Cuboctahedron Designs

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Abstract

We prove that the complete graph K_v can be decomposed into cuboctahedra if and only if $v \equiv 1$ or 33 (mod 48).

1 Introduction

In this paper we bring together two current themes in Combinatorial Design Theory. The first of these concerns *graph decompositions*. A Steiner system $S(2, k, v)$ can be regarded as a decomposition of the complete graph K_v into complete subgraphs K_k. Similarly, an n-cycle system is a decomposition of K_v into n-cycles C_n. The study of both of these types of design is a major activity in the field. But there is also interest in decompositions of K_v into other subgraphs, particularly those which are regular and have some degree of symmetry. Prime candidates here are the *Platonic graphs* and the current state of knowledge appears to be as follows.

1. Tetrahedron designs are equivalent to Steiner systems $S(2, 4, v)$. The necessary and sufficient condition is $v \equiv 1$ or 4 (mod 12), [7].

2. Octahedron designs are equivalent to Steiner triple systems $S(2, 3, v)$ which can be decomposed into Pasch configurations. The necessary and sufficient condition is $v \equiv 1$ or 9 (mod 24), [6], [1].

3. The admissibility condition for icosahedron designs is $v \equiv 1$, 16, 21 or 36 (mod 60). They are known to exist for $v \equiv 1$ (mod 60), [2].

4. Cube designs exist if and only if $v \equiv 1$ or 16 (mod 24), [8], [4].

5. The admissibility condition for dodecahedron designs is $v \equiv 1$, 16, 25 or 40 (mod 60). They are known to exist for $v \equiv 1$ (mod 60), [2].

A natural extension of the above is to consider decompositions into the *Archimedean graphs.* But there are two infinite fanilies of these (the prisms and antiprisms), as well as thirteen further examples. It is clearly necessary to choose which of these will be of most interest.

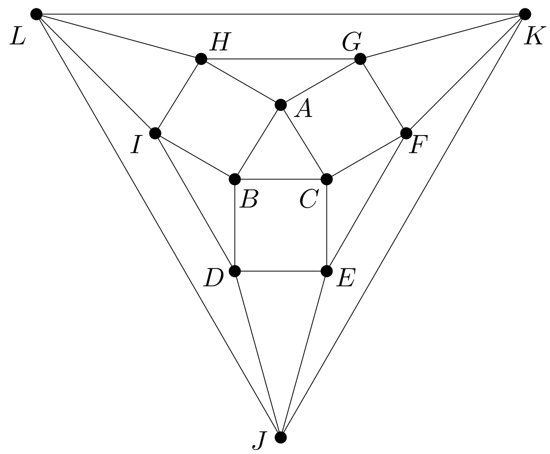
This leads to the second theme which is that of *metamorphosis.* This is a less well-defined concept but in the papers [9], (resp. [10]) is used to describe a decomposition of the complete graph K_v into either subgraphs K_4 or subgraphs K_3 (resp. C_4). The metamorphosis is achieved by removing appropriate edges from the subgraphs K_4 so as to form subgraphs K_3 (resp. C_4) and re-assembling these removed edges to form further subgraphs K_3(resp. C_4). In this paper, we consider the decomposition of the complete graph into *cuboctahedra.* By doing this we achieve a decomposition of the complete graph K_v into either subgraphs K_3 or subgraphs C_4, although the metamorphosis is different from that described above. In fact it is even simpler. No edges of the subgraphs need to be removed and re-assembled. The metamorphosis is achieved simply by regarding either the triangular faces or the square faces as the blocks into which each cuboctahedron can be further decomposed. In fact a cuboctahedron design might be more accurately described as a *simultaneous decomposition* of the complete graph K_v into subgraphs K_3 and cycles C_4.

It is easy to show that the admissibility condition for a cuboctahedron design on v points is $v \equiv 1$ or 33 (mod 48). We prove the existence of such systems for all these orders of v. In short, in this paper we prove the following result.

Theorem 1.1 *Cuboctahedron designs exist if and only if $v \equiv 1$ or 33 (mod 48).*

2 Construction

We first present cuboctahedron designs of orders 33 and 49, both of which were obtained by a computer search assuming appropriate cyclic automorphisms. The system for $v = 33$ was found by Peter Adams and Darryn Bryant, [3], of University of Queensland and we thank them for allowing us to include it in this paper. With regard to terminology, we will represent cuboctahedra by ordered twelve-tuples $(A, B, C, D, E, F, G, H, I, J, K, L)$ where the co-ordinates represent vertices as follows.



