Configurations in Steiner triple systems

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1 Introduction

The study of configurations in block designs is a topic which has recently come of age. The primary question in Design Theory is that of existence. In the usual notation, for a given admissible set of parameters t, k, v, λ does such a design exist? More generally, for fixed t, k, λ the goal is to determine the spectrum of v for which a t - (v, k, λ) design exists by exhibiting the required constructions. After this has been achieved, the emphasis changes to the consideration of more structural questions. Designs having prescribed automorphisms or with a certain subdesign structure, or lack of it, become important areas of investigation. The enumeration of pairwise non-isomorphic designs on the same parameter set leads naturally to the search for design invariants. The analysis of a design in terms of the configurations it contains becomes crucial. However only in the past few years has work been done, mainly for Steiner triple systems, in which configurations play a central role or feature prominently. It is the aim of this paper to present a survey of this work and we begin with the basic definitions.

A t - (v, k, λ) design is an ordered pair (V, B) where V is a base set of cardinality v, called *elements* or *points*, and B is a collection of k-element subsets of V, called *blocks* or *lines*, which collectively have the property that every t-element subset of V is contained in precisely λ blocks. Repeated blocks are allowed. When $t = 2$ and $k = 3$ the design is known as a *triple system* and denoted by $TS(v, \lambda)$. If in addition $\lambda = 1$ it is a *Steiner triple system* and is denoted by STS(v). Such systems exist if and only if the *order* of the system, $v \equiv 1$ or 3 (mod 6), a fact first proved by Kirkman [33] a century and a half ago. An *n-line configuration* is simply a partial design i.e. a collection of n k-element subsets which collectively have the property that every t-element subset is contained in at most λ blocks. In an n-line configuration the *degree* of a point is the number of lines which contain it.

This survey, which relates mainly but not exclusively to the study of configurations in Steiner triple systems, is partitioned into four sections. The first of these is concerned with the enumeration of pairwise non-isomorphic n-line configurations for small values of n, the identification of such configurations and their inter-relation. It also deals with the important question of how many of each configuration can occur in an STS(v). In the second section the focus is on the decomposition of STS(v)s into configurations all isomorphic to a given configuration. The third section deals with avoidance results, i.e. the construction of STS(v)s which contain no copies of a certain configuration. This leads naturally to a discussion of other extremal systems and also colouring problems. This is the subject matter of the fourth section. In all four sections are to be found some fundamental and challenging questions. We do not give proofs. To do so would increase the length of the survey manyfold. The reader therefore is referred to the original papers. Although the results presented are mainly about Steiner triple systems the same ideas can just as readily be applied not only to other classes of block designs but also to other combinatorial structures. Relatively little has been attempted so far but to see what is possible reference must be made to papers by Beezer [5] on regular graphs, Danziger & Mendelsohn [15] on Latin squares and Francel & Sarvate [20] on balanced ternary designs.

2 Counting

Denote by $C(n,\lambda)$ the number of pairwise non-isomorphic n-line configurations which can occur as blocks of a triple system $TS(v,\lambda)$. Trivially $C(1,\lambda) = 1$ for all λ. A 2-line configuration comprises a pair of lines intersecting in 0,1,2 or 3 points, denoted by A_1, A_2, A_3 or A_4 respectively. Hence $C(2,1) = 2$ and $C(2,\lambda) = 4$ for $\lambda \geq 2$. The 3-line configurations, see also [27], together with their designations and names which have now become traditional are shown in Figure 2.1. From these it follows that $C(3,1) = 5, C(3,2) = 13$ and $C(3,\lambda) = 16$ for $\lambda \geq 3$.

For further values of n, the value of $C(n,1)$ is given in the table [11].

n	4	5	6	7	8
$C(n,1)$	16	56	282	1865	17100

For $\lambda = 1$ the 4-line configurations are exhibited in Figure 2.2.

To the best of our knowledge these are the only values of $C(n,\lambda)$ which have been specifically enumerated, although doubtless, with the aid of a computer, it would not be too difficult to extend these results. Of course the combinatorial explosion quickly takes over and a more fundamental and interesting question is to determine an asymptotic result about the value of $C(n,\lambda)$.

The study of counting the number of occurrences of each configuration in an STS(v) was initiated in [21]. In that paper a configuration C is termed as either *constant* or *variable*. A constant configuration is one which, for each admissible value of v, occurs the same number of times in every STS(v). A variable configuration means that for some value of v there are at least two non-isomorphic STS(v)s

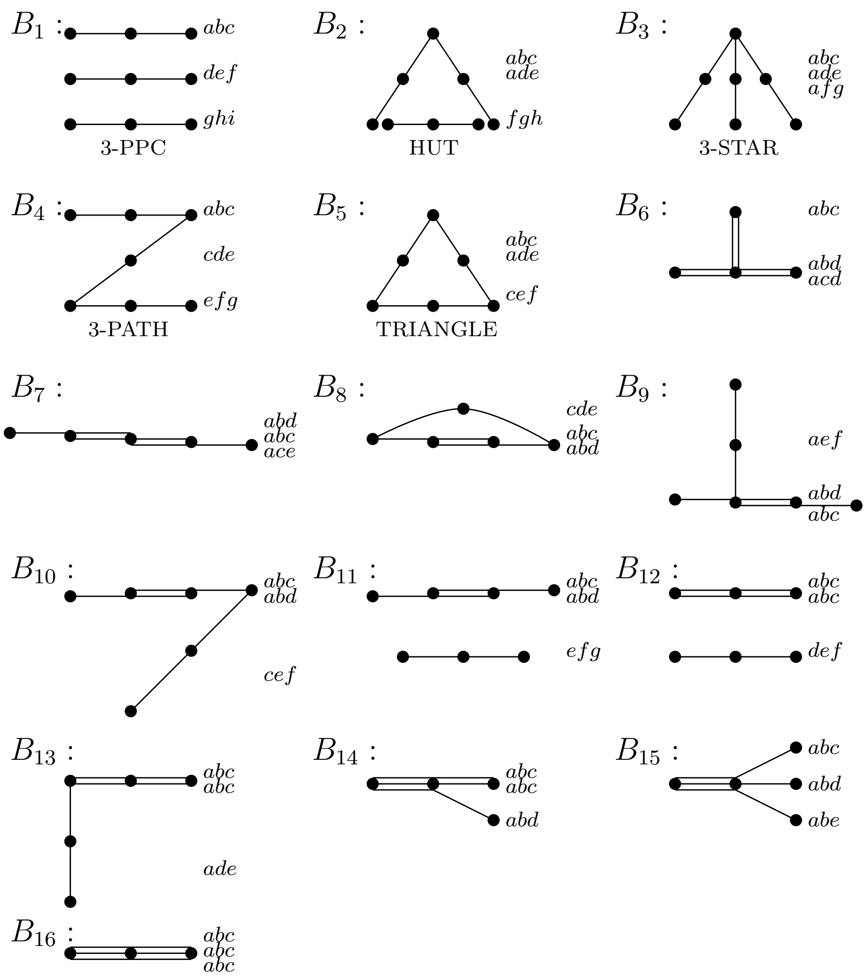

Figure 2.1. All 3-line configurations.

