Designs and Topology

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Abstract

An embedding of a graph in a surface gives rise to a combinatorial design whose blocks correspond to the faces of the embedding. Particularly interesting graphs include complete and complete multipartite graphs. Embeddings of these in which the faces are triangles, Hamiltonian cycles, or Eulerian cycles generate interesting designs. These designs include twofold, Mendelsohn and Steiner triple systems, and Latin squares. We examine some of these cases, looking at construction methods, structural properties and enumeration problems.

1 Context

Throughout this survey we will be predominantly concerned with triangular embeddings of graphs. These arise naturally in the context of the Heawood map-colouring conjecture. In its orientable form this asserts that the minimum number of colours required to colour a map on a surface S_g, the sphere with g handles, is given by

$$\chi(S_g) = \left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor, \quad g \geq 0.$$

For $g > 0$, the conjecture was finally established by Ringel, Youngs and others in 1968. The case $g = 0$ is the celebrated four colour theorem, finally established by Appel and Haken [7, 8] in 1976.

To see the connection between the Heawood conjecture and triangular embeddings, consider the dual problem obtained by placing a vertex in each region of the map and joining two vertices whenever the corresponding regions share a common border. We now require the minimum number of colours to vertex colour the resulting dual graph. The extremal case is the complete graph K_n requiring n colours. So it is natural to ask for the minimum genus g such that K_n may be embedded in S_g. Using Euler's formula $n + f - e = 2 - 2g$, where f denotes the number of faces and $e = \binom{n}{2}$ is the number of edges, we see that g is minimal when f is maximal and this will happen when the average number of edges per face is as small as possible. Euler's formula then gives $\lceil (n-3)(n-4)/12 \rceil$ as a lower bound for the genus. For $n \equiv 0$, 3, 4 or 7 (mod 12), this is achievable by taking all of the faces as triangles. When n does not lie in one of these congruence classes it is also achievable but a small number of non-triangular faces are required. The book by Ringel [78] gives the details and also deals with the nonorientable case of embedding K_n in N_γ, the sphere with γ crosscaps. In the nonorientable case Euler's formula is $n+f-e = 2-\gamma$ and a lower bound for the minimum genus is $\lceil (n-3)(n-4)/6 \rceil$. In the cases $n \equiv 0$ or 1 (mod 3) except $n = 7$ this is achievable with all the faces as triangles. The surface of minimum nonorientable genus in which K_7 can be embedded is N_3.

The connection between graph embeddings and combinatorial designs arises from the observation that, when a graph is embedded in a surface, the faces that result can be regarded as the blocks of a design. This design may be thought of as being embedded in the surface. The first person to observe the connection between combinatorial designs and graph embeddings was Heffter. In a paper dated November 1890 [57] he presents a partition of the integers $1, 2, \ldots, 12s + 6$, $s \geq 0$ into $4s + 2$ triples so that for each triple $\{a, b, c\}$, $a + b + c \equiv 0 \pmod{12s + 7}$. He then shows how, if $4s+3$ is prime and the order of 2 modulo $4s+3$ is either $4s+2$ or $2s+1$, these triples can be used to construct a twofold triple system (for the formal definition see Section 2) of order $12s + 7$, the blocks of which are the triangular faces of an embedding of the complete graph K_{12s+7} in an orientable surface. As observed in both [52] and [78] it is still not known if there are infinitely many such values of s. But the method is applicable for $s = 0, 1, 2, 4, 5, 11$ and 14, numbers given explicitly in [57].

The only other paper published before 1970 which explores the relationship between combinatorial designs and graph embeddings appears to be that by Emch [36]. Although mainly combinatorial in nature, it does contain diagrams of the embeddings of the twofold triple system of order 6 in the projective plane, the embedding of a pair of Steiner triple systems of order 7 in the torus, as well as an interesting embedding of a pair of Steiner triple systems of order 9 in a pseudosurface formed from a torus by identifying three pairs of points. We will meet all of these embeddings later in the paper; see Figures 6.1, 6.2 and 12.1 respectively.

2 Preliminaries

In this section we review terminology taken from combinatorial design theory and topological graph theory, and we summarize some of the basic results. The principal item required from design theory is the following definition. A *Steiner triple system of order n* is a pair $(V, \mathcal{B})$ where V is an n-element set (the *points*) and $\mathcal{B}$ is a collection of 3-element subsets (the *blocks*) of V such that each 2-element subset of V is contained in exactly one block of $\mathcal{B}$. It is well known that a Steiner triple system of order n (briefly STS(n)) exists if and only if $n \equiv 1$ or $3 \pmod 6$ [62]. If, in the definition, the words "exactly one block" are replaced by "exactly two blocks", then we have a *twofold triple system of order n*, TTS(n) for short. If a TTS(n) has no repeated blocks, it is said to be *simple*. A simple twofold triple system of order n exists if and only if $n \equiv 0$ or $1 \pmod 3$ [28]. A (possibly non-simple) TTS(n) may be obtained by combining the block sets of two STS(n)s which have a common point set. An STS(n) can be considered as a decomposition of the complete graph K_n into triangles (copies of K_3); likewise a TTS(n) can be considered as a decomposition of the twofold complete graph $2K_n$ (in which there are two edges between each pair of vertices) into triangles.

Up to isomorphism, there is just one STS(n) for $n = 3, 7, 9$, while there are two for $n = 13$, precisely one of which is cyclic (that is, has an automorphism of order 13). There are 80 STS(15)s [27], of which two are cyclic, and there are 11,084,874,829 STS(19)s [60], of which four are cyclic. The number of nonisomorphic STS(n)s is $n^{n^2/6-o(n^2)}$ [83] and, speaking asymptotically, almost all of these have only a trivial automorphism group [9].

A *Mendelsohn triple system of order n* is defined in a similar fashion to an STS(n) except that triples and pairs are taken to be ordered, so that the cyclically ordered triple (a, b, c) "contains" the ordered pairs $(a, b), (b, c)$ and (c, a). A Mendelsohn triple system of order n, MTS(n) for short, exists if and only if $n \equiv 0$ or 1 (mod 3) and $n \neq 6$ [73]. An MTS(n) may be considered as a decomposition of the complete directed graph on n vertices into directed 3-cycles. If the directions are ignored, then an MTS(n) gives a TTS(n).

A *transversal design of order n and block size* 3 is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into 3 parts (the *groups*) each of cardinality n, and $\mathcal{B}$ is a collection of 3-element subsets (the *blocks*) of V such that each 2-element subset of V is either contained in exactly one block of $\mathcal{B}$ or in exactly one group of $\mathcal{G}$, but not both. A transversal design of order n and block size 3 is denoted by TD$(3, n)$; since we only consider block size 3, we will simply speak of a transversal design of order n. A TD$(3, n)$ may be considered as a decomposition of the complete tripartite graph $K_{n,n,n}$ into triangles with the tripartition defining the groups of the design. A TD$(3, n)$ is equivalent to a Latin square of side n in which the triples are given by (row, column, entry).

To see the connection between design theory and graph embeddings, consider the case of an embedding of the complete graph K_n in an orientable surface in which all the faces are triangles. Taking these triangles with a consistent orientation to form a set of blocks, the faces of the embedding yield a Mendelsohn triple system of order n. Similarly, a triangular embedding of K_n in a nonorientable surface gives a twofold triple system of order n.

We note here that all the surfaces we consider will be, unless otherwise stated, closed, connected 2-manifolds, without a boundary. That is, in the orientable case, S_g the sphere with g handles and, in the nonorientable case, N_γ the sphere with γ crosscaps. The surfaces S_1 and S_2 are the *torus* and *double torus* respectively and the surfaces N_1 and N_2 are the *projective plane* and *Klein bottle* respectively. Given a surface embedding of some simple graph G with vertex set $V(G)$, the *rotation* at a vertex $v \in V(G)$ is the cyclically ordered permutation of vertices adjacent to v, with the ordering determined by the embedding. The set of rotations at all the vertices of G is called the *rotation scheme* for the embedding. In the case of an embedding of G in an orientable surface, the rotation scheme provides a complete description of the embedding. This is not generally the case for a nonorientable surface because the rotation scheme does not enable the faces of the embedding to be unambiguously reconstructed: some additional information is required. However, in the cases we consider this will not be an issue, since sufficient extra information to determine the faces will be known.

Ringel [78] gives the following tests to determine if a rotation scheme represents a triangular embedding.

Rule Δ: A rotation scheme represents a triangular embedding of a simple graph G if, for each vertex $a \in V(G)$, whenever the rotation at a contains the sequence $\dots bc \dots$, then the rotation at b contains either the sequence $\dots ac \dots$ or the sequence $\dots ca \dots$.

Rule Δ^*: If the rotations at each vertex can be directed in such a way that for each vertex $a \in V(G)$, whenever the rotation at a contains the sequence $\dots bc \dots$, then the rotation at b contains the sequence $\dots ca \dots$, then the embedding is in an orientable surface.