Lemma 2.1 *There exists a cuboctahedron design of order 33.*
Proof. Let the vertex set of the complete graph be $Z_{11} \times Z_3$. The decomposition consists of the cuboctahedra
((0,0),(1,0),(2,1),(7,1),(4,0),(3,1),(0,1),(6,0),(3,0),(3,2),(5,1),(0,2)) and
((0,0),(4,0),(4,2),(1,1),(9,2),(6,0),(1,2),(7,1),(0,2),(8,1),(8,2),(10,2)) under the action of the mapping $(i, j) \mapsto (i + 1, j)$ (mod 11). $\square$

Lemma 2.2 *There exists a cuboctahedron design of order 49.*
Proof. Let the vertex set of the complete graph be Z_{49}. The decomposition consists of the cuboctahedra (0,17,29,7,32,45,5,43,9,6,27,13) under the action of the mapping $i \mapsto i + 1$ (mod 49). $\square$

In general our method of proof uses a standard technique (Wilson's fundamental construction). For this we need the concept of a *group divisible design* (GDD). Recall therefore that a 3-GDD of type u^t is an ordered triple (V, G, B) where V is a base set of cardinality $v = tu$, G is a partition of V into t subsets of cardinality u called *groups* and B is a family of subsets of cardinality 3 called *blocks* which collectively have the property that every pair of elements from different groups occur in precisely one block but no pair of elements from the same group occur at all. We will also need 3-GDDs of type $u^t w^1$. These are defined analogously, with the base set V being of cardinality $v = tu + w$ and the partition G being into t subsets of cardinality u and one set of cardinality w.

Two of the main ingredients which we will need in applying Wilson's fundamental construction are given in the above lemmas and the third is a 3-GDD of type 4^3 which can be decomposed into two cuboctahedra. We present this next.

Lemma 2.3 *There exists a 3-GDD of type 4^3, decomposable into two cuboctahedra.*
Proof. Let the groups of the 3-GDD be $\{A_0, A_1, A_2, A_3\}$, $\{B_0, B_1, B_2, B_3\}$ and $\{C_0, C_1, C_2, C_3\}$.
The two cuboctahedra are $(A_0, B_0, C_0, C_2, B_1, A_1, C_3, B_3, A_2, A_3, B_2, C_1)$ and $(A_2, B_2, C_0, C_2, B_3, A_3, C_3, B_1, A_0, A_1, B_0, C_1)$. □

We are now in a position to present the main results.

Lemma 2.4 *There exists a cuboctahedron design of order $v = 48t + 1, t \geq 3$.*
Proof. There exists a 3-GDD of type $12^t, t \geq 3$, [11]. This is called the *master* 3-GDD. Replace each element of the base set V by 4 elements (i.e. inflate by a factor 4) and adjoin a further element ∞. On every inflated group of the 3-GDD, together with the element ∞, place the cuboctahedron design of order 49 from Lemma 2.2. Further replace each block of the master 3-GDD by the 3-GDD of type 4^3 from Lemma 2.3, called the *slave* 3-GDD. □

Lemma 2.5 *There exists a cuboctahedron design of order $v = 48t+33, t \geq 3$.*
Proof. There exists a 3-GDD of type $12^t 8^1, t \geq 3$, [5]. As in the previous lemma, replace each element of the base set V by 4 elements and adjoin a further element ∞. Again on every inflated group of the 3-GDD, together with the element ∞, place the cuboctahedron design of order 49 from Lemma 2.2 or, in the case of the inflated group of cardinality 32, the cuboctahedron design of order 33 from Lemma 2.1. Finally replace each block of the master 3-GDD by the slave 3-GDD of type 4^3 from Lemma 2.3. □

The above just leaves the orders $v = 81$, 97 and 129. Cuboctahedron designs for the last two of these orders can be constructed using respectively as the master 3-GDD, a 3-GDD of type 8^3 and 8^4, [11], and proceeding as in the proofs of Lemmas 2.4 and 2.5 using the cuboctahedron design of order 33 from Lemma 2.1 nad the slave 3-GDD of type 4^3 from Lemma 2.3. We state this formally.