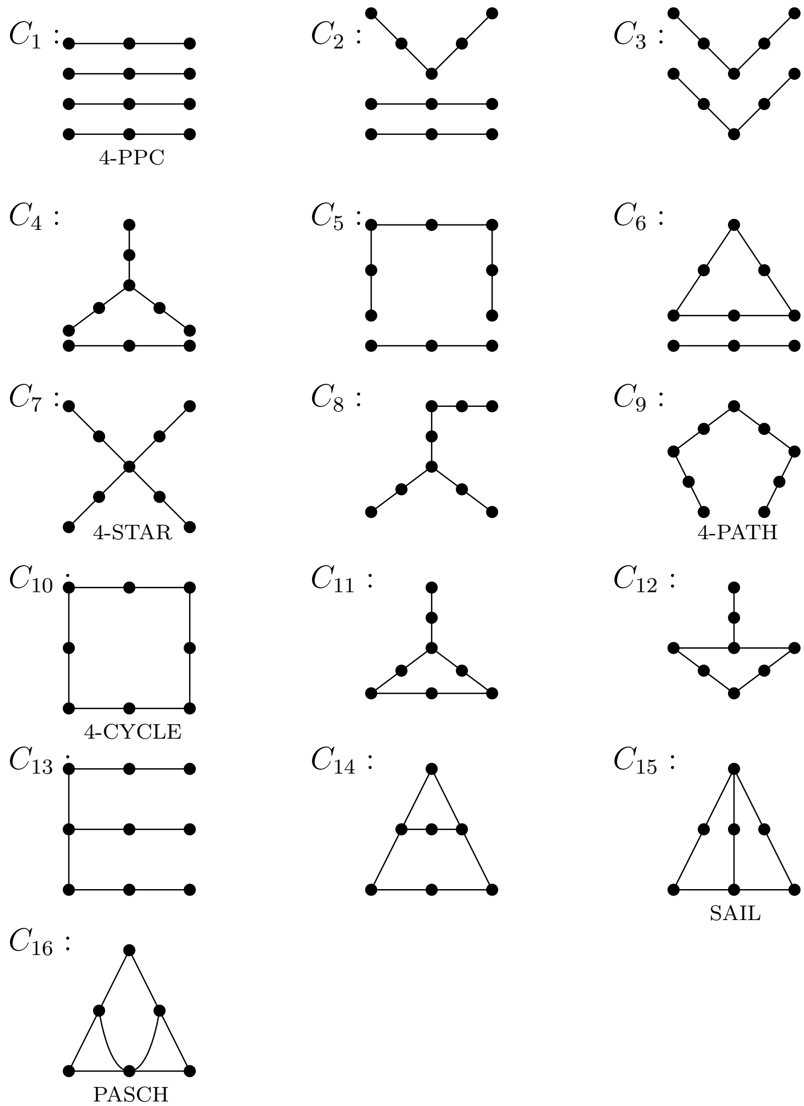

Figure 2.2. The 4-line configurations in Steiner triple systems.

containing different numbers of the configuration. It is easy to show that all n-line configurations for $n \leq 3$ are constant and the formulae are given in [21]. However for $n = 4$ it is immediately clear that this can not be the case. The configuration $C_{16} = P$, also known as the *quadrilateral* or *Pasch configuration* is variable. The two pairwise non-isomorphic STS(13)s contain 13 and 8 quadrilaterals respectively. The range of values for the 80 pairwise non-isomorphic STS(15)s is from zero, in a so-called *anti-Pasch* system $\sharp 80$ in the standard listing [39], to 105, the maximum possible, in $PG(3, 2)$. Most of [21] is devoted to determining formulae for the number of occurrences of each of the four-line configurations in an STS(v) by establishing inter-relations between them. It is shown that configurations C_4, C_7, C_8, C_{11} and C_{15} are constant and that all the others are variable. Further, the number of occurrences of all of the variable configurations can be expressed in terms of the order v and the number of occurrences of any one of them. The natural configuration to choose is the Pasch configuration if only because it is the "tightest" being the unique 4-line configuration containing the least number (6) of points. Observe also that this is the only n-line configuration, $1 \leq n \leq 4$, in which every point has degree at least 2. This result immediately raises two interesting and significant questions.

The first of these is to identify, for each n, an easily described subset of configurations with the property that the number of occurrences of every n-line configuration can be expressed in terms of the order v of the Steiner triple system and the number of occurrences of each member of the subset. This idea is considered by Horak, Phillips, Wallis & Yucas [31]. They make the following definitions.

Definition 2.1 A *generating set* M for n-line configurations is a set of m-line configurations, $1 \leq m \leq n$, such that for each admissible v the number of occurrences of any n-line configuration in an STS(v) can be expressed as a linear combination of the number of occurrences of the configurations in M, where the coefficients are polynomials in v. A *basis* is a minimal generating set.

So using this terminology, a single block is a basis for 1, 2 and 3-line configurations and the result of Grannell, Griggs & Mendelsohn [21] is that the single block and the Pasch configuration together form a basis for 4-line configurations. The main result in [31] is the following important theorem.

Theorem 2.1 (Horak, Phillips, Wallis & Yucas [31])
The single block together with all m-line configurations, $1 \leq m \leq n$, having all points of degree at least 2, form a generating set for the n-line configurations in a Steiner triple system.

The authors remark that it is possible to extend the definitions of generating set and basis to Steiner systems with block size $k > 3$ and that the proof of the theorem

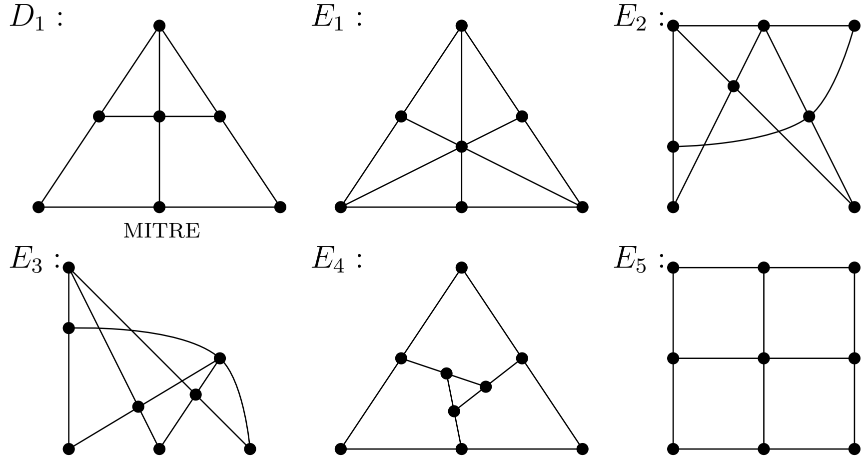

Figure 2.3. The 5- and 6-line configurations with all points of degree at least two.

requires only trivial modification. We observe that the result further extends to $\lambda > 1$. They also make the following conjecture.

