Cyclic bi-embeddings of Steiner triple systems on 12s+7 points

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Running head: Bi-embeddings of STS(12s+7)s

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Abstract. A cyclic face 2-colourable triangulation of the complete graph K_n in an orientable surface exists for $n \equiv 7 \pmod{12}$. Such a triangulation corresponds to a cyclic bi-embedding of a pair of Steiner triple systems of order n, the triples being defined by the faces in each of the two colour classes. We investigate in the general case the production of such bi-embeddings from solutions to Heffter's first difference problem and appropriately labelled current graphs. For $n = 19$ and $n = 31$ we give a complete explanation for those pairs of Steiner triple systems which do not admit a cyclic bi-embedding and we show how all non-isomorphic solutions may be identified. For $n = 43$ we describe the structures of all possible current graphs and give a more detailed analysis in the case of the Heawood graph.

1 Introduction

In 1967 Ringel and Youngs proved that the complete graph K_n can be embedded in an orientable surface of genus $\lceil (n-3)(n-4)/12 \rceil$ [10]. In the cases where $n \equiv 0, 3, 4$ or 7 (mod 12) the embeddings are triangulations and the faces form a Mendelsohn triple system. A *Mendelsohn triple system of order n*, MTS(n), is a pair (V, $\mathcal{B}$) where V is a set of points of cardinality n and $\mathcal{B}$ is a set of cyclically ordered triples of elements of V which collectively have the property that every ordered pair of elements of V is contained in one triple. (A triple $\langle a, b, c \rangle$ "contains" the pairs $\langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle$.) Such systems exist for $n \equiv 0$ or 1 (mod 3), $n \neq 6$. When $n \equiv 3$ or 7 (mod 12) there is potential for the Mendelsohn triple system to form two Steiner triple systems. A *Steiner triple system of order n*, STS(n), is a pair (V, $\mathcal{B}$) where V is a set of points of cardinality n and $\mathcal{B}$ is a set of triples of elements of V which collectively have the property that every unordered pair of elements of V is contained in one triple. Such systems exist for $n \equiv 1$ or 3 (mod 6).

If the graph K_n is embedded in an orientable surface and every triple of a Steiner triple system is a face of this embedding, then that system is also regarded as being embedded in the surface with these faces being coloured, say, black. If the remaining faces (white) also form a Steiner triple system, we then have a face two-colourable bi-embedding of the two Steiner triple systems.

An embedding of a graph (or design) in an orientable surface may be described by means of a rotation scheme. Given a vertex x of the graph (or a point of the design) the rotation about x comprises the cyclically ordered list of other vertices (points) which are adjacent to x taken in the order in which they appear around x in the embedding. The *rotation scheme* for the embedding is the set of all the vertices together with their rotations taken with a consistent orientation, i.e. all clockwise or all anticlockwise. A rotation scheme is *cyclic* if we can denote the vertices by $0, 1, ..., n-1$ in such a way that the rotation about x is obtained by adding x (mod n) to the rotation about 0. In the case where $n \equiv 3$ (mod 12), a cyclic STS(n) contains a unique short orbit and consequently there can be no cyclic bi-embeddings of such a system. In the case where $n \equiv 0$ or 4 (mod 12) the embedding will not be face two colourable.

In [12] Youngs gives at least one orientable cyclic bi-embedding for each case where $n \equiv 7$ (mod 12) and it is these that concern us in this paper. The cyclic bi-embeddings in the case $n = 19$ are given in [4] and those in the case $n = 31$ are given in [1]. Here we explore the construction of cyclic bi-embeddings in the general case $n = 12s+7$ and show how these are related to Heffter's first difference problem.

2 Cyclic bi-embeddings - the general case

We take as our starting point the result of [10] that every cyclic (or index 1) embedding of K_{12s+7} can be derived from an appropriate *current graph* having $4s + 2$ vertices. In our context, a current graph is a graph with *directions* (clockwise or anticlockwise) assigned at each vertex and whose edges are assigned both a direction (in the ordinary sense of the word) and a *current*, the current being a non-zero element of the group Z_{12s+7}. An example in the case $s = 2$ is shown in Figure 1.

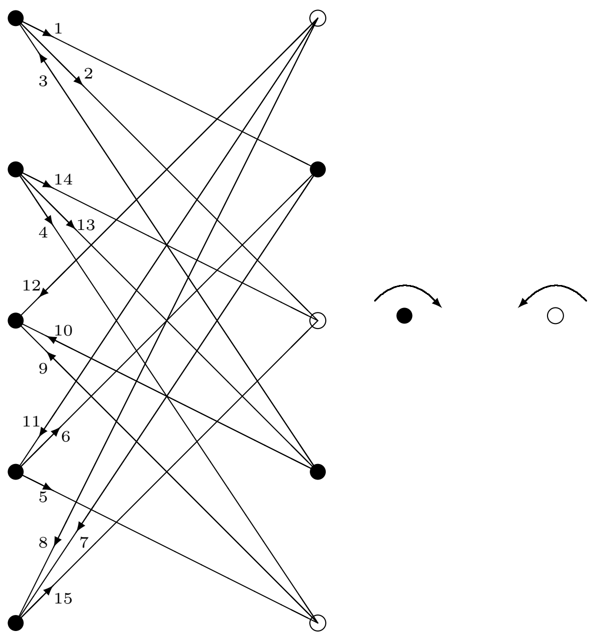

Figure 1: A current graph for $s = 2$.

The rotation about 0 in the resulting embedding of $K_{12s+7} = K_{31}$ is obtained by traversing the graph, recording the (directed) currents encountered on each edge, and taking the clockwise or anticlockwise exit from each edge as indicated at that vertex. Thus we obtain the permutation

1 7 15 29 28 10 22 26 20 8 24 25 5 27 14 16 23 12 21 18 4 9 19 11 6 30 2 17 13 3.

The rotation about x is then obtained by adding x (mod 31) to all the entries in this permutation. A full explanation of current graphs is given in [5]. In the case where we are seeking a bi-embedding of two STS($12s + 7$)s the current graph must have the following properties:

(i) Each vertex has valency 3 (trivalency).

(ii) At each vertex, the sum of the directed currents is zero (mod $12s + 7$) (Kirchoff's current law).

(iii) Each of the elements $1, 2, ..., 6s + 3$ of Z_{12s+7} appears exactly once as a current on one of the edges and each edge has exactly one of these currents.

(iv) The directions (clockwise or anticlockwise) assigned to each vertex are such that a *complete circuit* is formed, i.e. one in which each edge of the graph is traversed in both directions exactly once.

(v) The graph is bipartite.

Conditions (i) and (ii) ensure that the embedding is a triangulation, while conditions (iii) and (iv) ensure that the embedding is cyclic. See [10] and [5] for further details. Condition (v) ensures that the embedding is face two-colourable and therefore represents a bi-embedding of two STS($12s + 7$)s. Consideration of the valency and the currents shows that our current graphs have $4s + 2$ vertices. Furthermore, there can be no loops and, save for the exceptional case $s = 0$, there can be no multiple edges. This latter consideration stems from consideration of the configuration shown in Figure 2.