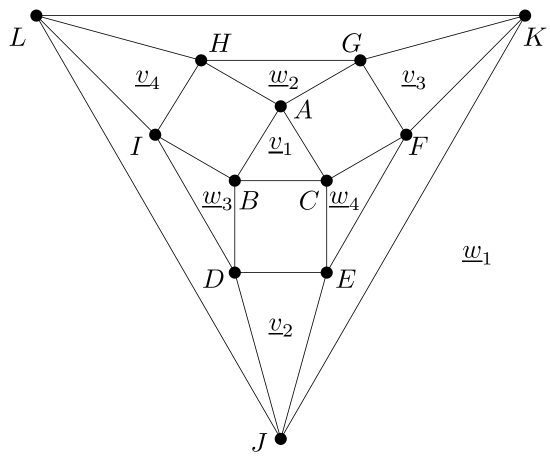
Lemma 2.6 *There exist cuboctahedron designs of orders 97 and 129.*

3 The case $v = 81$

In order to complete the proof of Theorem 1.1, it remains only to construct a decomposition of the complete graph K_{81} into 135 cuboctahedra. We do this in this Section.

Lemma 3.1 *There exists a cuboctahedron design of order 81.*
Proof. Let $V = GF(3^4)$, represented in the usual way as the vector space of polynomials of the form $a\lambda^3 + b\lambda^2 + c\lambda + d$ over $GF(3)$. We label the vertices of the complete graph K_{81} with the elements of V. If $\underline{v}$ and $\underline{w}$ are distinct elements of V, we will say that the edge joining $\underline{v}$ and $\underline{w}$ has length $\underline{v} - \underline{w}$, or alternatively $\underline{w} - \underline{v}$.

Let $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ be a basis for V and let $\underline{u} = \underline{v}_1 + \underline{v}_2 + \underline{v}_3 + \underline{v}_4$. Then it is easily verified that $\underline{w}_1 = \underline{v}_1 + \underline{u}$, $\underline{w}_2 = \underline{v}_2 + \underline{u}$, $\underline{w}_3 = \underline{v}_3 + \underline{u}$, $\underline{w}_4 = \underline{v}_4 + \underline{u}$ is also a basis for V. Consider the following planar drawing of a cuboctahedron and label the triangular faces as indicated.



We now describe an algorithm for labelling the vertices. Label vertex A with the zero vector $\underline{0}$. If XY is an edge and vertex X has label $\underline{x}$, then vertex Y receives label $\underline{x} + \underline{z}$ or $\underline{x} - \underline{z} = \underline{x} + 2\underline{z}$ respectively as to whether the direction from X to Y is anticlockwise or clockwise around the triangular face with label $\underline{z}$. This gives a consistent labelling of the vertices with each vertex receiving a distinct label. In terms of the basis $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ the labels are A:(0,0,0,0), B:(1,0,0,0), C:(2,0,0,0), D:(0,2,1,2), E:(0,1,1,2), F:(1,2,2,1), G:(1,2,1,1), H:(2,1,2,2), I:(2,1,2,1), J:(0,0,1,2), K:(1,2,0,1), L:(2,1,2,0). From this basic cuboctahedron we can construct 27 cuboctahedra as follows. For each vector $\underline{p} = a\lambda^3 + b\lambda^2 + c\lambda + d$

where $a+b+c+d=0$, add vector $\underline{p}$ to each label of the basic cuboctahedron. We obtain 27 cuboctahedra which collectively contain each edge of length $\pm\underline{v}_i$ or $\pm\underline{w}_i$, $i=1,2,3,4$, precisely once.

Let α be a primitive element of $V=GF(3^4)$. Denote by G, the multiplicative group of non-zero elements of V generated by α and let H be the cyclic subgroup of G generated by $\beta=\alpha^8$. Now assume that it is possible to choose the basis $\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$, $\underline{v}_4$ in such a way that the vectors $\underline{v}_i$ and $\underline{w}_i$, $i=1,2,3,4$ occur, one in each coset of H in G. Then we will be able to use as bases the sets $\{\beta^j\underline{v}_1,\beta^j\underline{v}_2,\beta^j\underline{v}_3,\beta^j\underline{v}_4\}$, $j=0,1,2,3,4$ which collectively contain precisely three edges of every length. From these five basic cuboctahedra we will then be able to construct $5\times 27=135$ cuboctahedra as described above and these are the required decomposition of the complete graph K_{81}.