Conjecture 2.1 *The single block together with all m-line configurations, $1 \leq m \leq n$, having all points of degree at least 2, form a basis for the n-line configurations in a Steiner triple system.*

The only 5-line configuration having all points of degree at least 2 is the so-called *mitre*. Horak et alia [31] also determine all 6-line configurations having this property and these are all shown in Figure 2.3.

By generating 8 random STS(19)s and counting the numbers of Pasch and mitre configurations and the five 6-line configurations $E_1, ..., E_5$ in each of them, it is easily determined that the generating sets for 5 and 6-line configurations as defined in Theorem 2.1 are also bases. Recently, Urland [50] has shown that Conjecture 2.1 is true for $n = 7$. There are precisely nineteen 7-line configurations having all points of degree at least 2.

The proof of Theorem 2.1 gives, in theory, a procedure for deriving, for any n-line configuration, a formula for the number of occurrences in terms of the number of occurrences of configurations in the generating set. However, in practice, the complexity of so doing is immense. Formulae for the number of occurrences of the 56 5-line configurations in terms of three variables, v the order of the Steiner triple system, p the number of Pasch configurations and m the number of mitres are listed in [16]. Already these are becoming complex. For example that for the 5-partial parallel class (5-PPC) i.e. 5 non-intersecting lines is

$v(v-1)(v-3)\times$

$$(v^7 - 91v^6 + 3588v^5 - 79510v^4 + 1069873v^3 - 8742231v^2 + 40167162v - 80101224)/933120$$
$$+(v-16)(v-21)p/6 + 2m.$$

Also listed for all of the 5-line configurations are the five 4-line configurations which they contain. Since publishing these results some errors have come to light and we wish to thank Gill Barber, a student at the University of Central Lancashire, for correcting these. We take this opportunity to list the corrections below in the same format as in [16].

30) 0 1 2 0 3 4 1 5 6 3 5 7 0 8 9
The 4-line subconfigurations are $C_8, C_8, C_9, C_9, C_{10}$.

31) 0 1 2 0 3 4 1 5 6 3 5 7 2 8 9
The 4-line subconfigurations are $C_5, C_9, C_9, C_{10}, C_{13}$.

42) 0 1 2 0 3 4 1 5 6 2 7 8 3 9 10
The 4-line subconfigurations are $C_2, C_5, C_9, C_9, C_{13}$.

The second of the questions is: what are the constant configurations? We are in little doubt what the answer to this is, though proving it certainly doesn't appear easy and may in fact be quite difficult. Define an *n-star* to be an n-line configuration in which all n lines intersect at a common point called the *centre*. The following conjecture is made in [31].

Conjecture 2.2 *For $n \geq 4$, an n-line configuration in a Steiner triple system is constant if and only if it can be obtained from an $(n-1)$-star by adjoining a further block.*

In general this can be done in precisely five ways. The"further block" can be disjoint from the $(n-1)$-star, intersect at the centre or intersect at one, two or three other points. The proof of the sufficiency of the conjecture is straightforward and formulae for the number of occurrences of all of the five such n-line configurations are given in [31]. Note that the conjecture is not true for $n < 4$. The configuration B_1, the 3-partial parallel class (3-PPC), is the sole exception.

Finally in this section we mention a third fundamental question. It is straightforward to verify that the four 3-line configurations obtained by removing each of the 4 blocks in turn from a 4-line configuration uniquely determine the 4-line configuration, and the results in [16] with the corrections given in this paper show that the same is true for the five 4-line configurations obtained from a 5-line configuration. Although of course this is not true for the 2-line configurations or 3-line configurations (both the 3-star and triangle giving three pairs of intersecting lines), we believe that these are the only exceptions. We therefore make the following conjecture.

Conjecture 2.3 *For $n \geq 4$, the collection of n $(n-1)$-line configurations obtained from a given n-line configuration by removing each of the n lines in turn uniquely characterizes that n-line configuration.*

Given that this conjecture is analogous to the graph reconstruction conjecture this too may be difficult to prove but progress on any of the questions raised in this section would be of interest.

3 Decomposition

The seminal paper on the decomposition of Steiner triple systems is by Horak & Rosa [32]. The fundamental question which is addressed is, given a configuration C, whether the blocks of an STS(v) can be partitioned into copies of C. Strictly speaking this implies that if C is an n-line configuration then n divides b, the number of blocks of the STS(v). However we will extend the definition of decomposition to include the situation where the lines of an STS(v) from which less than n lines have been deleted can be decomposed into copies of C. In the case where n divides b we will refer to an *exact decomposition*. Horak & Rosa [32] prove a number of theorems of which the following are the most general.

Theorem 3.1 (Horak & Rosa [32])
Every STS(v) can be decomposed into n-PPCs for $n < v/9$.

Theorem 3.2 (Horak & Rosa [32])
Every STS(v) can be decomposed into n-stars for $n < v/6$.

Then immediately by using these results and handling the small cases separately a complete answer for the 2-line configurations can be deduced.

Theorem 3.3 (Horak & Rosa [32])

(i) Every STS(v) with $v \neq 7$ or 9 can be decomposed into configurations A_1 (2-PPCs).

(ii) Every STS(v) can be decomposed into configurations A_2 (2-stars).

Similarly, complete or nearly complete results are obtained for the 3-star and the 3-PPC as well as for the hut, this latter configuration being a more complicated situation to handle.

Theorem 3.4 (Horak & Rosa [32])

(i) Every STS(v) with $v \geq 27$ can be decomposed into configurations B_1 (3-PPCs).

(ii) Every STS(v) with $v \geq 55$ can be decomposed into configurations B_2 (huts).

(iii) Every STS(v) with $v \neq 7$ or 9 can be decomposed into configurations B_3 (3-stars).

This just leaves the 3-line configurations B_4 (3-paths) and B_5 (triangles) but here much less is known. For the former configuration we have the following.

Theorem 3.5 (Horak & Rosa [32])
Every cyclic STS(v) with $v \neq 7$ can be decomposed into configurations B_4 (3-paths).

For triangles the best that is known is an existential result.

Theorem 3.6 (Mullin, Poplove & Zhu [42])
For every $v \equiv 1$ or 3 (mod 6), there exists an STS(v) decomposable into configurations B_5 (triangles).

It is known that all STS(v)s with $v \leq 15$ are decomposable into triangles and, except for $v = 7$, decomposable into 3-paths. It therefore seems appropriate to make the following conjectures.