We refer the reader to [52, 78] for an explanation of current and voltage graphs which are used to construct graph embeddings. In Sections 3, 4, 5 and 10 we make extensive use of these methods. The origin of current graphs lies in the work of Gustin [53] who regarded these as combinatorial tools. Voltage graphs were introduced by Gross [51].

In a surface embedding of K_n, the rotation at each vertex will comprise a single cycle of length $n-1$. As described in [31] this provides a test for an MTS(n), or a TTS(n), to be embeddable in an orientable, or a nonorientable surface, respectively. Let $(V, \mathcal{B})$ be a TTS(n). For each $x \in V$, define the *neighbourhood graph* G_x: its vertex set is $V \setminus \{x\}$, and two vertices y, z are joined by an edge if $\{x, y, z\} \in \mathcal{B}$. Clearly, G_x is a union of disjoint cycles. A TTS(n) occurs as a triangulation of a surface if and only if every neighbourhood graph consists of a single cycle. If the blocks of the TTS(n) can be ordered to form an MTS(n), then the surface is orientable, otherwise it is nonorientable.

Of much more interest is the relationship between embeddings of complete graphs and Steiner triple systems. Suppose that we have an embedding, not necessarily a triangular embedding, of the complete graph K_n with vertex set V in a surface S with the property that the faces can be properly 2-coloured, that is, no two faces with a common edge have the same colour. We will take the colour classes to be *black* and *white*. If either colour class consists entirely of triangles, then these triangles necessarily form the blocks of an STS(n) on the point set V. We will say that the STS(n) is *embedded* in the surface S. If both colour classes consist entirely of triangles, then we have two STS(n)s , black and white, *biembedded* in S. Slightly more generally, we will say that two STS(n)s, say B and W, are *biembeddable* in a surface S if there is a face 2-colourable triangular embedding of the complete graph K_n in S with the black (respectively white) faces forming a system isomorphic with B (respectively W).

The first obvious question is whether, given an STS(n), it has an embedding in an orientable and in a nonorientable surface. It turns out that this question has a positive answer, and the proof is not difficult. We will show in Section 8 how to construct a maximum genus embedding of an STS(n) where the faces comprise a set of black triangles representing the Steiner system, together with a single white face.

A sequence of deeper questions concerns biembeddings of STS(n)s, that is, face 2-colourable triangular embeddings of K_n. We list these in increasing order of difficulty.

1. For each $n \equiv 3$ or $7 \pmod{12}$ is there a biembedding of some pair of STS(n)s in an orientable surface? Similarly for each $n \equiv 1$ or $3 \pmod 6$ is there such a biembedding in a nonorientable surface?

2. If such biembeddings exist, how many are there?

3. Given an STS(n), does it have a biembedding with some other STS(n) in an orientable and in a nonorientable surface?

4. Given a pair of STS(n)s, do they have a biembedding in an orientable and in a nonorientable surface?

Of course, a necessary condition for a positive answer to questions 3 and 4 in the orientable case is that $n \equiv 3$ or $7 \pmod{12}$. A complete answer to question 1 is given in Section 3. In subsequent sections, principally Sections 4, 5 and 6, we describe progress with questions 2, 3 and 4. The remaining sections are devoted to other related aspects such as Hamiltonian embeddings, biembeddings of Latin squares, and biembeddings of symmetric configurations.

3 Existence

In this section we establish the existence of biembeddings of STS(n)s. The orientable case $n \equiv 3 \pmod{12}$ and the nonorientable case $n \equiv 3 \pmod 6$ come from Ringel [78]. For the orientable case $n \equiv 7 \pmod{12}$ we turn to graphs first constructed by Youngs [86]. In each of these cases we present the general solution either by specifying appropriate current graphs or by giving the logs obtained from such graphs. For the nonorientable case $n \equiv 1 \pmod 6$ we refer the reader to [49] which gives explicit current graphs. In both the orientable case $n \equiv 3 \pmod{12}$ and the nonorientable case $n \equiv 3 \pmod 6$, we relate these solutions to the Bose construction for Steiner triple systems.

We first consider the orientable case $n \equiv 3 \pmod{12}$. The current graphs constructed by Ringel for this case are index 3 Möbius ladders, and the general form is shown in Figure 3.1. The ends labelled A should be identified, and likewise the ends labelled B. The graph is bipartite, which ensures that the resulting embedding is face 2-colourable. The vertex directions are indicated by solid and hollow circles, representing clockwise and anticlockwise respectively. Taking account of these directions we form the *logs* of the three circuits denoted by [0], [1] and [2] in the figure.

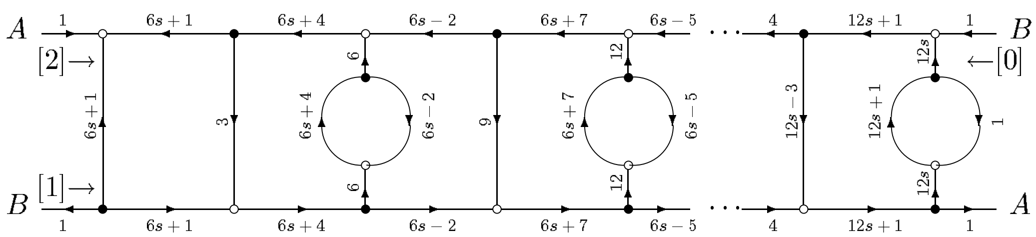

Figure 3.1: Orientable current graph for $n = 12s + 3$.

For the particular case $n = 15$, the logs are as follows.

[0] :	1	13	9	11	5	12	7	14	2	6	4	10	3	8
[1] :	14	7	8	5	9	4	10	6	11	2	3	1	13	12
[2] :	1	8	7	10	6	11	5	9	4	13	12	14	2	3

From an index 3 current graph with currents in $\mathbb{Z}_n$, we may obtain a rotation scheme for an embedding of K_n with vertex set $\mathbb{Z}_n$. The rotation at $i \in \mathbb{Z}_n$ is determined by adding i modulo n to each element of the log of $[a]$, where $i \equiv a \pmod 3$, $a \in \{0, 1, 2\}$.

An alternative approach to obtaining biembeddings of STS(n)s, where $n \equiv 3 \pmod{12}$ in orientable surfaces is given in [45] and uses the Bose construction.

Bose construction

Let $(G,+)$ be an Abelian group of odd order. Thus if $i,j \in G$ then $i * j = (i+j)/2$ is a well defined element of G. Let $V = G \times \mathbb{Z}_3$. On V form a collection $\mathcal{B}$ of triples as follows.

(1) $2s+1$ triples of the form $\{(i,0),(i,1),(i,2)\}$, $i \in G$,

(2) $3s(2s+1)$ triples of the form $\{(i,k),(j,k),(i*j,k+1)\}$, $i,j \in G$, $i \neq j$, $k \in \mathbb{Z}_3$.

Then it is easily verified that $(V,\mathcal{B})$ is a Steiner triple system of order $3|G|$.

A biembedding of STS(n)s where $n \equiv 3 \pmod{12}$ can now be obtained as follows. Build a Steiner triple system $(V,\mathcal{B})$, where $V = \mathbb{Z}_{4s+1} \times \mathbb{Z}_3$, by the Bose construction as above. Now define two Steiner triple systems $(\mathbb{Z}_n,\mathcal{B}_0)$ and $(\mathbb{Z}_n,\mathcal{B}_1)$, both isomorphic to $(V,\mathcal{B})$ using the bijections $f_m : V \mapsto \mathbb{Z}_n$, $m = 0,1$ given by $f_m(i,k) = 3i+(-1)^m ks$ where $s = 6t+1$. It is easy to prove that $\mathcal{B}_0 \cap \mathcal{B}_1 = \emptyset$, that is, the two STS($n$)s are disjoint. To show that the pair is biembeddable in an orientable surface, consider the triples in $\mathcal{B}_0$ (respectively $\mathcal{B}_1$) as the black (respectively white) triangles of a biembedding. For each pair of distinct points $u,v \in \mathbb{Z}_n$, we take the corresponding black and white triangles, both containing u and v as vertices, and glue these triangles together along the side uv. Let S be the resulting topological space; then S is certainly a generalized pseudosurface. We need to prove that, in fact, S is an orientable surface. This is done by exhibiting the rotation scheme and showing that it satisfies Ringel's Rule Δ^*. This is straightforward, though tedious, and details are given in [45]. Thus, use of the Bose construction provides a proof of the orientable case $n \equiv 3 \pmod{12}$ of the Heawood map-colouring conjecture by exclusively design-theoretic methods. In fact, the biembeddings so obtained are isomorphic to those obtained from Ringel's index 3 current graph construction.