An irreducible polynomial of degree 4 over $GF(3)$ is $\lambda^4+\lambda^2+2$ and with respect to this $\alpha=\lambda^2+\lambda$ is an element of order 80. Then $\beta=\alpha^8=2\lambda^3+\lambda^2+\lambda$. A suitable basis for $\underline{v}_1$,$\underline{v}_2$,$\underline{v}_3$,$\underline{v}_4$ found by computer search (in fact there are 3320 such bases) is $\underline{v}_1=\alpha=\lambda^2+\lambda$, $\underline{v}_2=\alpha^2=2\lambda^3+1$, $\underline{v}_3=\alpha^{13}=2\lambda^2+\lambda+1$, $\underline{v}_4=\alpha^{62}=\lambda^3+2\lambda^2+\lambda+1$. This gives $\underline{u}=2\lambda^2$ and $\underline{w}_1=\lambda=\alpha^{75}$, $\underline{w}_2=2\lambda^3+2\lambda^2+1=\alpha^{79}$, $\underline{w}_3=\lambda^2+\lambda+1=\alpha^{28}$, $\underline{w}_4=\lambda^3+\lambda^2+\lambda+1=\alpha^{16}$. □

4 Conclusion

In the same vein as the present paper, is the determination of the spectrum of icosidodecahedron designs. This would provide a simultaneous decomposition of the complete graph K_v into subgraphs K_3 and cycles C_5. The admissibility condition for an icosidodecahedron design is $v\equiv 1$, 25, 81 or 105 (mod 120) and it may be that the standard technique used in this paper will also succeed in the case of the icosidodecahedron. However, our initial investigations indicate that the main ingredients required to implement the construction may be more difficult to construct.

Also of some interest would be the construction of Steiner triple systems $S(2,3,v)$ which can be decomposed into the Desargues configuration. The latter is a collection of ten blocks of cardinality 3 isomorphic to $\{1,2,5\}$, $\{1,3,6\},\{1,4,7\},\{2,3,8\},\{2,4,9\},\{3,4,10\},\{5,6,8\},\{5,7,9\},\{6,7,10\}$, $\{8,9,10\}$. The configuration has the property that it is the smallest collection of blocks of cardinality 3 which contain precisely the same pairs as a collection of blocks of cardinality 4. In short, it is the smallest trade between blocks of different cardinality. The five blocks of cardinality 4 are $\{1,2,3,4\},\{1,5,6,7\},\{2,5,8,9\},\{3,6,8,10\},\{4,7,9,10\}$. Desarguesian designs provide a metamorphosis between Steiner systems $S(2,3,v)$ and $S(2,4,v)$. The admissibility condition is $v\equiv 1$ or 25 (mod 60).

References

[1] P. Adams, E.J. Billington and C.A. Rodger, Pasch decompositions of lambda-fold triple systems, *J. Combin. Math. & Combin. Comput.* **15** (1994), 53-63.

[2] P. Adams and D.E. Bryant, Decomposing the complete graph into Platonic graphs, *Bull. Inst. Combin. & Appl.* **17** (1996), 19-26.

[3] P. Adams and D.E. Bryant, A cuboctahedron design of order 33, *personal communication* (2000).

[4] D.E. Bryant, S. El-Zanati and R. Gardner, Decompositions of $K_{m,n}$ and K_n into cubes, *Australas. J. Combin.* **9** (1994), 285-290.

[5] C.J. Colbourn, D.G. Hoffman and R. Rees, A new class of group divisible designs with block size three, *J. Combin. Theory A* **59** (1992), 73-89.

[6] T.S. Griggs, M.J. deResmini and A. Rosa, Decomposing Steiner triple systems into four-line configurations, *Ann. Discrete Math.* **52** (1992), 215-226.

[7] H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* **11** (1975), 255-369.

[8] A. Kotzig, Decompositions of complete graphs into isomorphic cubes, *J. Combin. Theory B* **31** (1981), 292-296.

[9] C.C. Lindner and A. Rosa, The metamorphosis of λ-fold block designs with block size four into λ-fold triple systems, *J. Statist. Plann. & Inf.* to appear.

[10] C.C. Lindner and A.P. Street, The metamorphosis of λ-fold block designs with block size four into λ-fold 4-cycle systems, *Bull. Inst. Combin. & Appl.* **28** (2000), 7-18.

[11] L. Zhu, Some recent developments on BIBDs and related designs, *Discrete Math.* **123** (1993), 189-214.