Conjecture 3.1 *Every STS(v) with $v \neq 7$ can be decomposed into 3-paths.*

Conjecture 3.2 *Every STS(v) can be decomposed into triangles.*

Extending the work to 4-line configurations was done by Griggs, deResmini & Rosa [26]. In this paper only exact decompositions were considered so that 4 divides b. This implies that $v \equiv 1$ or 9 (mod 24). Griggs et alia gave an almost complete answer to the determination of the spectrum of v for which there exists an STS(v) exactly decomposable into copies of each 4-line configuration $C_i, i = 1, 2, ..., 16$. They left just a single value, $v = 81$, undetermined for six of the configurations. This missing value was later supplied by Adams, Billington & Rodger [2]. So the definitive result can be stated.

Theorem 3.7 (Griggs, deResmini & Rosa [26],)
(Adams, Billington & Rodger [2])

(i) Let $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\}$. Then there exists an STS(v) exactly decomposable into configurations C_i if and only if $v \equiv 1$ or 9 (mod 24), $v \geq 25$.

(ii) Let $i \in \{11, 13, 14, 15\}$. Then there exists an STS(v) exactly decomposable into configurations C_i if and only if $v \equiv 1$ or 9 (mod 24), $v \geq 9$.

Unlike the situation which may apply for 3-line configurations, it is not possible for almost every STS(v), $v \equiv 1$ or 9 (mod 24) to be exactly decomposable into all 4-line configurations, if only because the Pasch configuration C_{16} can be avoided (see Section 4). Nevertheless, Griggs, deResmini & Rosa do prove some universal results.

Theorem 3.8 (Griggs, deResmini & Rosa [26])
Let $i \in \{1, 3, 4, 7\}$. Then there exists an integer $v_0(i)$ such that every STS(v), $v \equiv 1$ or 9 (mod 24), $v \geq v_0(i)$ is exactly decomposable into configurations C_i.

It is currently known that $v_0(1) \leq 49,\ v_0(3) \leq 57,\ v_0(4) \leq 169$ and $v_0(7) = 25$.

This naturally leads to the question of the existence of Steiner triple systems which are *simultaneously decomposable* into copies of each configuration of a given set. This question was considered by Griggs, Mendelsohn & Rosa [23].

Using very similar techniques (Wilson's fundamental construction) to those employed in the earlier paper, they were able to obtain an almost complete answer in the case where $v \equiv 1$ (mod 24) and, in arithmetic set density terms, 5/14ths of the case where $v \equiv 9$ (mod 24).

Theorem 3.9 (Griggs, Mendelsohn & Rosa [23])
For every $v \equiv 1$ (mod 24) or $v \equiv 33$ (mod 96) or $v \equiv 57$ (mod 168), $v \neq 97$, there exists an STS(v) simultaneously decomposable into configurations $C_i, 1 \leq i \leq 16$.

The difficulty in the residue class $v \equiv 9$ (mod 24) lies with the configuration C_{15}. If we try to colour the points of each of the 4-line configurations so that each block receives three different colours, then three colours suffice except for C_{15} where four are needed. (This is easily verified by the reader.) This implies that the group divisible designs (GDDs) which are needed as the master GDDs in Wilson's fundamental construction are 4-GDDs rather than 3-GDDs but the former do not exist for all the required cases. It is likely therefore that new ideas will be needed to make further progress on this problem. However when the configuration C_{15} is ignored, Griggs et alia prove the following.

Theorem 3.10 (Griggs, Mendelsohn & Rosa [23])
For every $v \equiv 1$ or 9 (mod 24), $v \notin \{81, 97, 105, 153\}$, there exists an STS(v) simultaneously decomposable into configurations $C_i, 1 \leq i \leq 16, i \neq 15$.

The four missing values in Theorem 3.10 and $v = 97$ in Theorem 3.9 are not definite exceptions but values for which the results are undecided. It would be tidy to resolve these cases. The STS(81) given by Adams, Billington & Rodger [2] was shown to be decomposable into configurations $C_i, i \in \{6, 10, 11, 12, 14, 16\}$ and since all STS(81)s are decomposable into configurations C_1, C_3 and C_7, (Theorem 3.8 above) it would only remain to determine decompositions into configurations $C_i, i \in \{2, 4, 5, 8, 9, 13, 15\}$. The construction of a cyclic STS(97) simultaneously decomposable into all 4-line configurations should also not present any real problems. For the other two values, perhaps the simplest structures to consider are an STS(105) invariant under the cyclic group C_{35} (52 orbits) and an STS(153) invariant under C_{51} (76 orbits). Investigation of these and of alternatives is likely to be extremely tedious!

The significance of the Pasch configuration has already been referred to in Section 2. The paper by Adams, Billington & Rodger [2] places the configuration at the heart of the investigation and considers the question of constructing triple systems TS(v, λ) exactly decomposable into configurations C_{16}. Their main result is the following.

Theorem 3.11 (Adams, Billington & Rodger [2])
There exists a TS(v, λ) exactly decomposable into Pasch configurations if and only if λ and v satisfy the necessary conditions in the Table below and $(\lambda, v) \neq (1, 9)$.

Values of λ (mod 12)	Values of v
1, 5, 7 or 11	1 or 9 (mod 24)
2 or 10	1 or 9 (mod 12)
3 or 9	1 (mod 8)
4 or 8	0 or 1 (mod 3)
6	1 (mod 4)
0	all $v \geq 6$

Although not strictly within the scope of this survey we mention that an extension of this work to block designs with $k = 4$ and $\lambda = 1$ is contained in a further paper [7].

A different viewpoint on Steiner triple systems decomposable into Pasch configurations is given by Adams & Bryant [3]. They consider the question of the exact decomposition of the complete graph K_v into edge-disjoint copies of each of the five Platonic graphs. For the tetrahedron this is equivalent to the construction of a 2 - $(v, 4, 1)$ design and the necessary and sufficient condition is well-known to be $v \equiv 1$ or 4 (mod 12) [29]. The necessary condition for a cube decomposition is $v \equiv 1$ or 16 (mod 24). Kotzig [34] dealt with the former residue class and more recently, Bryant, El-Zanati & Gardner [10], the latter. Octahedron decomposition is equivalent to the exact decomposition of an STS(v) into Pasch configurations and hence, by Theorem 3.7, exists if and only if $v \equiv 1$ or 9 (mod 24), $v \geq 25$ (the reference given in [3] is incorrect). Partial results exist for the dodecahedron and icosahedron, and the reader is referred to the original paper.

Finally in this section we deal with a question which we were once asked. Steiner triple systems may contain other Steiner triple systems as subsystems. Indeed this is at the heart of many recursive constructions. But is it possible for a Steiner triple system to be exactly decomposable into isomorphic copies of some smaller system? In particular do there exist STS(v)s decomposable into STS(7)s or *Fano configurations*? The answer is in the affirmative. Such a system is equivalent to a 2 - $(v, 7, 1)$ design and is obtained by replacing each 7-block by a Fano configuration on the 7 points of the block. It is known that 2 - $(v, 7, 1)$ designs exist for all $v \equiv 1$ or 7 (mod 42) apart from a small number of values for which the existence is undecided [1].