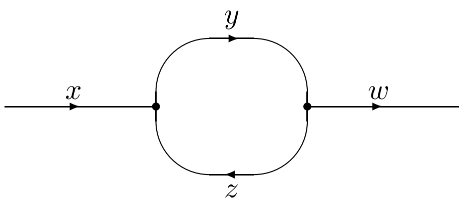

Figure 2: A possible multiple edge.

If this forms part of a current graph then $w \equiv x$ and so the whole current graph comprises two vertices with a triply repeated edge.

There is a close connection between current graphs and solutions of *Heffter's first difference problem* (HDP). In 1897 Heffter [6] posed the following question: can the integers $1, 2, ..., 3k$ be partitioned into k triples $\{a, b, c\}$ such that, for each triple, $a + b \pm c \equiv 0 \pmod{6k+1}$? Examination of the triples formed by the directed currents at each vertex in either of the two vertex sets of a bipartite current graph shows that they form a solution to HDP for $k = 2s + 1$.

In view of the above observations, the problem of constructing cyclic bi-embeddings of STS($12s + 7$)s ($s > 0$) may be reduced to three steps:

(a) Identifying trivalent bipartite connected simple graphs having $4s + 2$ vertices.

(b) Assigning directions (clockwise or anticlockwise) at each of the vertices which then give rise to a complete circuit.

(c) Taking two solutions of HDP and labelling the edges of the graph in such a way that the triples arising from each of the two vertex sets of the bipartition correspond to these two solutions.

These three steps have a large measure of independence from one another. However, we can not exclude the possibility that for a particular graph it may be impossible to assign vertex directions to give a complete circuit and, even if this is possible, it may not be possible to assign the HDP solutions to the edges. We note that a test for the existence of a complete circuit in a graph G is given by Xuong [11]. This asserts the existence of such a circuit (equivalent to a one-face orientable embedding of G) if and only if G has a spanning tree whose co-tree has no component with an odd number of edges. It is also appropriate at this point to note recent results of Korzhik and Voss

[7] which show that from one cyclic bi-embedding of a pair of STS($12s+7$)s it is possible to produce at least 4^s non-isomorphic cyclic bi-embeddings by varying the vertex directions.

Before proceeding with the strategy outlined in the three steps above, it is appropriate to recall how Steiner triple systems arise from solutions to HDP. Given a difference triple $\{a, b, c\}$ with $a + b \pm c \equiv 0 \pmod{6k+1}$, we may form a cyclic orbit by developing the starter $\{0, a, a+b\}$ or the starter $\{0, b, a+b\}$. Taking the union of such orbits corresponding to a solution of HDP yields a cyclic STS($6k+1$). The converse is also true, given a cyclic STS($6k+1$) we may obtain a solution to HDP by taking from each orbit a block $\{0, \alpha, \beta\}$ and forming the triple $\{\hat{\alpha}, \widehat{\beta-\alpha}, \hat{\beta}\}$ where

$$\hat{x} = \begin{cases} x & \text{if} \quad 1 \le x \le 3k \\ 6k+1-x & \text{if} \quad 3k+1 \le x \le 6k \end{cases}$$

Each solution to HDP produces 2^k different STS($6k+1$)s; however, there may be isomorphisms between these systems. In addition, for a given value of k, there will generally be many distinct solutions to HDP. For example, in [2] it is shown that for $k = 3$ there are four solutions to HDP producing 4×2^3 distinct STS(19)s which lie in four isomorphism classes.

Two solutions to HDP are said to be *multiplier equivalent* if one set of triples may be obtained from the other by first multiplying by a constant factor (mod $6k+1$) and then reducing any residue x in the range $3k+1 \le x \le 6k$ to $6k+1-x$. Clearly, two solutions to HDP which are multiplier equivalent will produce isomorphic sets of Steiner triple systems, the isomorphism being carried by the multiplier. Since it is known that, for certain values n, there exist cyclic STS(n)s which are isomorphic but where no isomorphism is carried by a multiplier (see [8]), we shall be careful to avoid the converse implication in the general case. However, for prime values of n it is known (see [8]) that isomorphic cyclic STS(n)s are multiplier equivalent. Consequently, for such values of n, two solutions to HDP which are <u>not</u> multiplier equivalent will give rise to non-isomorphic sets of Steiner triple systems. With this in mind, we define a *Heffter class* to be an equivalence class of solutions of HDP under all possible multipliers. The cases which we examine in detail, namely $n = 19$, 31 and 43, are all primes.

3 The case $n = 19$

In this case $s = 1$ and the current graph has $4s + 2 = 6$ vertices. The only bipartite trivalent simple graph on 6 vertices is $K_{3,3}$ (the complete bipartite graph with three vertices in each vertex set). In attempting to add directions (clockwise or anticlockwise) at each vertex in order to form a complete circuit we may fix arbitrarily the direction at one vertex. We find that there are 12 ways to add the remaining directions to give a complete circuit and these are shown in Figure 3.

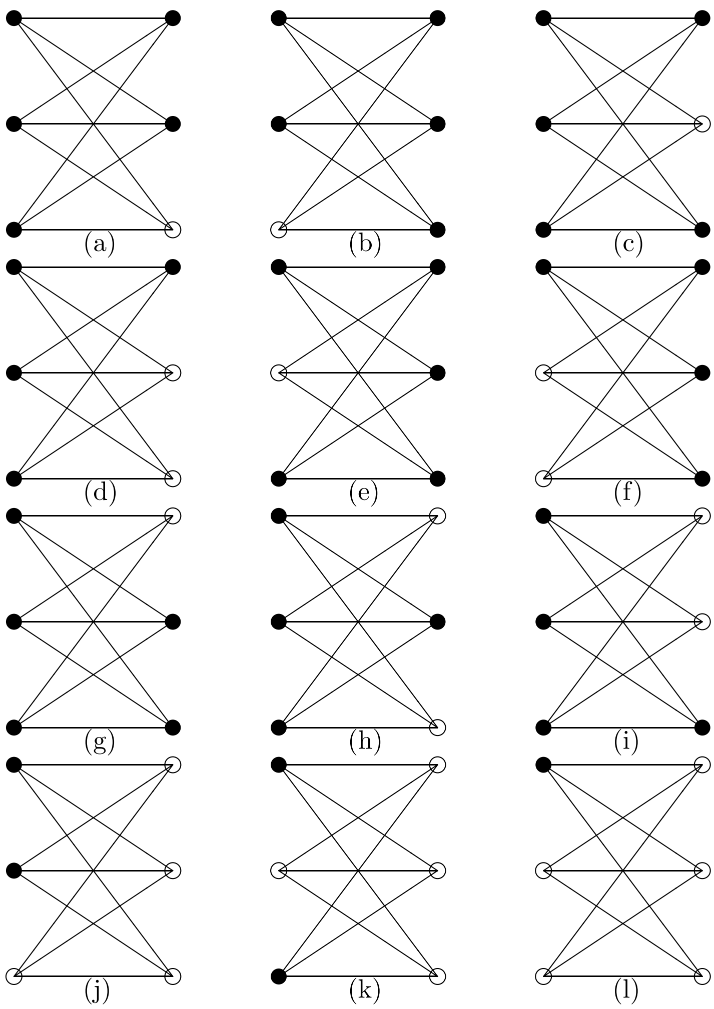

Figure 3: Directions at the vertices of $K_{3,3}$ that give a complete circuit.