The current graphs constructed by Ringel for the orientable case $n \equiv 7$ (mod 12) of the Heawood map-colouring conjecture are not bipartite. Nor are the graphs used in an alternative solution given by Youngs [85]. Hence the embeddings are not face 2-colourable and are consequently not biembeddings of Steiner triple systems. As recorded in Section 1, Heffter [57] had already in 1891 shown the existence of orientable biembeddings of STS(n)s for <u>some</u> $n \equiv 7 \pmod{12}$ but the case was not completed until nearly 80 years later. In [86] Youngs uses what he calls "zigzag diagrams" to construct index 1 bipartite current graphs, and hence biembeddings of Steiner triple systems for this case. In this context, index 1 means that there is a single circuit of the graph which traverses every edge precisely once in each direction and whose log contains every nonzero element precisely once. Hence for each $i \in \{1,2,\ldots,(n-1)/2\}$ either i or $-i$ must appear as a current on one of the edges and each edge has exactly one of these $(n-1)/2$ currents. The biembeddings thus constructed are cyclic. The general forms of these current graphs are shown in Figures 3.2 and 3.3 for $n = 24m+7$, $m \geq 2$, and $n = 24m+19$, $m \geq 3$ respectively. In each case the ends labelled A should be identified, and likewise the ends labelled B. For the values $n = 7, 19, 31, 43$ and 67, Youngs gives specific diagrams.

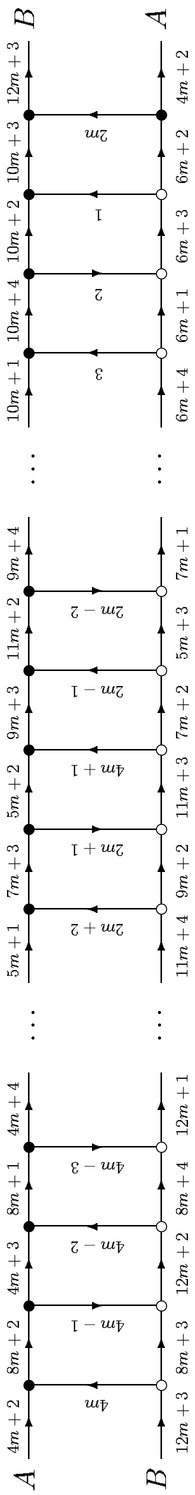

Figure 3.2: Orientable current graph for $n = 24m + 7$, $m \geq 2$.

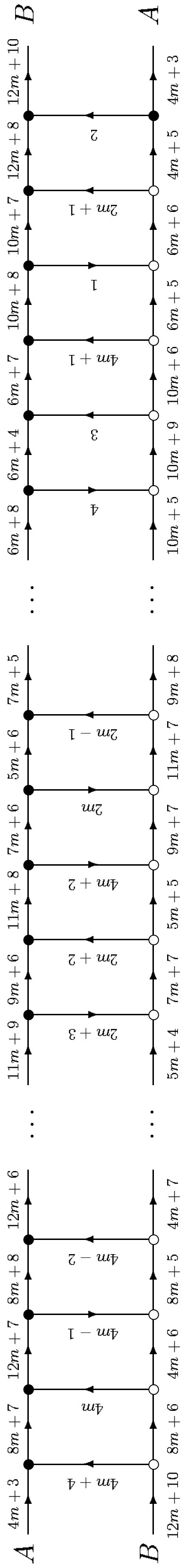

Figure 3.3: Orientable current graph for $n = 24m + 19$, $m \geq 3$.

Turning now to biembeddings of STS(n)s in nonorientable surfaces, the case $n \equiv 9 \pmod{12}$ can also be found in [78]. The solution involves another class of index 3 current graphs which Ringel calls "cascades", and the remark is made that the method also works for the nonorientable case $n \equiv 3 \pmod{12}$, although no details are given. These were later worked out and are given in [10]. A simpler description is the following where, as above, [0], [1] and [2] are the logs of the three circuits.

$$\begin{array}{llllllllll}
[0]: & 1 & 2 & [24t+12 & 12t+8 & 24t+24 & 12t+14] & [-(6t+2) & 6t+4] & \\
[1]: & 1 & -1 & [-(12t+6) & -(6t+4) & -(12t+12) & -(6t+7)] & [-(6t+2) & 6t+4] & \\
[2]: & -2 & -1 & [-(12t+6) & -(6t+4) & -(12t+12) & -(6t+7)] & [12t+4 & -(12t+8)]\,. &
\end{array}$$

Here the terms inside the square brackets are repeated for $t = 0, 1, \dots, 2s-1$ in the case of $n = 12s+3$ and for $t = 0, 1, \dots, 2s$ in the case of $n = 12s+9$, with arithmetic in each case modulo n. In both cases the rotation scheme obtained gives two isomorphic Steiner triple systems again generated by the Bose construction with the group $G = \mathbb{Z}_n$. To see this, map each $i \in \mathbb{Z}_n$ to (a, b) where $a = \lfloor i/3 \rfloor$ and $b = i - 3a$. One of the two STS(n)s is then very clearly a Bose system and by applying the mapping $f((a, b)) = (a + b, b)$ it is seen that the second system is also a Bose system.

An alternative proof from the Bose construction for $n \equiv 3 \pmod 6$ is given in [30] and is very similar to the construction given above for the orientable case. Build a Steiner triple system $(V, \mathcal{B})$, where $V = \mathbb{Z}_{2s+1} \times \mathbb{Z}_3$, by the Bose construction and define two Steiner triple systems $(V, \mathcal{B}_0)$ and $(V, \mathcal{B}_1)$, both isomorphic to $(V, \mathcal{B})$, using the bijections $f_m : V \mapsto V,\ m = 0, 1$, defined as follows.

$$\begin{aligned}
f_m((i,0)) &= (i,0)\\
f_m((i,1)) &= (i+m,1)\\
f_m((i,2)) &= (i-m+2s,2).
\end{aligned}$$

Verification that this gives a biembedding of the two Steiner triple systems in a nonorientable surface follows the same procedure as outlined in the orientable case.

Perhaps surprisingly, the existence of a nonorientable biembedding of STS(n)s for $n \equiv 1 \pmod 6$ was not established until fairly recently [49]. Much of the spectrum can be obtained from recursive constructions given in [19, 44, 46]. The cases $n \equiv 7$ or $25 \pmod{36}$, $n \neq 7$, follow immediately from Construction 4.2 given in Section 4 and the known biembeddings for $n \equiv 3$ or $9 \pmod{12}$. The case $n \equiv 13 \pmod{36}$ is more complex but comes from a nonorientable version of Construction 4 of [46] using a face 2-colourable triangular embedding of the complete tripartite graph $K_{6,6,6}$ having a parallel class in one of the colour classes, see [40], a nonorientable face 2-colourable triangular embedding of K_{13}, see [38], and the $n \equiv 3 \pmod{12}$ case. The case $n \equiv 1 \pmod{36}$ then follows from the Construction 4.2 using an inductive argument. This leaves the cases $n \equiv 19$ or $31 \pmod{36}$ but the former would follow immediately if a method for dealing with the latter was known. But in [49] direct constructions using index 1 current graphs are given in all cases. There are four general subcases corresponding to $n \equiv 1$, 7, 13 or $19 \pmod{24}$, as well as a number of particular cases. Limitations of space preclude us from giving details. We refer the reader to the original paper where all the current graphs are given in the same format as in this paper.

4 Growth estimates

We present two main recursive constructions. These have a degree of flexibility that enable us to obtain a lower bound on the number of biembeddings of STS(n)s for values of n lying in certain residue classes. Our first construction is new and produces biembeddings of STS($3n$)s from a biembedding of STS(n)s.

Construction 4.1

Take any biembedding of STS(n)s in either an orientable or a nonorientable surface. Pick a preferred point ∞ of these designs. Define *the cap at* ∞ to comprise all the triangles, both black and white, incident with ∞ in the embedding. Next pick a preferred white triangle T incident with ∞. We distinguish three categories of white triangles:

(i) those not on the cap at ∞,

(ii) those on the cap at ∞ other than the preferred triangle T,

(iii) the triangle T.

Next take three copies of the given biembedding on three disjoint surfaces S^0, S^1 and S^2. We use superscripts in a similar way to identify corresponding points on these surfaces.

For each white triangular face (uvw) of type (i), we "bridge" S^0, S^1 and S^2 by gluing a torus to the three triangles $T^i = (u^i v^i w^i)$ in the following manner. Take a face 2-colourable triangular embedding in a torus of the complete tripartite graph $K_{3,3,3}$ having three vertex parts $\{u^i\}, \{v^i\}$ and $\{w^i\}$ and having black faces $(u^i w^i v^i)$ for $i = 0, 1, 2$ (see Figure 4.1). We use the same labels for the vertices of this graph as we do for the vertices of the three triangles T^i, but initially think of them as distinct points. (A similar gloss will be used on several occasions.) Now glue the black faces $(u^i w^i v^i)$ on the torus to the white faces $(u^i v^i w^i)$ on S^0, S^1 and S^2 respectively, so that points on the torus and on the surfaces S^i with the same label are identified.

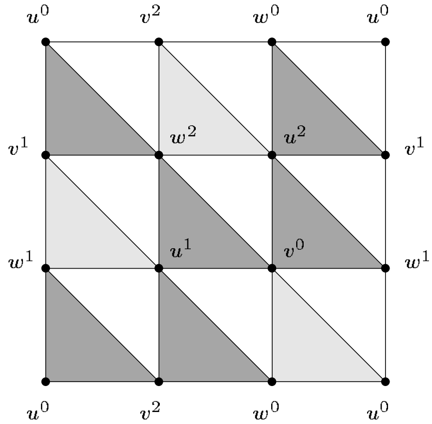

Figure 4.1: Toroidal embedding of $K_{3,3,3}$.