4 Avoidance

In this section we discuss the question of the construction of triple systems which avoid, i.e. contain no copies of, a specified configuration or configurations. Of particular interest is the determination, for a given configuration C, of the spectrum of admissible values v for which there exists a $TS(v, \lambda)$ avoiding C. For all 2-line and 3-line configurations an almost complete answer to this question is given in the paper by Griggs & Rosa [27] and we follow their notation. Denote by $B(\lambda)$, the set of *admissible values* of v for given λ, i.e. the set of values of v for which there exists a $TS(v, \lambda)$.

Let C be a configuration. The *avoidance set* $\Omega(C, \lambda)$ for C and given λ is defined by

$$\Omega(C, \lambda) = \{v : v \in B(\lambda) \text{ and there exists a } TS(v, \lambda) \text{ without } C\}.$$

Further, if Σ is a set of configurations, the *simultaneous avoidance set* $\Omega(\Sigma, \lambda)$ is defined by

$$\Omega(\Sigma, \lambda) = \{v : v \in B(\lambda) \text{ and there exists a } TS(v, \lambda) \text{ without } C \text{ for all } C \in \Sigma\}.$$

The results for the 2-line configurations are given in the following theorem.

Theorem 4.1 (Griggs & Rosa [27])

(i) $\Omega(A_1, \lambda) = B(\lambda) \cap \{3, 4, 5, 6, 7\}$;

(ii) $\Omega(A_2, \lambda) = B(\lambda) \cap \{3, 4\}$;

(iii) $\Omega(A_3, \lambda) = B(1)$;

(iv) $\Omega(A_4, \lambda) = B(\lambda) \cap \{v : v \geq \lambda + 2\}$.

The results in the above theorem for configuration A_1 (a pair of non-intersecting blocks) and for configuration A_2 (a pair of blocks intersecting in a single point) indicate that these can only be avoided in triple systems with a small number of elements. They are completely straightforward to prove. The result for configuration A_3 (a pair of blocks intersecting in two points) is simply a statement that this can be avoided only by assembling a $TS(v, \lambda)$ as λ copies of an STS(v). The result for configuration A_4 (a pair of repeated blocks) is certainly not straightforward and is a restatement of the result on the existence of triple systems without repeated triples which was first proved by Dehon [17].

Griggs & Rosa [27] continue by determining the simultaneous avoidance sets for all combinations of 2-line configurations and the avoidance sets for 14 of the 16 3-line configurations. The latter relies on a wide range of known design-theoretic techniques as well as enumeration results and other constructions which the authors supply. It is therefore a problem which has become feasible to tackle only recently. As noted in the Introduction, it is an example of the study of configurations coming of age. We refer the reader to the original paper for the exact results, but for the two configurations (B_1 and B_6) where the avoidance set is not completely determined, we give the incomplete result and hence indicate what still needs to be done.

Theorem 4.2 (Griggs & Rosa [27])

(i) $\Omega(B_1, 1) = \{3, 7\}$;

(ii) $\{3, 7\} \subseteq \Omega(B_1, \lambda) \subseteq \{3, 7, 9\}$ *if* $\lambda \equiv 1$ *or* 5 *(mod* 6*),* $\lambda > 1$*;*

(iii) $\Omega(B_1, 2) = \{3, 4, 6, 7, 9\}$;

(iv) $\{3, 4, 6, 7, 9\} \subseteq \Omega(B_1, \lambda) \subseteq \{3, 4, 6, 7, 9, 10\}$ *if* $\lambda \equiv 2$ *or* 4 *(mod* 6*);* $\lambda > 2$*;*

(v) $\{3, 5, 7\} \subseteq \Omega(B_1, 3) \subseteq \{3, 5, 7, 11\}$*;*

(vi) $\{3, 5, 7\} \subseteq \Omega(B_1, \lambda) \subseteq \{3, 5, 7, 9, 11\}$ *if* $\lambda \equiv 3$ *(mod* 6*),* $\lambda > 3$*;*

(vii) $\{3,4,5,6,7,8,9\} \subseteq \Omega(B_1, \lambda) \subseteq \{3,4,5,6,7,8,9,10,11\}$ *if* $\lambda \equiv 0$ *(mod 6).*

Theorem 4.3 (Griggs & Rosa [27])

(i) If $\lambda \not\equiv 0$ *(mod 6) then*

$$\Omega(B_6, \lambda) = \begin{cases} B(1) \text{ if } \lambda \equiv 1 \text{ or } 5 \pmod 6, \\ B(2)\backslash\{4\} \text{ if } \lambda \equiv 2 \text{ or } 4 \pmod 6, \\ B(3)\backslash\{5\} \text{ if } \lambda \equiv 3 \pmod 6. \end{cases}$$

(ii) If $\lambda \equiv 0$ *(mod 6) then*

$$B(6)\backslash\{4,5,8,14,20\} \subseteq \Omega(B_6, \lambda) \subseteq B(6)\backslash\{4,5\}.$$

It would be useful to tidy up this area by resolving the undetermined cases. This is not likely to be particularly easy since they seem to fall between what is achievable by enumeration and what is achievable by known techniques.

The determination of simultaneous avoidance sets for 3-line configurations and avoidance sets for 4-line configurations appears, in general, to be a formidable undertaking. However for the latter it is possible to proceed further in the case where $\lambda = 1$, and this is discussed next.

Recall (Section 2) that formulae for the number of occurrences of all of the 4-line configurations in an STS(v) in terms of v and p, the number of Pasch configurations C_{16}, is known [21].

An easy counting argument gives that the maximum value which can be taken by p is $v(v-1)(v-3)/24$. Using this fact together with the formulae, it immediately follows that the avoidance sets $\Omega(C_i, 1), i = 1, 2, ..., 13$ and $i = 15$ are finite, consisting of a small number of small values. We leave the determination of the exact membership as an easy exercise for the interested reader. For C_{14}, the formula implies that this can be avoided only if p actually takes its maximum value. This is achieved only in the projective spaces $PG(n, 2)$, see [49], and hence $\Omega(C_{14}, 1) = \{2^{n+1} - 1 : n \geq 1\}$. This just leaves the Pasch configuration itself.

The interest in Steiner triple systems which avoid the Pasch configuration, so-called *anti-Pasch* or *quadrilateral-free* STS(v)s, pre-dates the recent work on configurations. The unique STS(7) and both of the two pairwise non-isomorphic STS(13)s contain quadrilaterals. But apart from these two exceptional orders it is conjectured that there exists an anti-Pasch STS(v) for all admissible v. Certainly the unique STS(9) is anti-Pasch, as is precisely one of the 80 pairwise non-isomorphic STS(15)s. A study of anti-Pasch STS(19)s, [24], revealed that such systems are not particularly rare. Nevertheless the conjecture remains unproven in its entirety although there are substantial partial results.