There are four solutions to Heffter's first difference problem for $k = 3$, [2], and these are given below.

$$\begin{array}{lcl} \text{D1} &=& \{\{1, 3, 4\},\ \ \{2, 7, 9\},\ \ \{5, 6, 8\}\} \\ \text{D2} &=& \{\{1, 4, 5\},\ \ \{2, 6, 8\},\ \ \{3, 7, 9\}\} \\ \text{D3} &=& \{\{1, 5, 6\},\ \ \{2, 8, 9\},\ \ \{3, 4, 7\}\} \\ \text{D4} &=& \{\{1, 7, 8\},\ \ \{2, 3, 5\},\ \ \{4, 6, 9\}\} \end{array}$$

These four solutions of HDP fall into two Heffter classes. D4 is invariant under all multipliers modulo 19. However,

$4\times$D2 (mod 19) gives $\{\ \{1, 3, 4\}, \{5, 6, 8\}, \{2, 7, 9\}\}$, and
$2\times$D3 (mod 19) gives $\{\ \{2, 7, 9\}, \{1, 3, 4\}, \{5, 6, 8\}\}$.

Thus D1, D2 and D3 are multiplier equivalent. We may take as our representatives of the Heffter classes the following two sets of difference triples:

$$\begin{array}{lcl} \text{A} &=& \{\ \{1, 3, 4\},\ \ \{2, 7, 9\},\ \ \{5, 6, 8\}\} \\ \text{B} &=& \{\ \{1, 7, 8\},\ \ \{2, 3, 5\},\ \ \{4, 6, 9\}\} \end{array}$$

It can be shown that, up to isomorphism, there is only one way in which any two of the four solutions of HDP may be imposed upon $K_{3,3}$. To see this, consider first an attempt to impose B on one of the two vertex sets. It is immediately clear that D4 (= B) cannot be applied to the other vertex set as this would imply the existence of the configuration shown in Figure 4.

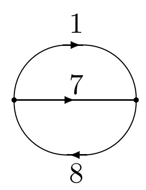

Figure 4: Illegal configuration.

Thus we see that B can only be combined with one of D1, D2 or D3. By considering multiplier equivalence we may assume that any solution involving B also involves A (= D1). Further elementary arguments lead to the solution shown in Figure 5 (together with the solution obtained by reversing all the currents).

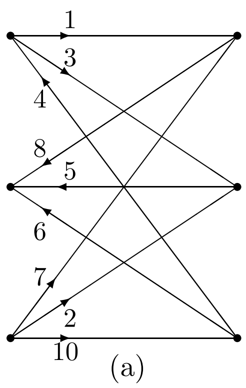

Figure 5: A with B.

Next consider an attempt to impose A on one of the two vertex sets with a copy of A on the other. Since A (= D1) contains the triple $\{1, 3, 4\}$ and D2 contains the triple $\{1, 4, 5\}$ and D3 contains the triple $\{3, 4, 7\}$ it is immediately clear that A cannot be combined with D2 or D3, or indeed itself.

Combining the twelve possible allocations of vertex directions (clockwise or anticlockwise) with the unique allocation of HDP solutions gives twelve bi-embeddings of cyclic STS(19)s. We find that these lie in eight isomorphism classes corresponding to Figure 3 (a), (b), (c), (d), (f), (g), (h) and (j). This result is in agreement with the results of [4].

The eight non-isomorphic rotations about zero generated by the graphs are:

(a)	1	12	2	5	13	9	7	8	14	16	15	6	11	18	3	17	10	4
(b)	1	12	10	6	11	18	3	17	7	8	14	16	15	9	2	5	13	4
(c)	1	12	2	16	15	9	7	8	14	17	10	6	11	18	3	5	13	4
(d)	1	12	2	16	15	6	11	18	3	5	13	9	7	8	14	17	10	4
(f)	1	12	10	6	14	16	15	9	2	5	11	18	3	17	7	8	13	4
(g)	1	8	14	16	15	9	7	18	3	17	10	6	11	12	2	5	13	4
(h)	1	8	14	16	15	6	11	12	2	5	13	9	7	18	3	17	10	4
(j)	1	8	14	17	10	6	11	12	2	16	15	9	7	18	3	5	13	4

In the eight corresponding cyclic bi-embeddings we only have Steiner triple systems from Heffter class A bi-embedded with Steiner triple systems from Heffter class B. Neither Heffter class A systems nor Heffter class B systems cyclically bi-embed with themselves. Examining the reasons for this we can state a general theorem.

Theorem 1 *Suppose S_1 and S_2 are sets of triples representing solutions to HDP for a given value of $k = 2s + 1$ ($s > 0$). If any triple of S_1 has two (or three) elements in common with any triple of S_2, then no Steiner triple system obtained from S_1 may be cyclically bi-embedded with any Steiner triple system from S_2.*

Proof. The two (or three) common elements can only be accommodated by multiple edges in the current graph, but these are forbidden. □

Corollary 1.1 *If C_1 and C_2 are Heffter classes for a given value of $k = 2s + 1$ ($s > 0$), and every pair (S_1, S_2) $\in C_1 \times C_2$ satisfies the conditions of Theorem 1 then no Steiner triple system obtained from C_1 may be cyclically bi-embedded with any Steiner triple system obtained from C_2. In particular, if C is a Heffter class containing a single solution to HDP (which is equivalent to saying that this solution is invariant under all multipliers modulo $6k+1$), then no pair of Steiner triple systems obtained from C may be cyclically bi-embedded together.* □

Remark If $n = 12s + 7$ is not prime then it is possible for isomorphic copies of a given STS(n) to correspond to different Heffter classes. In such a circumstance, although it remains true that no realisation of a system obtained from C_1 may be cyclically bi-embedded with any realisation obtained from C_2, there is still the possibility that a system obtained from C_1 may be cyclically bi-embedded with an isomorphic copy of a realisation obtained from C_2. If $n = 12s + 7$ is prime, this complication cannot arise and Corollary 1.1 can be used to assert that no Steiner triple system from C_1 may be cyclically biembedded with any isomorphic copy of a Steiner triple system obtained from C_2.

4 The case $n = 31$

In this case $s = 2$ and the current graph has $4s + 2 = 10$ vertices. There are two non-isomorphic connected bipartite trivalent simple graphs on ten vertices. These may be obtained from $K_{5,5}$ by removing either a 10-cycle, or a 6-cycle plus a disjoint 4-cycle. The two graphs are shown in Figure 6 which also specifies a set of vertex directions which in each case results in a complete circuit.