For each white triangular face $(uv\infty)$ of type (ii), we carry out a similar bridging operation but using a different type of bridge. For this we take a face 2-colourable triangular embedding of the graph $K_9 - K_3$ in the nonorientable surface N_4, defined by the following rotation scheme, where the colouration is determined by taking each $(u^i v^i \infty^i)$ as a black triangle.

∞^0	:	u^0	v^0	v^2	u^1	u^2	v^1		
∞^1	:	u^1	v^1	v^0	u^2	u^0	v^2		
∞^2	:	u^2	v^2	u^0	v^1	u^1	v^0		
u^0	:	u^1	u^2	∞^1	v^2	∞^2	v^1	∞^0	v^0
u^1	:	u^0	u^2	∞^0	v^2	∞^1	v^1	∞^2	v^0
u^2	:	u^0	u^1	∞^0	v^1	v^2	∞^2	v^0	∞^1
v^0	:	v^1	v^2	∞^0	u^0	u^1	∞^2	u^2	∞^1
v^1	:	v^0	v^2	u^2	∞^0	u^0	∞^2	u^1	∞^1
v^2	:	v^0	v^1	u^2	∞^2	u^0	∞^1	u^1	∞^0

Table 4.1: N_4 embedding of $K_9 - K_3$.

We glue these bridges to S^0, S^1 and S^2 as before. Note that none of these bridges contain any edge $\infty^i\infty^j$.

To complete the construction, we construct a single bridge to join the three copies of the type (iii) triangle T. For this bridge we take a face 2-colourable triangular embedding of K_9 in the nonorientable surface N_5. Such an embedding, a biembedding of STS(9)s, is given in Section 3 and we can label the vertices so that the black faces include the triangles $(v^i u^i \infty^i)$ for $i = 0, 1, 2$. As before, we glue the white triangle $T^i = (u^i v^i \infty^i)$ on S^i to the black triangle $(v^i u^i \infty^i)$ on the bridge. Note that this bridge contains the three edges $\infty^i\infty^j$.

It is now routine to check that the resulting embedding represents a biembedding of two STS($3n$)s in a nonorientable surface. □

We now make some observations about the construction that enable us to extend it. Firstly, the toroidal embedding of $K_{3,3,3}$ given in Figure 4.1 may be replaced by one in which the cyclic order of the three superscripts is reversed. The reversed embedding of $K_{3,3,3}$ is isomorphic to the original but is labelled differently (see Figure 4.2). For each white triangular face (uvw) of S we may carry out the bridging operation across S^0, S^1, S^2 using either the original $K_{3,3,3}$ embedding or the reversed embedding. The choice of which of the two to use can be made independently for each white triangle (uvw). Replacing one bridge by the reversed bridge is an example of a *surface trade*; these are discussed more generally in Section 7.

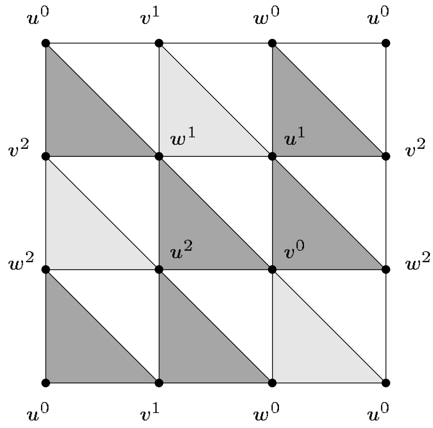

Figure 4.2: Reversed toroidal embedding of $K_{3,3,3}$.

As a consequence of this observation, we have the following result.

Theorem 4.1 *For $n \equiv 3$ or 9 (mod 18), there are at least $2^{n^2/54-o(n^2)}$ nonisomorphic face 2-colourable triangular embeddings of the complete graph K_n, and hence biembeddings of STS(n)s, in a nonorientable surface.*

Proof Take three fixed copies of the same face 2-colourable triangular embedding of K_m, $m \equiv 1$ or 3 (mod 6), and apply Construction 4.1 while varying the toroidal bridges. Since there are $(m-1)(m-3)/6$ toroidal bridges and two choices for each bridge, we may construct $2^{(m-1)(m-3)/6}$ differently labelled face 2-colourable embeddings of K_{3m}. The maximum possible size of an automorphism class of these is $(3m)!$. Hence there are at least $2^{m^2/6-o(m^2)}$ nonisomorphic face 2-colourable triangular embeddings of K_{3m}, and replacing $3m$ by n gives the result. □

Our second observation about the construction is that it is not necessary for S^0, S^1 and S^2 to contain three copies of the same embedding of K_n. All that the construction requires is that the three embeddings have the "same" white triangular faces. To be more precise, by the term "same" we mean that there are mappings from the vertices of each surface onto the vertices of each of the other surfaces that preserve the white triangular faces. The sceptical reader may feel dubious that we can satisfy this requirement without in fact having three identically labelled copies of a single embedding. However, if we examine the black triangles of the embeddings generated as described in Theorem 4.1, we will see that it is indeed possible. We claim that in any two such embeddings, the black triangles are identical. To see this, note that the black triangles come from four sources, the original surfaces and the three types of bridges. Those lying on the surfaces S^0, S^1 and S^2 are unaltered during the construction and therefore are common to both embeddings. Those lying on the $K_{3,3,3}$ bridges are the same whether or not the bridges are reversed (see Figures 4.1 and 4.2). Those lying on N_4 bridges and on the N_5 bridge are common to both embeddings. It follows that the $2^{n^2/54-o(n^2)}$ nonisomorphic embeddings of K_n generated by Theorem 4.1 all contain identical black triangles. In each of these embeddings, by reversing the colours, we produce a plentiful supply of nonisomorphic embeddings in surfaces S^i on which to base a reapplication of the construction. All of these embeddings of K_n have the "same" white triangles.

We can select three surface embeddings from this collection to form S^0, S^1, S^2 in N^3 ways, where $N = 2^{n^2/54 - o(n^2)}$. The $K_{3,3,3}$ bridges may be selected in $2^{(n-1)(n-3)/6}$ different ways. Any two of the resulting embeddings of K_{3n} (obtained by varying the surfaces S^0, S^1 and S^2, and the $K_{3,3,3}$ bridges) will be differently labelled. These results lead easily to the next theorem.

Theorem 4.2 *For $n \equiv 9$ or 27 (mod 54), there are at least $2^{2n^2/81 - o(n^2)}$ nonisomorphic face 2-colourable triangular embeddings of the complete graph K_n, and hence biembeddings of STS(n)s, in a nonorientable surface.*

Our second construction was first given by Širáň and ourselves in [44]. It produces biembeddings of STS($3n-2$)s from a biembedding of STS(n)s. It uses many of the same ingredients as Construction 4.1, and we will be brief in our description of these common features. However, unlike Construction 4.1, this second construction can be used to produce both orientable and nonorientable biembeddings.

Construction 4.2

Take any biembedding of STS(n)s in either an orientable or a nonorientable surface S. Pick a preferred point ∞ and define the cap at ∞ as before. Delete this cap from S by removing the point ∞, all (open) edges incident with ∞ and all (open) triangular faces incident with ∞ to give an embedding of K_{n-1} in a surface S^* with a boundary $D = (u_1 u_2 \dots u_{n-1})$. Each alternate edge of this Hamiltonian cycle is incident with a white triangle in S^*; suppose that these edges are $u_2u_3, u_4u_5, \dots, u_{n-1}u_1$. Next take three copies of this embedding in three disjoint surfaces S^{*i}, $i = 0, 1, 2$, each with a boundary $D^i = (u_1^i u_2^i \dots u_{n-1}^i)$. The white triangles on S^{*0}, S^{*1} and S^{*2} are bridged as before using toroidal face 2-colourable triangular embeddings of $K_{3,3,3}$.

After all the white triangles have been bridged we are left with a new connected triangulated surface with a boundary. We denote this surface by Σ. It has $3n-3$ vertices and the boundary comprises the three disjoint cycles D^i, each of length $n-1$. In order to complete the construction to obtain a face 2-colourable triangular embedding of K_{3n-2}, which gives a biembedding of two STS($3n-2$)s, we must construct an auxiliary triangulated bordered surface T^* and paste it to Σ so that the three holes of Σ will be capped.

The bordered surface T^* is constructed from a surface T which has, as vertices, the points u_j^i for $i = 0, 1, 2$ and $j = 1, 2, \dots, n-1$ together with one additional point which we call ∞^*. The construction of T uses voltage assignments. Suppose initially that $n \equiv 3 \pmod 6$.

Let ν be the plane embedding of the multigraph L with faces of length 1 and 3 coloured black and white, as depicted in Figure 4.3.