A particularly elegant construction of Steiner triple systems of order $v \equiv 3 \pmod 6$ is that first given by Bose [8]. Let G be a finite Abelian group of odd order and $V = G \times \{0, 1, 2\}$. The blocks of an STS(v) are obtained by choosing the following triples:

(i) $\{(x, 0), (x, 1), (x, 2)\}, x \in G$;

(ii) $\{(x, i), (y, i), (z, i + 1)\}, x, y \in G, x \neq y, z^2 = xy, i \in \{0, 1, 2\}$.

Observe that z is defined uniquely and that addition is modulo 3. As was observed by Doyen [18], this construction yields anti-Pasch STS(v)s whenever v is not divisible by 7. The case where v is divisible by 7 was resolved by Brouwer [9], (see also [25]). Hence we can state the following result.

Theorem 4.4 (Brouwer [9])
There exists an anti-Pasch STS(v) for all $v \equiv 3 \pmod 6$.

The case where $v \equiv 1 \pmod 6$ seems more difficult to handle. There are a number of partial results throughout the literature. These include the following.

(A) If $v = p^n$ where p is prime and $p \equiv 19 \pmod{24}$, then there exists an anti-Pasch STS(v). This result, due to Robinson [45], follows from an analysis of the cycle structure of the so-called Netto systems, more of which later.

(B) Suppose $v \equiv 1 \pmod 6$ is a prime or prime power, say $v = p^n$, and $p \neq 7$ or 13. If either n is even or $p \equiv 1$ or $3 \pmod 8$ then there exists an anti-Pasch STS(v). This result is also due to Brouwer [9] and illustrates the complex conditions which have to be satisfied with the known constructions. The next result is in the same vein.

(C) If the order of $-2 \pmod p$ is singly even for every prime divisor p of $v - 2$, then there exists an anti-Pasch STS(v). This result, due to Grannell, Griggs & Phelan [22], is obtained from an analysis of the systems produced by a construction first given independently by Schreiber [48] and Wilson [52]. In fact this construction goes further. In these cases it can be used to produce a partition of the set of all triples into $v-2$ mutually disjoint anti-Pasch STS(v)s.

(D) If there exists an anti-Pasch STS(v) where $v \equiv 1 \pmod 4$ and $v - 1$ has an odd divisor greater than 3 then there exists an anti-Pasch STS($3v - 2$). This result, given in a paper by Stinson & Wei [49], yields a linear class of anti-Pasch STS(v)s. By applying the result to known anti-Pasch STS($24s + 21$)s, we have that there exists anti-Pasch STS(v)s for all $v \equiv 61 \pmod{72}$.

(E) If there exists an anti-Pasch STS(u) and an anti-Pasch STS(v) then there exists an anti-Pasch STS(uv). This is the usual product construction for Steiner triple systems [25], [49].

Very recently, Ling, Colbourn, and the present authors have generalized Stinson & Wei's construction (D, above) to remove the conditions on v, as well as producing further constructions. The paper [37] is at the time of writing still in preparation but by interplaying the results the following can be proved.

Theorem 4.5 (Ling, Colbourn, Grannell & Griggs [37])
There exists an anti-Pasch STS(v) for all $v \equiv 1, 7, 19, 25, 37, 43, 49, 55$ *or* 61 *(mod* 72*),* $v \neq 7$*.*

In fact somewhat more can be proved but unfortunately we are unable to complete a solution of the problem. Nevertheless the above does take care of 3/4ths of the $v \equiv 1 (\text{mod } 6)$ case in arithmetic set-density terms. To complete the problem it would suffice to produce anti-Pasch STS(v)s in the two cases (i) $v = 6p + 1, p \equiv 5$ (mod 6) and prime and (ii) $v = 12p + 1, p \equiv 1$ (mod 6) and prime. We leave this as a challenge for the reader.

However the problem of determining the spectrum of v for which there exists an anti-Pasch STS(v) is only the first of an infinite sequence of such avoidance questions. Define a *(p, l)-configuration* in a Steiner triple system to be an l-line configuration whose union contains precisely p points. We will be particularly concerned with the case where $p = l + 2$. Erdos (see [13]) conjectured that for every $m \geq 4$, there is an integer v_m such that for every admissible $v > v_m$, there exists an STS(v) avoiding $(l + 2, l)$-configurations for $4 \leq l \leq m$. We will call such an STS(v), *m-sparse*.

There are precisely two pairwise non-isomorphic (7,5)-configurations, the mitre (see Section 2), and the *mia* which is obtained from the Pasch configuration by extending the latter with an extra line through any of the three pairs of non-collinear points. *Anti-mia* systems are therefore exactly the same as anti-Pasch systems. The problem of constructing *anti-mitre* Steiner triple systems was first studied by Colbourn, Mendelsohn, Rosa & Siran [13] whose main result is the following.

Theorem 4.6 (Colbourn, Mendelsohn, Rosa & Siran [13])
There exists an anti-mitre STS(v) for all $v \equiv 3, 7, 9, 19, 21$ *or* 27 *(mod* 36*).*

With regard to 5-sparse STS(v)s, i.e. systems which are both anti-Pasch and anti-mitre, Colbourn et alia make the following observations. Let $v = p^n$ where p is prime and $p \equiv 7$ (mod 12). Further let ϵ_1 and ϵ_2 be the two primitive sixth roots of unity in $GF(q)$. For any two distinct elements $a, b \in GF(q)$ define $a < b$ if $b - a$

is a square in $GF(q)$, and let $c = f(a,b) = a\epsilon_1 + b\epsilon_2$. Then it is easy to check that $b < c,\ c < a,\ a = f(b,c)$ and $b = f(c,a)$. The *Netto system* $N(q)$ is the Steiner triple system (V,B) where $V = GF(q)$ and $B = \{\{a,b,c\} : a < b$ and $c = f(a,b)\}$. Every Netto system is anti-mitre. Using the known result that such systems are also anti-Pasch precisely in the case where $p \equiv 19 \pmod{24}$, (see (A), above) it follows that they are also 5-sparse. In a recent paper Ling [36] gives the following product construction.

Theorem 4.7 (Ling [36])
If there exists a transitive 5-sparse STS(v), $v \equiv 1 \pmod 6$ and a 5-sparse STS(w) (including $w = 3$), then there exists a 5-sparse STS(vw).

Since the Netto systems are transitive and an analysis of known cyclic STS(v)s reveal that there exist cyclic 5-sparse STS(v)s for all admissible v satisfying $33 \leq v \leq 57$ [13], Ling's result extends the known spectrum. However this still leaves much to be done even to prove Erdos' conjecture for $m = 5$. For convenience we list the next steps which need to be taken to make progress in this area.

(1) (Starred problem) Complete the proof of the Theorem that there exists an anti-Pasch (4-sparse) STS(v) for all admissible $v \equiv 1$ or $3 \pmod 6$, $v \neq 7$ or 13.