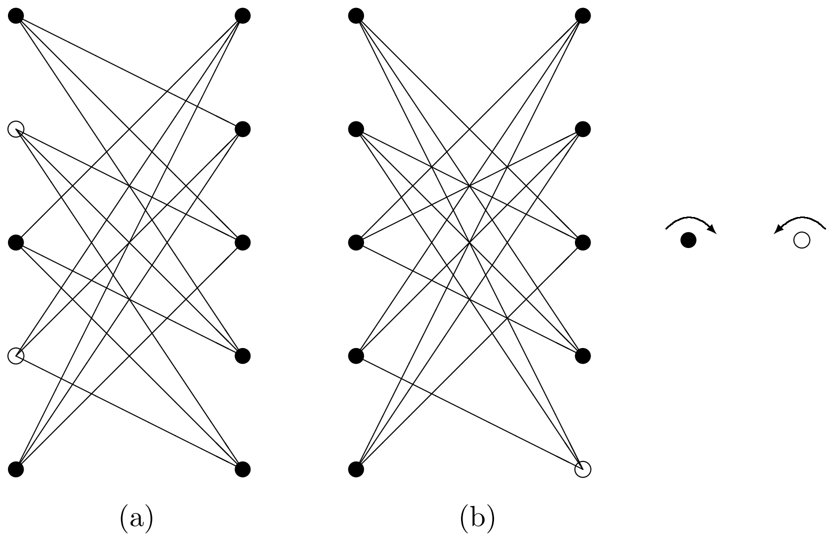

Figure 6: Current graphs for $n = 31$.

In each case fixing the direction at one vertex, computer analysis gives a total of 160 sets of vertex directions in case (a) and a total of 128 sets of vertex directions in case (b) which result in complete circuits.

There are eight Heffter difference classes corresponding to $k = 5$ (i.e. $n = 31$). These may be obtained from the complete listing of solutions of HDP given in [2] and are listed below:

A	=	{{1, 2, 3},	{4, 7, 11},	{5, 12, 14},	{6, 9, 15},	{8, 10, 13}}
B	=	{{1, 2, 3},	{4, 8, 12},	{5, 9, 14},	{6, 10, 15},	{7, 11, 13}}
C	=	{{1, 3, 4},	{2, 8, 10},	{5, 12, 14},	{6, 9, 15},	{7, 11, 13}}
D	=	{{1, 2, 3},	{4, 7, 11},	{5, 10, 15},	{6, 12, 13},	{8, 9, 14} }
E	=	{{1, 3, 4},	{2, 10, 12},	{5, 11, 15},	{6, 7, 13},	{8, 9, 14} }
F	=	{{1, 5, 6},	{2, 7, 9},	{3, 13, 15},	{4, 10, 14},	{8, 11, 12}}
G	=	{{1, 5, 6},	{2, 10, 12},	{3, 13, 15},	{4, 7, 11},	{8, 9, 14} }
H	=	{{1, 11, 12},	{2, 7, 9},	{3, 5, 8},	{4, 13, 14},	{6, 10, 15}}

The number of ways of imposing HDP solutions on the graph of Figure 6(a) was found as follows:

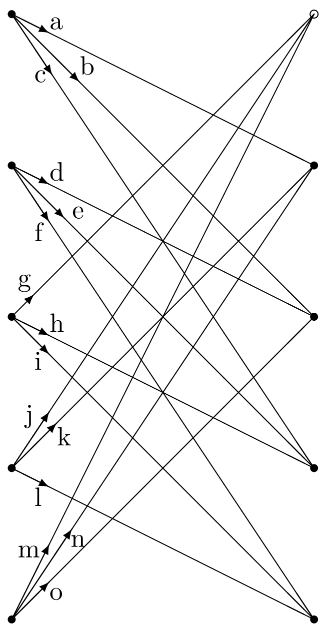

Figure 7: Assigning currents to the graph of Figure 6(a).

The currents at all the vertices obey Kirchoff's current law and so we obtain

$$\begin{array}{rr} a+b+c\equiv 0 & g+j+m\equiv 0 \\ d+e+f\equiv 0 & a+k+n\equiv 0 \\ g+h+i\equiv 0 & b+d+o\equiv 0 \\ j+k+l\equiv 0 & c+e+h\equiv 0 \\ m+n+o\equiv 0 & f+i+l\equiv 0 \end{array}$$

where all the congruencies are modulo 31.

If we select $\{a,b,c\}$ to correspond to a triple from a representative of a Heffter class then these congruencies give nine equations in twelve unknowns. The symmetry of the graph of Figure 6(a) (obtained from $K_{5,5}$ by deletion of a 10-cycle) is such that we may, without loss of generality, assume that a,b and c (in that order) correspond to the first triple listed against one of the eight Heffter classes A to H shown above. We find that there are 32 solution sets for $a,b,...,o$. In consequence we obtain $160 \times 32 = 5120$ rotation schemes from the graph of Figure 6(a). Analysis of these shows that they lie in precisely 1760 isomorphism classes.

In the case of Figure 6(b), the graph is less symmetric. There are two classes of vertex (corresponding to the 6- and 4-cycles deleted from $K_{5,5}$) and there are six possible orders in which the three elements of a difference triple may be selected to correspond to a,b and c. Taking account of these aspects we find 120 solution sets for the systems of equations relating to Figure 6(b). In consequence we obtain $128 \times 120 = 15360$ rotation schemes from the graph of Figure 6(b). Analysis of these shows that they lie in precisely 648 isomorphism classes.

Combining our results for Figures 6(a) and 6(b) gives at most $1760 + 648 = 2408$ isomorphism classes for cyclic bi-embeddings of the STS(31)s. In general, the automorphism group of a bi-embedding of two STS(n)s can have order at most $n(n-1)$ (or $2n(n-1)$ if we permit a reversal of the orientation). It follows that, for prime values of n, the automorphism group of such a bi-embedding has at most one cyclic subgroup of order n. Consequently, given a cyclic bi-embedding of two STS(31)s, the structure of the associated current graph may be uniquely determined from the embedding. Since the graphs of Figure 6(a) and 6(b) are non-isomorphic it follows that none of the 1760 bi-embeddings corresponding to Figure 6(a) can be isomorphic to any of the 648 bi-embeddings corresponding to Figure 6(b). We may therefore assert that there are precisely 2408 isomorphism classes for cyclic bi-embeddings of STS(31)s. This result is confirmed by [1]. In addition we are now able to answer the question raised in that paper concerning the reasons why certain STS(31)s cannot be bi-embedded.