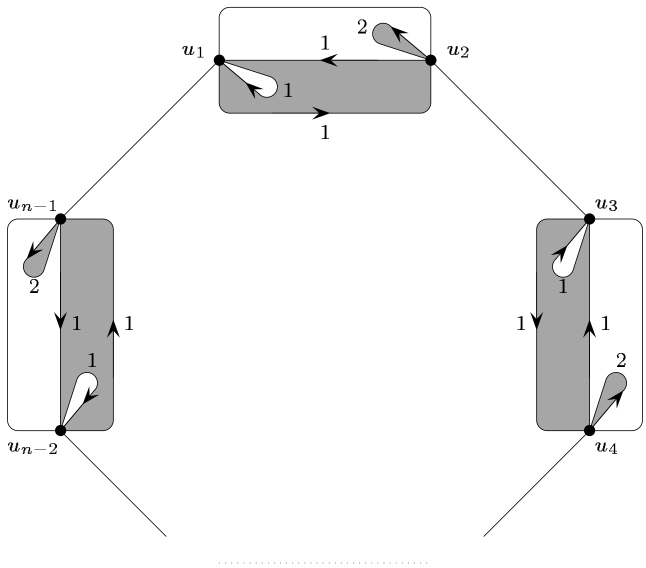

Figure 4.3: The plane embedding of the multigraph L.

Figure 4.3 also shows voltages α on directed edges of L, taken in the group $\mathbb{Z}_3 = \{0, 1, 2\}$. The edges with no direction assigned carry the zero voltage.

The lifted graph L^α has the vertex set $\{u_j^i;\ 1 \le j \le n-1,\ i \in \mathbb{Z}_3\}$. As before, we use the same letters for vertices of L^α as for vertices of our embedded graph in Σ, but we initially assume that these graphs are disjoint. The edge set of L^α can be described as follows. For each fixed $l = 1, 3, 5, \ldots, n-2$, the six vertices u_l^i, u_{l+1}^i, $i \in \mathbb{Z}_3$, induce a complete graph $J_l \simeq K_6$ in L^α. Moreover, two successive complete subgraphs J_l and J_{l+2} (indices mod $(n-1)$) are joined by three edges $u_{l+1}^i u_{l+2}^i$, $i \in \mathbb{Z}_3$. Thus we have a total of $15(n-1)/2 + 3(n-1)/2 = 9(n-1)$ edges in L^α, and there are neither loops nor multiple edges.

The lifted embedding $\nu^\alpha :\ L^\alpha \to T$ has $4(n-1)$ triangular faces: the white ones are bounded by the triangles $(u_l^0 u_{l+1}^0 u_{l+1}^2)$, $(u_l^1 u_{l+1}^1 u_{l+1}^0)$, $(u_l^2 u_{l+1}^2 u_{l+1}^1)$ and $(u_l^0 u_l^1 u_l^2)$, where $l = 1, 3, 5, \ldots, n-2$, and the black ones are bounded by $(u_l^0 u_{l-1}^1 u_{l-1}^2)$, $(u_l^1 u_{l-1}^2 u_{l-1}^0)$, $(u_l^2 u_{l-1}^0 u_{l-1}^1)$ and $(u_l^0 u_l^2 u_l^1)$, where $l = 2, 4, \ldots, n-1$. In addition, there are four more faces in the embedding ν^α; three faces, which we denote by F^i, bounded by $(n-1)$-gons of the form $(u_1^i u_2^i \ldots u_{n-1}^i)$, $i \in \mathbb{Z}_3$, and one face F' bounded by the $(3n-3)$-gon $(u_1^0 u_2^1 u_3^1 u_4^2 \ldots u_{n-2}^2 u_{n-1}^0)$; here we use the fact that $n-1$ is coprime with 3. Thus the boundary of F' is a Hamiltonian cycle, say B, in L^α.

Now cut out from T the three (open) faces F^i, $i \in \mathbb{Z}_3$, bounded by the above three disjoint $(n-1)$-gons, obtaining thereby an orientable bordered surface T^*. Let L^* be the graph obtained from L^α by adding a new vertex ∞^* and joining it to each vertex of L^α, and keeping all edges in L^α unchanged. We construct an embedding $\nu^* :\ L^* \to T^*$ from ν^α in an obvious way: in the embedding ν^α (after the removal of the three open faces), we insert the vertex ∞^* in the centre of the face F' bounded by the $(3n-3)$-gon and join this point by open arcs within F' to every vertex on the boundary of F' (that is, with every vertex of the Hamiltonian cycle B). Instead of F'

we now have $(3n-3)$ new triangular faces on T^*; they are bounded by 3-cycles of the form $\infty^* u^i_j u^{i'}_{j+1}$. We now colour the new triangular faces as follows: the face of ν^* bounded by the 3-cycle $\infty^* u^i_j u^{i'}_{j+1}$ will be black (respectively white) if the triangular face of the embedding ν^α containing the edge $u^i_j u^{i'}_{j+1}$ is white (respectively black). It is easy to check that this rule defines a 2-colouring of the triangular embedding $\nu^* : L^* \to T^*$. We thus have $4(n-1) + (3n-3) = 7(n-1)$ triangular faces on T^*, exactly half of which are black.

We are ready for the final step of the construction. The surface Σ has three holes with boundaries $D^i = (u^i_1 u^i_2 \dots u^i_{n-1})$. The bordered surface T^* has three holes as well, whose boundary cycles D^{*i} can be oriented in the form $D^{*i} = (u^i_{n-1} \dots u^i_2 u^i_1)$. It remains to do the obvious: namely, for $i = 0, 1, 2$ to paste together the boundary cycles D^i and D^{*i} so that corresponding vertices u^i_j get identified. As the result we obtain a surface $\Sigma \# T^*$, known as the *connected sum* of the bordered surfaces Σ and T^*, and a triangular embedding $\sigma : K \to \Sigma \# T^*$ of some graph K. It is then routine to check that $K \simeq K_{3n-2}$ and that the triangulation is face 2-colourable.

If $n \equiv 1 \pmod 6$ then we amend the voltage assignment on L as follows. We take one of the two-point subgraphs in Figure 4.3, say that containing u_1 and u_2, and replace the voltages 1 by 2 and vice versa, the remaining part of L being unaltered. The proof then proceeds on the same lines as before with the modified version of L. Note that this alteration ensures that the lifted embedding still has a $(3n-3)$-gon face even though $n-1$ is not coprime with 3. The order of the vertices around this face differs from that given previously, but it is still possible to insert a new vertex ∞^* and to complete a 2-colouring of the resulting triangular embedding. □

In Construction 4.2, the surface T^* is orientable, as are the toroidal bridges. Hence, if the original biembedding of STS(n)s is orientable, then the resulting biembedding of STS($3n-2$)s will be orientable. This is always possible for $n \equiv 3$ or 7 (mod 12).

As with Construction 4.1, we may obtain growth estimates as given in [19] by Bonnington, Širáň and ourselves.

Theorem 4.3 *For $n \equiv 1$ or 7 (mod 18), there are at least $2^{n^2/54-o(n^2)}$ nonisomorphic face 2-colourable triangular embeddings of the complete graph K_n, and hence biembeddings of STS(n)s, in a nonorientable surface.*

Theorem 4.4 *For $n \equiv 1$ or 19 (mod 54), there are at least $2^{2n^2/81-o(n^2)}$ nonisomorphic face 2-colourable triangular embeddings of the complete graph K_n, and hence biembeddings of STS(n)s, in a nonorientable surface.*

By starting with orientable embeddings, we also obtain the following results.

Theorem 4.5 *For $n \equiv 7$ or 19 (mod 36), there are at least $2^{n^2/54-o(n^2)}$ nonisomorphic face 2-colourable triangular embeddings of the complete graph K_n, and hence biembeddings of STS(n)s, in an orientable surface.*

Theorem 4.6 *For $n \equiv 19$ or 55 (mod 108), there are at least $2^{2n^2/81-o(n^2)}$ nonisomorphic face 2-colourable triangular embeddings of the complete graph K_n, and hence biembeddings of STS(n)s, in an orientable surface.*

Not all residue classes that permit face 2-colourable triangular embeddings are covered by the theorems of this section. In particular, results are not given for $n \equiv 13$ or 15 (mod 18). We remark that further generalizations of Constructions 4.1 and 4.2 are possible. Some details of these and additional constructions are given in [46] where more than three copies of the initial embedding are used. These allow some inroads to be made into these two remaining residue classes modulo 18, but we do not have full coverage of these values.

