists a* DTS(u) *with $f > \frac{2}{3} - \epsilon$.*

PROOF. Since $19^k \equiv 1 \pmod{18}$ for all $k \geq 1$, it follows from Theorem 8 that $\overline{f_{19^k}} \geq \frac{2}{3}$ for all $k \geq 1$. Now take $l = 19$ in Theorem 13. □

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