We will say that two Heffter classes C_1 and C_2 *bi-embed* if there exists a cyclic bi-embedding of two STS(n)s, one system, S_1, obtained from C_1 and the other system, S_2, obtained from C_2. From the results of [1] we know that, for $n = 31$, the Heffter difference classes bi-embed as in the following table:

A	bi-embeds with	A	C			
B	bi-embeds with	B	C			
C	bi-embeds with	A	B	C	D	F
D	bi-embeds with	C	D	E	F	
E	bi-embeds with	D	E	F	H	
F	bi-embeds with	C	D	E	F	
G	bi-embeds with	none				
H	bi-embeds with	E				

Table 1: Bi-embeddings of Heffter classes for $n = 31$.

The following pairs of Heffter classes satisfy the conditions of Corollary 1.1 and therefore do not bi-embed:

(A, G)	(A, H)	(B, E)	(B, G)	(B, H)	(C, E)	(C, G)	(C, H)
(D, G)	(D, H)	(E, G)	(F, G)	(F, H)	(G, G)	(G, H)	(H, H)

In particular, class G is invariant under all multipliers. However, Corollary 1.1 does not, of itself, provide a complete explanation for Table 1, for example it does not show that classes B and D do not bi-embed. Similarly Theorem 1 is not sufficient to explain in all cases why a particular pair of cyclic STS(31)s cannot be cyclically bi-embedded. In order to complete the explanation of the computational results of [1], we now obtain a further necessary condition which two solutions to Heffter's difference problem must satisfy if there exists a cyclic bi-embedding of two STS(n)s derived from these two solutions. We will refer to this condition as the "in/out" test and we illustrate it by reference to two representatives of Heffter classes B and D.

Denote these sets of triples by S_1 and S_2 where

S_1 ($= 4\times$B) $= \{\{1, 14, 15\}, \{4, 8, 12\}, \{5, 6, 11\}, \{2, 7, 9\}, \{3, 10, 13\}\}$,
S_2 ($=$ D) $= \{\{1, 2, 3\}, \{4, 7, 11\}, \{5, 10, 15\}, \{6, 12, 13\}, \{8, 9, 14\}\}$.

These two representatives do not generate a cyclic bi-embedding but this is not explained by Theorem 1. If S_1 and S_2 are to be placed at the two vertex sets of a bipartite graph and Kirchoff's current law is to be obeyed at each vertex, then the elements of the first triple of S_1 (which are currents on the edges of the graph) can, without loss of generality, be assigned directions $\{1^o, 14^o, 15^i\}$, where 1^o implies that the current is flowing out (away) from the vertex and 15^i implies that the current is flowing into (towards) the vertex. Having assigned these directions to a triple in S_1, the directions of the currents 1, 14 and 15 in S_2 are determined. Thus we have

$$\begin{aligned} S_1 &= \{\{1^o, 14^o, 15^i\}, \{4, 8, 12\}, \{5, 6, 11\}, \{2, 7, 9\}, \{3, 10, 13\}\}, \\ S_2 &= \{\{1^i, 2, 3\}, \{4, 7, 11\}, \{5, 10, 15^o\}, \{6, 12, 13\}, \{8, 9, 14^i\}\}. \end{aligned}$$

We are now able to assign directions to currents in three of the triples of S_2 depending upon those triples being of the form $a + b + c \equiv 0 \pmod{31}$ or $a + b - c \equiv 0 \pmod{31}$. This gives

$$\begin{aligned} S_1 &= \{\{1^o, 14^o, 15^i\}, \{4, 8, 12\}, \{5, 6, 11\}, \{2, 7, 9\}, \{3, 10, 13\}\}, \\ S_2 &= \{\{1^i, 2^i, 3^o\}, \{4, 7, 11\}, \{5^i, 10^i, 15^o\}, \{6, 12, 13\}, \{8^i, 9^i, 14^i\}\}. \end{aligned}$$

The 2^i in S_2 will determine the directions in the triple of S_1 that contains 2 as $\{2^o, 7^o, 9^i\}$. This leads to a contradiction between the directions of the current 9 in S_1 and S_2. Thus these two representatives of B and D do not generate a cyclic bi-embedding.

In general, two solutions to Heffter's difference problem must produce a consistent allocation of "in/out" directions if a corresponding cyclic bi-embedding is to be achieved. Note also that the graphical structure and edge labelling of the potential current graph are determined by the above process, although the vertex directions (clockwise or anticlockwise) are not.

We find that applying the "in/out" test in addition to Theorem 1 provides a full explanation for those pairs of Heffter classes that do not bi-embed in the case $n = 31$. This is not true in general for larger values of n; see the example for $n = 133$ given below.

To complete the analysis of the $n = 31$ case, consider a pair of Heffter classes which do bi-embed. From Tables 1 and 2 in the Appendix of [1] it can be observed that, within each such pair of classes, there are pairs of STS(31)s that still do not cyclically bi-embed. Consider such a pair of STS(31)s. The corresponding pair of HDP solutions will determine the structure of any potential current graph as described in the previous paragraph. However, the individual STS(31)s will determine the vertex directions (clockwise or anticlockwise) in this graph. In every case, the failure to achieve a cyclic bi-embedding of such a pair of STS(31)s is a consequence of the vertex directions in the associated current graph being incompatible in that they do not yield a complete circuit.

As remarked earlier, for larger values of n, the combination of Theorem 1 and the "in/out" test is generally not sufficient to guarantee the existence of a cyclic bi-embedding. For example, in the case $n = 133$ consider the two sets of Heffter difference triples:

S_1			S_2		
$\{7, 49, 56\}$	$\{14, 21, 35\}$	$\{28, 42, 63\}$	$\{7, 21, 28\}$	$\{14, 49, 63\}$	$\{35, 42, 56\}$
$\{18, 23, 41\}$	$\{10, 27, 37\}$	$\{19, 20, 39\}$	$\{8, 23, 31\}$	$\{10, 19, 29\}$	$\{6, 24, 30\}$
$\{8, 25, 33\}$	$\{15, 16, 31\}$	$\{12, 17, 29\}$	$\{9, 18, 27\}$	$\{4, 22, 26\}$	$\{5, 20, 25\}$
$\{2, 4, 6\}$	$\{22, 44, 66\}$	$\{26, 48, 59\}$	$\{36, 44, 53\}$	$\{38, 40, 55\}$	$\{34, 45, 54\}$
$\{24, 40, 64\}$	$\{30, 46, 57\}$	$\{32, 36, 65\}$	$\{33, 41, 59\}$	$\{15, 51, 66\}$	$\{17, 47, 64\}$
$\{34, 38, 61\}$	$\{11, 43, 54\}$	$\{3, 47, 50\}$	$\{13, 52, 65\}$	$\{16, 46, 62\}$	$\{11, 50, 61\}$
$\{13, 45, 58\}$	$\{1, 51, 52\}$	$\{9, 53, 62\}$	$\{12, 48, 60\}$	$\{1, 2, 3\}$	$\{37, 39, 57\}$
$\{5, 55, 60\}$			$\{32, 43, 58\}$		

It is easy to verify that a cyclic bi-embedding corresponding to the pair (S_1, S_2) is not excluded by either Theorem 1 or the "in/out" test. However, the current graph obtained from the "in/out" test is disconnected (note that the first three triples in each of the above sets consist of the same nine elements) and so cannot admit a complete circuit. Thus no cyclic bi-embedding can result from the pair (S_1, S_2).