An alternative approach is given by Korzhik and Voss [65, 66, 67, 63]. By starting with suitable current graphs and varying the vertex directions (see Section 5 for what this means), they construct for all suitably large n in each residue class modulo 12, $A2^{bn}$ nonisomorphic minimum genus embeddings of K_n in both orientable and nonorientable surfaces. The values of A and b vary with the residue class but in all cases $b > 1/12$. As observed in Section 1, in the nonorientable case, minimum genus embeddings of K_n are triangular embeddings when $n \equiv 0$ or 1 (mod 3), and in the orientable case when $n \equiv 0$, 3, 4 or 7 (mod 12). Since none of Korzhik and Voss' embeddings is face 2-colourable, they do not represent embeddings of Steiner triple systems but, rather, embeddings of twofold triple systems or, in the orientable case, Mendelsohn triple systems. Although these results cover all residue classes, the bound is a long way from 2^{an^2}. In a more recent development [64], Korzhik and Kwak combine the current graph approach with the cut-and-paste technique of Constructions 4.1 and 4.2 to prove that if $12s + 7$ is prime and if $n = (12s + 7)(6s + 7)$, then the number of nonorientable triangular embeddings of K_n is at least $2^{n^{3/2}(\sqrt{2}/72+o(1))}$.

5 Orientable cyclic biembeddings

By a *cyclic biembedding* we mean a biembedding of two STS(n)s, each of which has the same cyclic automorphism, and such that this cyclic automorphism extends to an automorphism of the biembedding. We will assume that this cyclic automorphism is $z \mapsto z + 1$ (mod n). A cyclic STS(n) exists for every $n \equiv 1$ or 3 (mod 6) apart from $n = 9$ [76], see [25] for details. In the case where $n \equiv 3$ (mod 6), a cyclic STS(n) contains a unique short orbit and consequently there can be no cyclic biembeddings. As detailed in Section 3, Youngs [86] produced orientable cyclic biembeddings for all $n \equiv 7$ (mod 12) constructed from index 1 current graphs, and it is this case that we consider in this section. We take as our starting point the fact that every such biembedding can be obtained in this way from a current graph having the following properties.

(i) Each vertex has degree 3.

(ii) At each vertex, the sum of the directed currents is 0 (mod $12s + 7$) (*Kirchoff's current law*).

(iii) For each $i \in \{1, 2, \ldots, 6s + 3\}$, either i or $-i$ appears exactly once as a current on one of the edges and each edge has exactly one of these $6s + 3$ currents.

(iv) The directions (clockwise or anticlockwise) assigned to each vertex are such that a *complete circuit* is formed, that is, one in which every edge of the graph is traversed in each direction exactly once.

(v) The graph is bipartite.

Consideration of the degree and the currents shows that these current graphs have $4s + 2$ vertices. Furthermore, there can be no loops and, save for the exceptional case $s = 0$, no multiple edges. This last fact follows from consideration of the configuration shown in Figure 5.1.

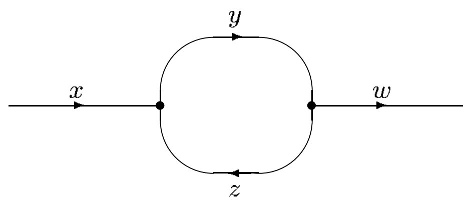

Figure 5.1: A possible multiple edge.

If this forms part of a current graph then $w \equiv x$ and so the whole current graph comprises two vertices with a triply repeated edge.

There is a close connection between current graphs and solutions of *Heffter's first difference problem* (HDP). In 1897, Heffter [58] posed the following question: can the integers $1, 2, \ldots, 3k$ be partitioned into k triples $\langle a, b, c\rangle$ such that, for each triple, $a + b \pm c \equiv 0 \pmod{6k + 1}$? Examination of the triples formed by the directed currents at each vertex in either of the two vertex sets of a bipartite current graph shows that they form a solution to HDP for $k = 2s + 1$.

In view of the above observations, the problem of constructing orientable cyclic biembeddings of a pair of STS$(12s + 7)$s, $s > 0$, may be reduced to three steps:

(a) Identifying simple connected cubic bipartite graphs having $4s + 2$ vertices.

(b) Assigning directions (clockwise or anticlockwise) at each of the vertices which then give rise to a complete circuit.

(c) Taking two solutions of HDP and labelling the edges of the graph in such a way that the triples arising from each of the vertex sets of the bipartition correspond to these two solutions.

These three steps have a large measure of independence from one another. However, we cannot exclude the possibility that for a particular graph it may be impossible to assign vertex directions to give a complete circuit, and, even if this is possible, it may not be possible to assign the HDP solutions to the edges. We note that a test for the existence of a complete circuit in a graph G is given by Xuong [84]. It asserts the existence of such a circuit, equivalent to a one-face orientable embedding of G, if and only if G has a spanning tree whose co-tree has no component with an odd number of edges.

Before proceeding further, it is appropriate to recall how Steiner triple systems arise from solutions to HDP. Given a difference triple $\langle a, b, c\rangle$ with $a+b\pm c \equiv 0 \pmod{6k+1}$, we may form a cyclic orbit by developing the starter $\{0, a, a+b\}$ or the starter $\{0, b, a+b\}$. By taking all the difference triples from a solution of HDP and forming a cyclic orbit from each, a cyclic STS$(6k+1)$ is obtained. The converse is also true: given a cyclic STS$(6k+1)$, we may obtain a solution to HDP by taking from each orbit a block $\{0, \alpha, \beta\}$ and forming the difference triple $\langle \hat{\alpha}, \widehat{\beta-\alpha}, \hat{\beta}\rangle$, where

$$\hat{x} = \begin{cases} x & \text{if} \quad 1 \le x \le 3k \\ 6k+1-x & \text{if} \quad 3k+1 \le x \le 6k \end{cases}$$

Each solution to HDP produces 2^k different STS$(6k+1)$s; however, there may be isomorphisms between these systems. In addition, for a given value of k, there will generally be many distinct solutions to HDP. In this context, we say that two solutions to HDP for $k = 2s+1$ are *multiplier equivalent* if one set of difference triples may be obtained from the other by first multiplying by a constant factor $\pmod{6k+1}$ and then reducing any residue x in the range $3k+1 \le x \le 6k$ to $6k+1-x$. Further, we define a *Heffter class* to be a class of all solutions to HDP that are multiplier equivalent. The significance of this definition is that STS$(6k+1)$s obtained from multiplier equivalent solutions to HDP are themselves multiplier equivalent and hence isomorphic.

For $n = 19$, all the computations may be done by hand. The only cubic bipartite graph is $K_{3,3}$. Fixing the rotation at one vertex of $K_{3,3}$ there are twelve ways of assigning vertex directions to produce a complete circuit [15]. There are four solutions to HDP for $k = 3$ [22], but only two Heffter classes, namely:

$$\begin{array}{llll} \text{I}: & \langle 1,3,4\rangle & \langle 2,7,9\rangle & \langle 5,6,8\rangle \\ & \langle 1,4,5\rangle & \langle 2,6,8\rangle & \langle 3,7,9\rangle \\ & \langle 1,5,6\rangle & \langle 2,8,9\rangle & \langle 3,4,7\rangle \\ \text{II}: & \langle 1,7,8\rangle & \langle 2,3,5\rangle & \langle 4,6,9\rangle \end{array}$$

It is then easy to show that there is only one pair of solutions to HDP with which to label the edges of $K_{3,3}$ as described above; one solution coming from Heffter class I and the other from Heffter class II. The resulting orientable cyclic biembeddings of STS(19)s are then found to lie in just eight isomorphism classes. The rotations at 0 of these biembeddings together with an identification of the cyclic systems so biembedded are given in the Table below. They were first listed in [45]. The references to the cyclic STS(19)s, A1, A2, A3, A4, are as given in [72]. The rotation at i is obtained by adding $i \pmod{19}$ to the rotation at 0.

(1)	1	12	10	6	14	16	15	9	2	5	11	18	3	17	7	8	13	4	$A1$	$A3$
(2)	1	8	13	9	2	16	15	6	14	17	7	18	3	5	11	12	10	4	$A1$	$A3$
(3)	1	12	2	16	15	9	7	8	14	17	10	6	11	18	3	5	13	4	$A2$	$A3$
(4)	1	12	2	5	13	9	7	8	14	16	15	6	11	18	3	17	10	4	$A2$	$A3$
(5)	1	8	14	16	15	6	11	12	2	5	13	9	7	18	3	17	10	4	$A2$	$A3$
(6)	1	8	14	17	10	6	11	12	2	16	15	9	7	18	3	5	13	4	$A2$	$A3$
(7)	1	12	2	16	15	6	11	18	3	5	13	9	7	8	14	17	10	4	$A2$	$A4$
(8)	1	8	14	16	15	9	7	18	3	17	10	6	11	12	2	5	13	4	$A2$	$A4$

Table 5.1: Rotations at 0 of the eight STS(19) cyclic biembeddings.

All four cyclic STS(19)s are cyclically biembeddable but none cyclically biembeds with itself. Only STS(19)s from Heffter class I (A1 and A2) may be cyclically embedded with STS(19)s from Heffter class II (A3 and A4). The first of these cyclic biembeddings was also previously given in [70] as well as two further cyclic embeddings of K_{19} corresponding to TTS(19)s.