(2) Complete the determination of the spectrum of v for which there exists an anti-mitre STS(v). In particular it is conjectured that such systems exist for all admissible $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$.

(3) Find further examples of and constructions for 5-sparse STS(v)s. It is known that no 5-sparse STS(v)s exist for $v = 7, 9, 13$ or 15 but $N(19)$ is 5-sparse. The next order for which it is known that a 5-sparse STS(v) exists is $v = 33$. The cases $v = 21, 25, 27$ and 31 are undetermined.

(4) Find a 6-sparse STS(v). None is known. The Netto systems are not 6-sparse nor are any of the cyclic STS(v)s, $v \leq 57$.

Finally in this section we mention a result in the opposite direction.

Theorem 4.8 (Lefmann, Phelps & Rodl [35])
There exists a constant $c > 0$ such that every STS(v) contains a $(p, p-2)$-configuration for some p such that $6 \leq p \leq (c \log v)/(\log \log v)$.

5 Colouring

Problems connected with the colouring of blocks of a design may be viewed as extensions of avoidance results and we adopt this approach in this section. Let T be a triple system $\mathrm{TS}(v, \lambda)$ and C a configuration. The minimum number of colours required to colour the blocks of T avoiding monochromatic copies of C will be denoted by $\chi(C, T)$. Thus for example the statement that a particular STS(v), S, is anti-Pasch is equivalent to $\chi(P, S) = 1$.

Most attention in this area has focused on the case where the configuration $C = A_2$, a pair of intersecting lines. This is the well-known *chromatic index*, most usually denoted by $\chi'(T)$. Further we make the following definitions.

(i) $\underline{\chi'}(v, \lambda) = \min \{\chi'(T) : T \text{ is a } TS(v, \lambda)\}$,

(ii) $\overline{\chi'}(v, \lambda) = \max \{\chi'(T) : T \text{ is a } TS(v, \lambda)\}$.

When $\lambda = 1$, ie in the case of a Steiner triple system, we simplify the terminology to $\underline{\chi'}(v)$ and $\overline{\chi'}(v)$.

Clearly, in colourings which avoid monochromatic copies of A_2, the colour classes consist of sets of parallel lines. For $v \equiv 3 \pmod 6$, the lowest value of the chromatic index, namely $(v-1)/2$, is achieved by taking an STS(v) which can be partitioned into parallel classes each consisting of $v/3$ blocks. Such a system is said to be *resolvable* and is called a *Kirkman triple system*, KTS(v). In a classic paper, Ray-Chaudhuri & Wilson [44] determined that Kirkman triple systems exist for all $v \equiv 3 \pmod 6$, and hence we can state the following.

Theorem 5.1 (Ray-Chaudhuri & Wilson [44])
For $v \equiv 3 \pmod 6$, $\underline{\chi'}(v) = (v-1)/2$.

In the case where $v \equiv 1 \pmod 6$, $v \geq 19$, a system which attains the minimum possible value of the chromatic index, namely $(v+1)/2$, is known as an *Hanani triple system*, HATS(v). The existence of these systems was determined in [51] and we therefore have the following.

Theorem 5.2 (Vanstone et alia [51])
For $v \equiv 1 \pmod 6$, $v \geq 19$, $\underline{\chi'}(v) = (v+1)/2$.

Results for *twofold triple systems*, TS(v,2) or TTS(v) are also available.

Theorem 5.3 (Hanani [28], [30])

(i) For $v \equiv 0 \pmod 3$, $v \geq 9$, $\underline{\chi}'(v,2) = v - 1$;

(ii) For $v \equiv 1 \pmod 3$, $\underline{\chi}'(v,2) = v$.

To complete the picture in the above two theorems, $\underline{\chi}'(7) = 7, \underline{\chi}'(6,2) = 10$ (any two blocks of the unique STS(7) and unique TTS(6) intersect) and $\underline{\chi}'(13) = 8$ [39].

Determination of the exact upper bound seems much more difficult. Colbourn & Colbourn [12] proved that $\overline{\chi'}(v) \leq 3v/2$ and if CS is a cyclic STS(v) then $\chi'(CS) \leq v$. But these bounds may be weak. No STS(v), S, is known for which $\chi'(S) > \underline{\chi}'(v)+2$. We observe that an STS(v), S, without a parallel class of $v/3$ blocks in the case $v \equiv 3 \pmod 6$ and without an almost parallel class of $(v-1)/3$ blocks in the case of $v \equiv 1 \pmod 6$, $v \neq 7$, has $\chi'(S) \geq \underline{\chi}'(v) + 2$. But knowledge of such systems is very fragmentary; examples are known only for $v = 15$ [39], 19 [38] and 21 [40]. Nevertheless it is conjectured that they exist for all $v \equiv 1$ or $3 \pmod 6$, $v \geq 15$. Further information on the chromatic index can be found in the survey by Rosa & Colbourn [46].

Some work has also been done for the configuration $C = A_1$, a pair of parallel lines. Danziger, Grannell, Griggs & Rosa [14] call the resulting chromatic index the *2-parallel chromatic index* and denote it by $\chi''(T)$. In this case the colour classes consist of sets of pairwise intersecting lines. It follows that in a Steiner triple system they consist of either n-stars or any subconfiguration of the unique STS(7), *Fano subconfigurations*. Colourings using only the former, *star-colourings*, are related to the concept of an *independent set*, which is defined to be a subset $I \subset V$ in an STS(v), $S = (V, B)$ such that no block of B is contained in I. Sauer & Schonheim [47] proved that the cardinality of a maximum independent set is given by

$$|I| = \begin{cases} (v-1)/2 \text{ if } v \equiv 1 \text{ or } 9 \pmod{12}, \\ (v+1)/2 \text{ if } v \equiv 3 \text{ or } 7 \pmod{12}. \end{cases}$$

A star colouring is obtained by firstly assigning a distinct colour to each point of the set $V \setminus I$. Then each block can be assigned the colour of the point from $V \setminus I$ (or any one of the points if there are more than one) with which it is incident. Since I is a maximum independent set, this is the optimum colouring using only n-stars. It remains to consider the possible effect of being able to use Fano subconfiguration colour classes and this is the major part of [14]. Using the fact that these can contain at most 7 blocks, careful analysis reveals that for $v \geq 27$ no colouring with less colours than are used in the star-colouring can be achieved. The remaining cases are dealt with individually. The definitive result is as follows.

Theorem 5.4 (Danziger, Grannell, Griggs & Rosa [14])

(i) $\underline{\chi''}(3) = \underline{\chi''}(7) = 1$; $\underline{\chi''}(9) = 3$; $\underline{\chi''}(13) = \underline{\chi''}(15) = 6$; $\underline{\chi''}(19) = 8$; $\underline{\chi''}(21) = 9$; $\underline{\chi''}(25) = 12$.