5 The case $n = 43$

In this case $s = 3$ and the current graph has $4s + 2 = 14$ vertices. There are thirteen non-isomorphic connected bipartite trivalent simple graphs on fourteen vertices (see [9]). The graphs are shown in Figure 8 which also specifies a set of vertex directions which in each case results in a complete circuit.

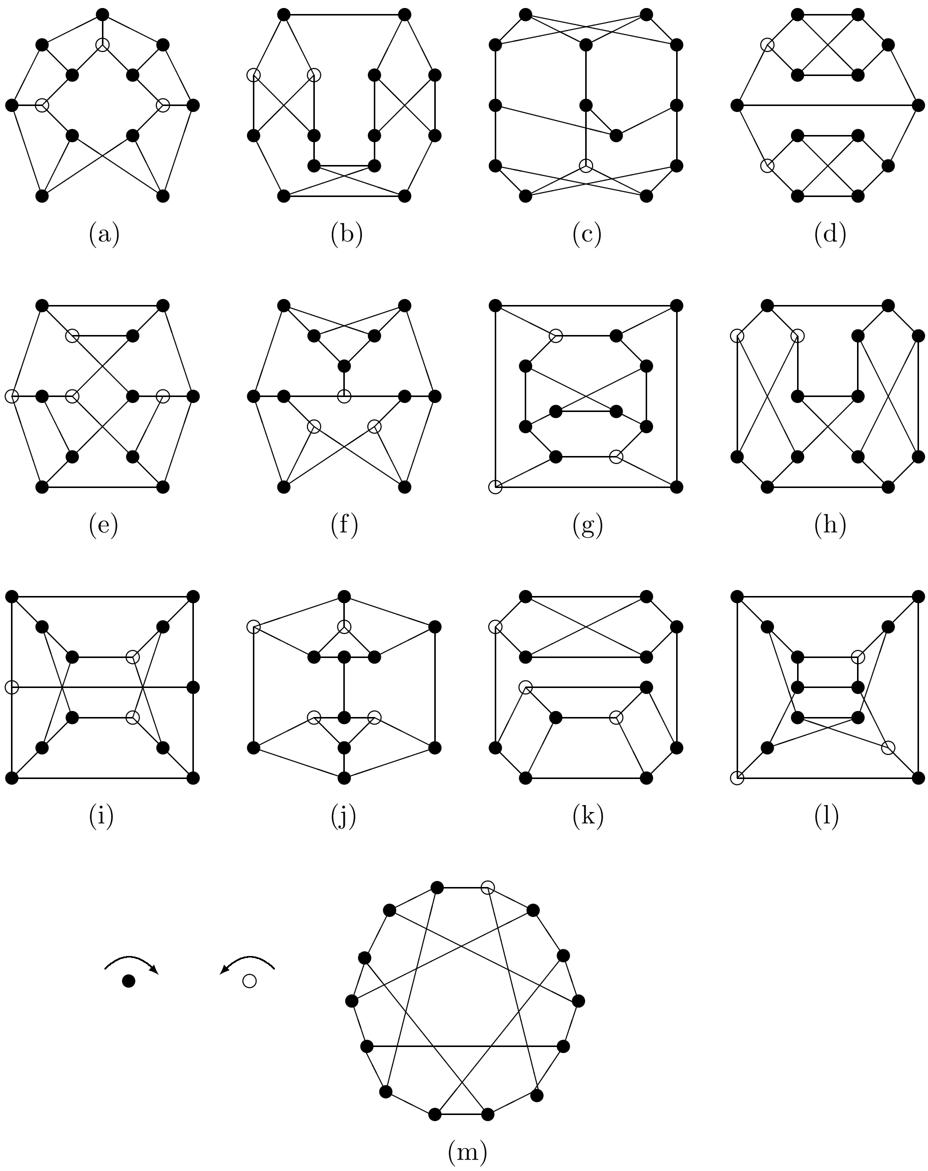

Figure 8: Current graphs for $n = 43$.

For each case (fixing the direction at one vertex), the number of sets of vertex directions which result in complete circuits is given in the following table:

(a)	(b)	(c)	(d)	(e)	(f)	(g)
1792	1280	1408	1152	2048	1664	1920

(h)	(i)	(j)	(k)	(l)	(m)
1984	1920	1728	1152	2178	2240.

There are 159 Heffter difference classes corresponding to $k = 7$ (i.e. $n = 43$) which are listed in the Appendix.

The large number of orientations for each of these graphs along with the potentially large number of ways of imposing the Heffter triples at the vertices renders analysis similar to that in the case $n = 31$ beyond the scope of this paper. However there are a number of observations that can be made:

1. Graphs (d) and (k) are two-edge connected. This implies that they cannot have currents along their edges that are unique as required by property (ii) of Section 2 since the current in one of the two edges of the cutset would have to be equal (but opposite in direction) to the current along the other of these two edges. However, for each of the remaining graphs the edges can be labelled in accordance with (c) in Section 2.

2. Graph (m) is the Heawood graph. Its symmetry reduces the problem of assigning the triples at the vertices in a similar way to case (a) of $n = 31$ as explained in Section 4. In this case there are 14 congruencies in 21 unknowns. Fixing an initial triple yields 13 congruencies in 18 unknowns. Each of the 159 Heffter classes has a representative containing one of the six triples

$$\{1, 2, 3\}, \{1, 3, 4\}, \{1, 4, 5\}, \{1, 5, 6\}, \{1, 6, 7\}, \text{ or } \{1, 9, 10\}.$$

Computer analysis then gives the following results:

Initial triple of the Heffter class	Number of solutions produced by this triple
{1, 2, 3}	72
{1, 3, 4}	96
{1, 4, 5}	108
{1, 5, 6}	24
{1, 6, 7}	200
{1, 9, 10}	68

The number of pairs of Heffter difference classes (when $n = 43$) that can be bi-embedded using the Heawood graph is sixteen and these pairs are listed below, the number of each Heffter difference class corresponding to that given in the Appendix.

$(1, 1)$	$(1, 158)$	$(6, 109)$	$(30, 31)$	$(40, 40)$	$(40, 155)$
$(61, 126)$	$(68, 138)$	$(71, 114)$	$(83, 111)$	$(114, 151)$	$(114, 152)$
$(125, 127)$	$(151, 152)$	$(151, 155)$	$(152, 158)$		

3. If $n = 12s + 7$ is a prime and ω is a primitive root of unity, then with $t = 2s + 1$, the set of triples

$$\{\{\omega^{2i} + j, \omega^{2t+2i} + j, \omega^{4t+2i} + j\} : 0 \leq i < t,\ 0 \leq j \leq n - 1\}$$

forms the blocks of a cyclic Steiner triple system on n points. Such systems are known as *Netto systems* (see, for example, [3]). It is known that, in the case $n = 19$, the Netto system does cyclically bi-embed with other systems. In fact the Netto system for $n = 19$ is generated by the Heffter class B given in Section 3 and cyclic bi-embeddings of this Netto system are given by rotations (d) and (g) of that Section. In the case $n = 31$ the Netto system is generated by the Heffter class G given in Section 4 and there are no cyclic bi-embeddings of this system. In the case $n = 43$ the Netto system corresponds to Heffter class number 157 in the Appendix. This class is invariant under all multipliers (mod 43) and thus this system will not cyclically bi-embed with itself. Whether this system cyclically bi-embeds with another system was determined as follows.