For $n = 31$, the computations require a computer. There are two cubic bipartite graphs on 10 vertices and they may be obtained from $K_{5,5}$ by either removing a single 10-cycle, or a 6-cycle and a 4-cycle. Fixing the direction at one vertex gives a total of 160 sets of vertex directions in the former case and 128 sets of vertex directions in the latter case which result in complete circuits. There are 64 solutions to HDP for $k = 5$ [22], which lie in eight Heffter classes. Altogether there are 2,408 isomorphism classes of orientable cyclic biembeddings of STS(31)s, involving 76 of the 80 cyclic STS(31)s [26]. Of these classes, 64 are cyclic biembeddings of a system with itself, representing 44 distinct systems. These were first given in [12] and further details of the argument again appear in [15]. As with $n = 19$, only systems from certain pairs of Heffter classes are cyclically biembeddable. The four STS(31)s which are not cyclically biembeddable all come from one particular Heffter class, represented by the difference triples $\langle 1, 5, 6\rangle$, $\langle 2, 10, 12\rangle$, $\langle 3, 13, 15\rangle$, $\langle 4, 7, 11\rangle$, $\langle 8, 9, 14\rangle$. These STS(31)s are not cyclically biembeddable with any STS(31) from any Heffter class. Of course, this does not imply that these four systems have no biembeddings at all.

For $n = 43$, there are 13 cubic bipartite graphs on 14 vertices to consider [77]. Of these, two have edge-connectivity 2, and so cannot have currents assigned along their edges that are different as required by property (iii) above. This is because the current in one of the two edges of the cutset would have to be equal (but opposite in direction) to that in the other. The 11 remaining graphs admit direction and current assignments. Further details are given in [10, 15].

Before leaving this section we remark that [15] gives theoretical reasons, based on the above analysis, why certain pairs of cyclic STS(n)s cannot be cyclically biembedded together in an orientable surface. These are sufficient to give a complete explanation of cyclic biembeddability in orientable surfaces for $n = 19$ and $n = 31$, but not for all $n \equiv 7 \pmod{12}$.

6 Enumeration

Our purpose in this section is to briefly summarize the current state of knowledge about triangular embeddings of the complete graph K_n and hence of embeddings and biembeddings of designs, for small values of n. Specifically we will consider the cases $n = 3$, 4, 7, 12 and 15 for embeddings in an orientable surface and $n = 6$, 7, 9, 10, 12, 13 and 15 for embeddings in a nonorientable surface. We give enumeration results, by which we mean the number of nonisomorphic embeddings of the specified type. Automorphisms include those that, for face 2-colourable embeddings, exchange the colour classes and, in the orientable case, those that reverse the orientation. The first two cases are trivial. The STS(3) has a unique biembedding in the sphere which has automorphism group S_3 of order 6. There is a unique MTS(4), the embedding of which in the sphere is also unique. The automorphism group is S_4 of order 24 and odd permutations reverse the orientation.

The next two cases are less trivial but well-known. There is a unique TTS(6) and its unique embedding in the projective plane is shown below. The automorphism group is $PSL_2(5) \simeq A_5$ realized as $\langle z \mapsto (az+b)/(cz+d),\ a,b,c,d \in GF(5),\ ad-bc = 1\rangle$. This has order 60, the maximum possible, and acts transitively on *flags*, that is ordered triples (v,e,f) where e is an edge incident to vertex v and face f.

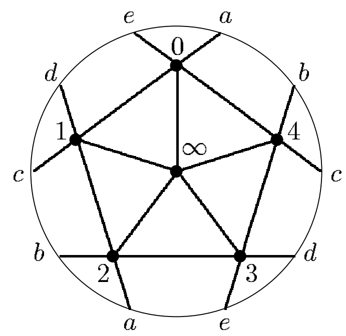

Figure 6.1: Embedding of TTS(6) in the projective plane.

The unique biembedding of the STS(7) in the torus has for its automorphism group the affine linear group AGL(1,7) of order 42. In the realization shown below, this is $\langle z \mapsto az+b,\ a,b \in GF(7),\ a \neq 0\rangle$. The automorphisms of even order exchange the colour classes but preserve the orientation. There is no embedding of the complete graph K_7 in the Klein bottle.

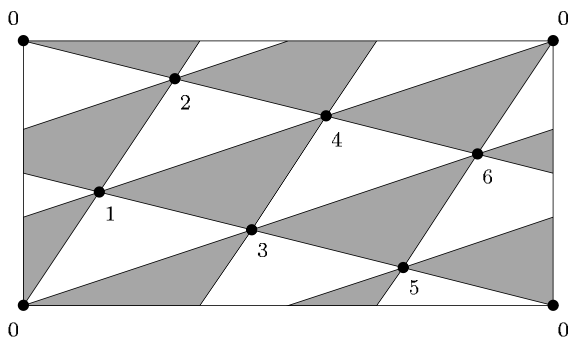

Figure 6.2: Biembedding of STS(7) in the torus.

Triangulations for $n = 9$ and $n = 10$ are necessarily nonorientable. In the former case there are precisely two embeddings. One of these is a biembedding of STS(9)s and has automorphism group $C_3 \times S_3$ of order 18. A realization is obtained by taking one system with block set {012, 345, 678, 036, 147, 258, 048, 156, 237, 057, 138, 246} and the other obtained from this by applying the permutation $\pi = (0\ 1)(2\ 6)(4\ 7)(3)(5)(8)$. In this realization, the permutations π and

(0 6 7)(1 8 4 3 2 5) generate the automorphism group. The automorphisms of even order exchange the colour classes. The other embedding is not face 2-colourable and is the TTS(9) having the following block set {BC0, CA1, AB2, BC3, CA4, AB5, A05, B10, C21, A32, B43, C54, A04, B15, C20, A31, B42, C53, 013, 124, 235, 340, 451, 502}. These embeddings were found by Altshuler and Brehm [6], and rediscovered by Bracho and Strausz [20], from which the given realization is taken. The automorphism group is C_6 of order 6 and is generated by the permutation (0 1 2 3 4 5)(A B C). The two embeddings of K_9 correspond to the twofold triple systems #36 and #35 respectively of the listing of the 36 nonisomorphic TTS(9)s as given in [25]. Using this listing it is not difficult for the reader to verify these results independently by examining the neighbourhood graphs of the systems as explained in Section 2. There are 394 nonisomorphic TTS(10)s without repeated blocks [23]. Of these, precisely 14 can be embedded. Four have trivial automorphism group, four have C_2 and there is one each with groups C_3, C_5, S_3, C_9, A_4 and A_5, [20].

The next two cases to consider are $n = 12$ and $n = 13$. There are 59 nonisomorphic embeddings of MTS(12)s in an orientable surface [5], and 182,200 nonisomorphic embeddings of TTS(12)s in a nonorientable surface [35]. There are two STS(13)s, one is cyclic and the other is not. We will refer to these here as C and N respectively. There are 615 biembeddings of C with C, of which 36 have an automorphism group of order 2 and four an automorphism group of order 3; the rest have only the trivial automorphism. There are 8,539 biembeddings of C with N, of which ten have an automorphism group of order 3 and the rest have only the trivial automorphism. Finally, there are 29,454 biembeddings of N with N, of which 238 have an automorphism group of order 2 and the rest have only the trivial automorphism. In each case, automorphisms of order 2 exchange the colour classes. These results come from [38] and were confirmed in [35] where all 243,088,286 nonorientable triangular embeddings of K_{13} were determined.

The final case which we consider is $n = 15$, and is of particular interest. To quote Ellingham and Stephens, [35], "it is probably infeasible to generate all triangular embeddings of K_{15} in N_{22}" and "it may be possible to generate all orientable embeddings of K_{15} in S_{11}" but "finding them may be a feasible (if still long-term) project on a large many-processor system". But the importance of this case is that $n = 15$ is the smallest value, apart from the well-known cases of the trivial STS(3) and unique STS(7), for which biembeddings of STS(n)s in an orientable surface can be investigated. There are 80 nonisomorphic STS(15)s; a standard numbering and some of their structural properties are given in [72]. They provide a laboratory for experimentation and for framing conjectures. However, before we consider orientable embeddings we first of all deal with the nonorientable case.

In [16], it was shown that every pair of the 80 isomorphism classes of STS(15) may be biembedded in a nonorientable surface. There are precisely three such biembeddings of system #1 with itself and five such biembeddings of system #1 with system #2 [11, 14]. System #1 is the point-line design of the projective geometry PG(3, 2) and system #2 is obtained from system #1 by making a Pasch trade, see Section 7. As a consequence of the results concerning the biembeddings of STS(n)s for $n = 9, 13$ and 15, we believe that there is reasonable evidence to support the following conjecture.

Conjecture 6.1 *Every pair of STS(n)s, $n \equiv 1$ or 3 (mod 6) and $n \geq 9$, can be biembedded in a nonorientable surface.*

Turning to orientable biembeddings of the STS(15)s, we firstly observe that there are precisely three systems having an automorphism of order 5. Each of these systems has a biembedding with itself having an automorphism group of order 10. One of these is the one originally given by Ringel [78], and which can also be obtained from the Bose construction, see Section 3 for details. The other two may be obtained by Ringel's method from index 3 current graphs [13]. In each case an automorphism of order 2 with a single fixed point, exchanges the colour classes but preserves the orientation. In [17] a computer search for biembeddings of the 80 systems, each with itself, was based on examining all possible automorphisms of order 2 having a single fixed point and exchanging the colour classes. As a result, it was shown that 78 of the 80 systems have orientable biembeddings of this type. The exceptions are systems numbered #2 and #79 in the standard listing. In the case of #2, it is further shown in [17] not to have an orientable biembedding with itself. It was also shown that, in the case of #79, any such biembedding can only have the trivial automorphism group. However more recent and, at the time of writing, unpublished work by the present authors and Martin Knor has disposed of this possibility. Hence we can state the following theorem.