(ii) $\underline{\chi''}(v) = (v+1)/2, v \equiv 1$ *or* 9 *(mod* 12*)*, $v \geq 33$.

(iii) $\underline{\chi''}(v) = (v-1)/2, v \equiv 3$ *or* 7 *(mod* 12*)*, $v \geq 27$.

Again results concerning the upper bound seem more difficult to obtain and the best that Danziger et alia were able to show was that $\overline{\chi''}(v) \leq v - c\sqrt{v \log v}$ for some absolute constant c. This is based on a result of Phelps and Rodl [43] on the cardinality of an independent set in an STS(v).

Much remains to be done in this area. For example it may be possible to extend the above work to twofold triple systems. The present authors have some partial results concerning the 3-line configurations of which the most interesting is B_5, the triangle. We hope to make these the subject of further papers. But it would also be of interest to obtain results for other configurations, particularly the Pasch configuration. Four questions of interest are the following.

(1) Do there exist triple systems $TS(v, \lambda)$ with $\lambda > 1$ which are anti-Pasch, other than those obtained by assembling λ copies of an anti-Pasch STS(v)? In particular, consider the case of $\lambda = 2$ and the possible existence of anti-Pasch $TS(v, 2)$ without repeated blocks.

(2) Related to the above is the question of how many triples of a base set V can be chosen without introducing a Pasch configuration. To our knowledge no investigation of either of these two questions has been undertaken.

(3) If S is a Steiner triple system what is the range of values which can be taken by $\chi(P, S)$? Anti-Pasch systems have $\chi(P, S) = 1$, the lower bound. What is the upper bound as a function of v, the order of S? The answer to this is clearly related to the next question.

(4) What is the maximum number of Pasch configurations which can occur in a Steiner triple system? Anti-Pasch STS(v)s are extremal systems at one end of the spectrum. What are the systems which lie at the other end of the spectrum?

Denote the number of Pasch configurations in an STS(v), S, by $P(S)$. Define $\overline{P}(v) = \max\{P(S) : S \text{ is an STS}(v)\}$. An STS($v$), S, is then said to be *maxi-Pasch* if $P(S) = \overline{P}(v)$. A preliminary investigation of maxi-Pasch Steiner triple systems was undertaken by Stinson & Wei [49]. An elementary counting argument shows that $\overline{P}(v) \leq v(v-1)(v-3)/24$. Stinson & Wei then go on to show that this bound is achieved only by the projective spaces $PG(n,2)$. They continue by deriving some recursive lower bounds for $\overline{P}(v)$ and end with a computational study of systems containing "large" numbers of quadrilaterals. Yet apart from the trivial case of $v = 9$, and also $v = 13$ where there are precisely two non-isomorphic systems containing respectively 13 and 8 quadrilaterals, no other (non-projective) maxi-Pasch STS(v)s have been positively identified. Even a definitive answer in some small cases would be of interest. It seems that, apart from the orders of projective spaces, it is not possible to come "close" to the theoretical bound. Indeed there is no known STS(v), S, with $v \neq 2^n - 1$ for which $\frac{1}{2}(v(v-1)(v-3))/24 \leq P(S) < v(v-1)(v-3)/24$, and we would not be too surprised if no such system exists. However it is possible to determine the order of the function $\overline{P}(v)$. The Doyen-Wilson theorem [19] states that any STS(w) can be embedded in an STS(v) for all $v \geq 2w+1$ and admissible i.e. $v \equiv 1$ or 3 (mod 6). So for given v, choose w such that $w = 2^{n+1} - 1$ and $(v-\alpha)/4 < w \leq (v-\alpha)/2$ where $\alpha = 1$ when $v \equiv 3$ or 7 (mod 12), $\alpha = 3$ when $v \equiv 9$ (mod 12) and $\alpha = 7$ when $v \equiv 1$ (mod 12). Then there exists an STS(v) containing the projective space $PG(n,2)$ having $w(w-1)(w-3)/24$ quadrilaterals. Thus $\overline{P}(v) = O(v^3)$.

Similar questions may be asked concerning the mitre configuration. Denote the number of mitre configurations in an STS(v), S, by $M(S)$ and define $\overline{M}(v) = \max\{M(S) : S \text{ is an STS}(v)\}$. A *maxi-mitre* STS($v$), S, satisfies $M(S) = \overline{M}(v)$. The following is known.

Theorem 5.5 (Assmus [4])
$\overline{M}(v) \leq v(v-1)(v-3)/12$ *with equality if and only if* $v = 3^n$, *and the system achieving this bound is a Hall triple system.*

A *Hall triple system* is a Steiner triple system in which every triple which is not a line of the system generates a subsystem isomorphic to the affine plane $AG(2,3) =$ STS(9). The class of Hall triple systems contains the class of affine spaces $AG(n,3)$ as a subclass. More information on Hall triple systems can be found in the survey by Beneteau [6]. But apart from Assmus' result this entire area is still open to investigation.

Finally, this survey would not be complete without some mention of ubiquity. This was introduced by Mendelsohn & Rosa [41] as an intermediate concept between the two extremes of decomposition (Section 3) and avoidance (Section 4). A Steiner triple system, S, is said to be *C-ubiquitous* for a configuration C if every line of S is contained in a copy of C. Further S is said to be *n-ubiquitous* if it is C-ubiquitous for every n-line configuration C. For $n = 2$ and $n = 3$ universal results are obtained, and for $n = 4$ an existential one.

Theorem 5.6 (Mendelsohn & Rosa [41])

(i) Every STS(v) except the unique STS(7) is 2-ubiquitous.

(ii) Every STS(v) except the unique STS(7) and the unique STS(9) is 3-ubiquitous.

(iii) There exists a 4-ubiquitous STS(v) for all admissible v except for $v = 3, 7$ and 9.

But 5-ubiquity seems much more difficult to handle. Although most of the 56 5-line configurations cause no problem, Mendelsohn & Rosa identify six which do. They verify that there is no 5-ubiquitous STS(v) for $v \leq 15$ although some systems come "close". Of the 80 STS(15)s, systems ♯19 and ♯61 are C-ubiquitous for all 5-line configurations except the mia, and systems ♯20 and ♯40 for all 5-line configurations except the 5-PPC. However they do show that the cyclic STS(19) with base blocks $\{0, 1, 4\}, \{0, 2, 9\}$, and $\{0, 5, 11\}$ is 5-ubiquitous.

Conjecture 5.1 *There exists a 5-ubiquitous STS(v) for all admissible v except for $v = 3, 7, 9, 13$ and 15.*

This concludes the survey. As we observed in the introduction all four main sections contain ideas for future research. Undoubtedly some of the problems are very difficult whilst others should yield to attack. We have attempted to indicate our opinion of which questions might repay further study. But we could be hopelessly wrong. Our hope is that readers will be sufficiently interested to begin work in this developing area. We wish you success.

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