Each graph in Figure 8, apart from (d) and (k), was considered separately. Graphs (a), (b), (e), (g), (h), (i), (j) and (l) are symmetric with respect to the bipartition while (c) and (f) are not. In the symmetric cases a single vertex was arbitrarily chosen from one of the two sets of the bipartition and in the unsymmetric cases a single vertex was arbitrarily chosen from each of the two sets of the bipartition. In turn, each difference triple from Heffter class 157 was imposed on the three edges incident with the chosen vertex in each of the six possible permutations. The resulting congruencies were solved in each case, as outlined above. Only in the case of graph (c) did the solutions corresponding to one of the vertex sets of the bipartition give the complete set of difference triples from Heffter class 157. In every case, these solutions admitted the Netto system when appropriate directions that give a complete circuit were imposed on the vertices of the graph. The corresponding Steiner triple systems given by the other vertex set of the bipartition belong to Heffter class 4, 12, 69 or 111. A sample cyclic bi-embedding of the Netto system of order 43 is given by the following permutation which specifies the rotation about 0.

1 37 12 41 40 18 6 7 24 28 20 11 33 16 29 31 25 22 32 9 5

35 15 39 34 23 8 13 27 17 36 42 2 14 30 38 4 19 26 10 21 3

The other Heffter class involved in this particular bi-embedding is number 4.

Acknowledgement: We thank Professor Josef Širáň of the Slovak Technical University, Bratislava, for his helpful comments and for pointing out reference [11].

Appendix: Heffter difference classes for $n = 43$.

1:	1	2	3	4	5	9	6	12	18	7	14	21	8	15	20	10	16	17	11	13	19
2:	1	2	3	4	5	9	6	12	18	7	14	21	8	16	19	10	13	20	11	15	17
3:	1	2	3	4	5	9	6	12	18	7	16	20	8	13	21	10	14	19	11	15	17
4:	1	2	3	4	5	9	6	12	18	7	17	19	8	15	20	10	11	21	13	14	16
5:	1	2	3	4	5	9	6	13	19	7	15	21	8	12	20	10	16	17	11	14	18
6:	1	2	3	4	5	9	6	14	20	7	15	21	8	11	19	10	16	17	12	13	18
7:	1	2	3	4	5	9	6	15	21	7	11	18	8	16	19	10	13	20	12	14	17
8:	1	2	3	4	5	9	6	15	21	7	16	20	8	10	18	11	13	19	12	14	17
9:	1	2	3	4	5	9	6	15	21	7	17	19	8	10	18	11	12	20	13	14	16
10:	1	2	3	4	5	9	6	16	21	7	13	20	8	11	19	10	15	18	12	14	17
11:	1	2	3	4	5	9	6	17	20	7	11	18	8	13	21	10	14	19	12	15	16
12:	1	2	3	4	5	9	6	17	20	7	14	21	8	10	18	11	13	19	12	15	16
13:	1	2	3	4	6	10	5	12	17	7	16	20	8	13	21	9	15	19	11	14	18
14:	1	2	3	4	6	10	5	13	18	7	15	21	8	16	19	9	11	20	12	14	17
15:	1	2	3	4	6	10	5	13	18	7	17	19	8	14	21	9	11	20	12	15	16
16:	1	2	3	4	6	10	5	15	20	7	11	18	8	16	19	9	13	21	12	14	17
17:	1	2	3	4	6	10	5	17	21	7	11	18	8	12	20	9	15	19	13	14	16
18:	1	2	3	4	6	10	5	18	20	7	14	21	8	9	17	11	13	19	12	15	16
19:	1	2	3	4	7	11	5	12	17	6	14	20	8	16	19	9	13	21	10	15	18
20:	1	2	3	4	7	11	5	13	18	6	16	21	8	15	20	9	10	19	12	14	17
21:	1	2	3	4	7	11	5	13	18	6	17	20	8	14	21	9	10	19	12	15	16
22:	1	2	3	4	7	11	5	14	19	6	12	18	8	15	20	9	13	21	10	16	17
23:	1	2	3	4	7	11	5	14	19	6	17	20	8	10	18	9	13	21	12	15	16
24:	1	2	3	4	7	11	5	15	20	6	16	21	8	9	17	10	14	19	12	13	18
25:	1	2	3	4	7	11	5	15	20	6	18	19	8	9	17	10	12	21	13	14	16
26:	1	2	3	4	7	11	5	17	21	6	12	18	8	15	20	9	10	19	13	14	16
27:	1	2	3	4	7	11	5	17	21	6	13	19	8	10	18	9	14	20	12	15	16
28:	1	2	3	4	7	11	5	18	20	6	10	16	8	13	21	9	15	19	12	14	17
29:	1	2	3	4	8	12	5	10	15	6	17	20	7	14	21	9	16	18	11	13	19
30:	1	2	3	4	8	12	5	11	16	6	14	20	7	17	19	9	13	21	10	15	18
31:	1	2	3	4	8	12	5	15	20	6	10	16	7	17	19	9	13	21	11	14	18
32:	1	2	3	4	8	12	5	16	21	6	9	15	7	17	19	10	13	20	11	14	18
33:	1	2	3	4	8	12	5	17	21	6	10	16	7	13	20	9	15	19	11	14	18
34:	1	2	3	4	8	12	5	17	21	6	14	20	7	9	16	10	15	18	11	13	19
35:	1	2	3	4	8	12	5	18	20	6	11	17	7	15	21	9	10	19	13	14	16
36:	1	2	3	4	9	13	5	11	16	6	17	20	7	12	19	8	14	21	10	15	18
37:	1	2	3	4	9	13	5	12	17	6	16	21	7	11	18	8	15	20	10	14	19
38:	1	2	3	4	9	13	5	14	19	6	10	16	7	15	21	8	17	18	11	12	20
39:	1	2	3	4	9	13	5	14	19	6	15	21	7	11	18	8	12	20	10	16	17
40:	1	2	3	4	9	13	5	15	20	6	12	18	7	14	21	8	11	19	10	16	17
41:	1	2	3	4	9	13	5	16	21	6	8	14	7	17	19	10	15	18	11	12	20
42:	1	2	3	4	9	13	5	18	20	6	10	16	7	12	19	8	14	21	11	15	17
43:	1	2	3	4	9	13	5	18	20	6	10	16	7	15	21	8	11	19	12	14	17
44:	1	2	3	4	10	14	5	8	13	6	15	21	7	17	19	9	16	18	11	12	20
45:	1	2	3	4	10	14	5	8	13	6	18	19	7	16	20	9	12	21	11	15	17
46:	1	2	3	4	10	14	5	15	20	6	11	17	7	12	19	8	13	21	9	16	18
47:	1	2	3	4	10	14	5	16	21	6	7	13	8	17	18	9	15	19	11	12	20
48:	1	2	3	4	10	14	5	17	21	6	9	15	7	16	20	8	11	19	12	13	18
49:	1	2	3	4	10	14	5	17	21	6	13	19	7	8	15	9	16	18	11	12	20
50:	1	2	3	4	10	14	5	18	20	6	7	13	8	16	19	9	12	21	11	15	17
51:	1	2	3	4	11	15	5	7	12	6	18	19	8	13	21	9	14	20	10	16	17
52:	1	2	3	4	11	15	5	8	13	6	14	20	7	17	19	9	16	18	10	12	21
53:	1	2	3	4	11	15	5	9	14	6	13	19	7	16	20	8	17	18	10	12	21
54:	1	2	3	4	11	15	5	12	17	6	18	19	7	9	16	8	14	21	10	13	20
55:	1	2	3	4	11	15	5	13	18	6	14	20	7	10	17	8	16	19	9	12	21
56:	1	2	3	4	11	15	5	14	19	6	10	16	7	13	20	8	17	18	9	12	21
57:	1	2	3	4	11	15	5	16	21	6	12	18	7	13	20	8	9	17	10	14	19
58:	1	2	3	4	11	15	5	17	21	6	8	14	7	16	20	9	10	19	12	13	18
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60:	1	2	3	4	12	16	5	6	11	7	17	19	8	13	21	9	14	20	10	15	18
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73:	1	2	3	4	14	18	5	8	13	6	11	17	7	16	20	9	15	19	10	12	21
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75:	1	2	3	4	14	18	5	8	13	6	16	21	7	10	17	9	15	19	11	12	20
76:	1	2	3	4	14	18	5	15	20	6	7	13	8	11	19	9	12	21	10	16	17
77:	1	2	3	4	14	18	5	16	21	6	7	13	8	12	20	9	10	19	11	15	17

78:	1	2	3	4	14	18	5	16	21	6	9	15	7	10	17	8	12	20	11	13	19
79:	1	2	3	4	15	19	5	6	11	7	16	20	8	10	18	9	13	21	12	14	17
80:	1	2	3	4	15	19	5	8	13	6	12	18	7	14	21	9	11	20	10	16	17
81:	1	2	3	4	15	19	5	8	13	6	17	20	7	9	16	10	12	21	11	14	18
82:	1	2	3	4	15	19	5	11	16	6	12	18	7	10	17	8	13	21	9	14	20
83:	1	2	3	4	15	19	5	12	17	6	10	16	7	11	18	8	13	21	9	14	20
84:	1	2	3	4	15	19	5	16	21	6	7	13	8	10	18	9	11	20	12	14	17
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86:	1	2	3	4	16	20	5	6	11	7	10	17	8	14	21	9	15	19	12	13	18
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88:	1	2	3	4	16	20	5	8	13	6	9	15	7	17	19	10	12	21	11	14	18
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91:	1	2	3	4	16	20	5	13	18	6	11	17	7	8	15	9	12	21	10	14	19
92:	1	2	3	4	17	21	5	6	11	7	9	16	8	15	20	10	14	19	12	13	18
93:	1	2	3	4	17	21	5	9	14	6	7	13	8	16	19	10	15	18	11	12	20
94:	1	2	3	4	18	21	5	6	11	7	10	17	8	12	20	9	15	19	13	14	16
95:	1	2	3	4	18	21	5	6	11	7	13	20	8	9	17	10	14	19	12	15	16
96:	1	2	3	4	18	21	5	7	12	6	11	17	8	15	20	9	10	19	13	14	16
97:	1	2	3	4	18	21	5	9	14	6	13	19	7	8	15	10	16	17	11	12	20
98:	1	2	3	4	18	21	5	12	17	6	10	16	7	8	15	9	14	20	11	13	19
99:	1	2	3	4	19	20	5	6	11	7	9	16	8	13	21	10	15	18	12	14	17
100:	1	2	3	4	19	20	5	7	12	6	9	15	8	13	21	10	16	17	11	14	18
101:	1	2	3	4	19	20	5	12	17	6	10	16	7	8	15	9	13	21	11	14	18
102:	1	2	3	4	19	20	5	13	18	6	8	14	7	9	16	10	12	21	11	15	17
103:	1	3	4	2	5	7	6	13	19	8	15	20	9	12	21	10	16	17	11	14	18
104:	1	3	4	2	5	7	6	15	21	8	11	19	9	14	20	10	16	17	12	13	18
105:	1	3	4	2	5	7	6	15	21	8	17	18	9	10	19	11	12	20	13	14	16
106:	1	3	4	2	5	7	6	17	20	8	11	19	9	12	21	10	15	18	13	14	16
107:	1	3	4	2	5	7	6	17	20	8	13	21	9	10	19	11	14	18	12	15	16
108:	1	3	4	2	6	8	5	13	18	7	16	20	9	12	21	10	14	19	11	15	17
109:	1	3	4	2	6	8	5	17	21	7	13	20	9	10	19	11	14	18	12	15	16
110:	1	3	4	2	7	9	5	13	18	6	16	21	8	12	20	10	14	19	11	15	17
111:	1	3	4	2	7	9	5	17	21	6	14	20	8	10	18	11	13	19	12	15	16
112:	1	3	4	2	7	9	5	18	20	6	11	17	8	13	21	10	14	19	12	15	16
113:	1	3	4	2	8	10	5	12	17	6	16	21	7	13	20	9	15	19	11	14	18
114:	1	3	4	2	8	10	5	12	17	6	18	19	7	15	21	9	11	20	13	14	16
115:	1	3	4	2	9	11	5	12	17	6	13	19	7	16	20	8	14	21	10	15	18
116:	1	3	4	2	9	11	5	15	20	6	10	16	7	17	19	8	14	21	12	13	18
117:	1	3	4	2	9	11	5	16	21	6	17	20	7	8	15	10	14	19	12	13	18
118:	1	3	4	2	9	11	5	18	20	6	13	19	7	10	17	8	14	21	12	15	16
119:	1	3	4	2	10	12	5	9	14	6	15	21	7	16	20	8	17	18	11	13	19
120:	1	3	4	2	10	12	5	9	14	6	18	19	7	16	20	8	13	21	11	15	17
121:	1	3	4	2	10	12	5	13	18	6	11	17	7	16	20	8	14	21	9	15	19
122:	1	3	4	2	10	12	5	17	21	6	18	19	7	8	15	9	11	20	13	14	16
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158:	1	6	7	2	16	18	3	14	17	4	5	9	8	15	20	10	12	21	11	13	19
159:	1	9	10	2	18	20	3	4	7	5	17	21	6	8	14	11	13	19	12	15	16

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