Theorem 6.1 *Of the 80 nonisomorphic STS(15)s, 78 have a biembedding with themselves in an orientable surface. The two exceptions which have no such biembedding are #2 and #79 in the standard listing.*

An orientable biembedding of system #79 with system #77 having an automorphism of order 3 is also given in [17] and is the first known example of a biembedding of a pair of nonisomorphic STS(15)s, though of course, as described in Section 5, there are already many known biembeddings of pairs of nonisomorphic STS(n)s for $n = 19$ and $n = 31$.

Again, with Martin Knor, we have established a programme to find further such biembeddings. Of particular interest is whether there exists a biembedding of system #2 with some other system. In fact we have discovered such a biembedding and hence can state another theorem.

Theorem 6.2 *Each of the 80 nonisomorphic STS(15)s has a biembedding with some STS(15) in an orientable surface.*

7 Trades

The concept of a trade is well established in combinatorial design theory. Below we give definitions sufficient for our purposes. A good overview is given in [29] and the listings we make use of appear in [61]. In this section we describe surface trades in triangular embeddings. By this we mean replacing one set of triangular faces with another set that covers the same edges. By applying such trades one may generally move between nonisomorphic embeddings of the same graph. Referring to the constructions presented in Section 4, the replacement of one $K_{n,n,n}$ toroidal bridge by the reversed bridge provides an example of a surface trade. Underlying any such surface trade there is a combinatorial trade on some (possibly partial) twofold triple system. However, the existence of a combinatorial trade amongst the triples formed by a set of triangular faces does not ensure the existence of a corresponding surface trade since applying the trade may transform the surface into a generalized pseudosurface. The geometrical arrangement of the faces is important both for the feasibility of the trade and for questions of orientability.

One may also consider surface trades in the context of the "distance" between different triangular embeddings of a graph G, where distance is defined as the minimum number of faces in which two triangular embeddings of G can differ. We describe below various surface trades which were used in [43] to show that the minimum distance between two different nonorientable triangular embeddings of K_n is at least 4, a number that increases to 6 if one or both of the embeddings is orientable. Moreover, these distances are achievable for some values of n.

A triangular embedding of a graph G, with vertex set V of cardinality n, determines a *partial twofold triple system*, PTTS$(n) = (V, \mathcal{B})$, where $\mathcal{B}$ is the collection of triples of points of V formed by the vertices of the triangular faces; this has the property that every pair of points corresponding to an edge of G appears in precisely two triples (triangular faces of the embedding), but those corresponding to the edges of the complementary graph do not appear in any triple. When G is a complete graph K_n, the resulting PTTS(n) is a TTS(n). A combinatorial trade on a PTTS(n) may be defined as follows.

Suppose that T_1 and T_2 are disjoint sets of triples taken from a finite base set U. If every pair of points of U occurs in the triples of T_1 with precisely the same multiplicity (0, 1 or 2) with which it appears in the triples of T_2, then the pair $\mathcal{T} = \{T_1, T_2\}$ is called a *combinatorial trade*. The *volume* of the trade $\mathcal{T}$, vol$(\mathcal{T})$, is the common cardinality of T_1 and T_2, and the *foundation* of the trade $\mathcal{T}$, found$(\mathcal{T})$, is the set of points of U which appear amongst the triples of T_1 (or T_2).

The rationale for the above definition is that if $P_1 = (V, \mathcal{B}_1)$ is a PTTS(n) whose triples include those of T_1, then by replacing these triples with those of T_2, we form another PTTS(n), $P_2 = (V, \mathcal{B}_2)$ say, and the triples of $\mathcal{B}_1$ and $\mathcal{B}_2$ cover exactly the same pairs of points from V with the same multiplicities.

Now consider the effect of making a trade on an embedding. Suppose that M_1 is a triangular embedding of the simple connected graph G in some surface S and that $P_1 = (V, \mathcal{B}_1)$ is the associated PTTS(n). Further suppose that $\mathcal{T} = \{T_1, T_2\}$ is a trade with found$(\mathcal{T}) \subseteq V$ and that $T_1 \subseteq \mathcal{B}_1$. Put $\mathcal{B}_2 = (\mathcal{B}_1 \setminus T_1) \cup T_2$, so that $P_2 = (V, \mathcal{B}_2)$ is a PTTS(n) covering all the edges of G precisely twice and no other pairs from V. If we now regard the triples from $\mathcal{B}_2$ as triangular faces and sew these faces together along the common edges, then this operation may or may not result in a surface embedding M_2 of G; the reason that the process may fail to yield such an embedding is that the sewing operation may yield a generalized pseudosurface. However, when the operation succeeds in producing a surface embedding, then we say that $\mathcal{T}$ forms a *surface trade* between the embeddings M_1 and M_2 of the graph G.

A variety of interesting questions may be posed concerning trades and embeddings. For example, does every combinatorial trade on a PTTS(n) yield at least one surface trade? Is it possible to characterize those combinatorial trades which, no matter how they lie on the surface, always transform a surface embedding into a surface embedding (rather than into a generalized pseudosurface embedding)? Which surface trades are guaranteed to preserve orientability? How many different surface trades with foundation size less than n must a triangular embedding of K_n possess? And if $b = b(n)$ denotes the minimum integer such that any two triangular embeddings of K_n may be transformed into one another by a trade of volume at most b, how does b vary with n? In order to make progress with such questions it is helpful to have a catalogue of small surface trades.

Apart from the trivial case $G = K_3$, no triangular embedding of a simple connected graph G can give rise to a PTTS(n) with a repeated triple. Furthermore, in this trivial case, it is clear that no trade exists. We may therefore assume that $G \neq K_3$, and that the associated PTTS(n) does not contain any repeated triples. We consider here the case of trades $\mathcal{T}$ on PTTS(n)s with vol($\mathcal{T}$) ≤ 6. Up to isomorphism, there are precisely five such combinatorial trades, one having vol($\mathcal{T}$) $= 4$ and the other four having vol($\mathcal{T}$) $= 6$. These five trades are all given in [61], and it is shown in [18] that there are no further trades $\mathcal{T} = \{T_1, T_2\}$ having vol($\mathcal{T}$) ≤ 6.

The five trades are listed below. The first three have common names as given. In each case T_1 is isomorphic with T_2.

1. (Pasch or quadrilateral trade) $T_1 = \{123, 145, 624, 635\}$,
   $T_2 = \{124, 135, 623, 645\}$.

2. (6-cycle trade) $T_1 = \{123, 145, 167, 834, 856, 872\}$,
   $T_2 = \{134, 156, 172, 823, 845, 867\}$.

3. (Semihead trade) $T_1 = \{127, 136, 145, 235, 246, 347\}$,
   $T_2 = \{126, 135, 147, 237, 245, 346\}$.

4. (Trade-X) $T_1 = \{123, 124, 156, 256, 345, 346\}$,
   $T_2 = \{125, 126, 134, 234, 356, 456\}$.

5. (Trade-Y) $T_1 = \{124, 125, 136, 137, 267, 345\}$,
   $T_2 = \{126, 127, 134, 135, 245, 367\}$.

Surface trades are not new. For example, in Figure 1 of [21], which relates to triangulations of the projective plane, the pair {a, b} gives a geometrical realization of trade-X, the pair {c, d} a realization of a Pasch trade, and the pair {e, f} a realization of a semihead trade. Trade-X represents a sequence of diagonal flips. However, our purpose in this section is to show how one can determine the precise geometrical circumstances in which a surface trade results from a combinatorial trade. We give the details for Pasch trades and we summarize the other cases, leaving the interested reader to consult our joint paper with Bennett, Korzhik and Širáň [18] for further information.

So, consider the possibility of the triangular faces (defined by their vertex triples) 123, 145, 624, 635 of an embedding M being traded with the triangular faces 124, 135, 623, 645 to form an embedding M'. Initially we ignore the question of orientability. At the point 1, and up to reversal, there are two possibilities for the rotation in M, namely

(a) $1:\ 23\cdots45\cdots$ or

(b) $1:\ 23\cdots 54\cdots$,

where $\cdots$ denotes undetermined sections of the rotation.

In M' there are faces 124 and 135, but in case (b) the partial rotations $4\cdots 2$ and $3\cdots 5$ preclude these unless the undetermined sections of these partial rotations are empty, that is, case (b) has the form $1:\ 2354$. In this case M also contains the faces 124 and 135, and so M' would have two copies of each of these faces. So we may exclude case (b). Returning to case (a) and applying similar reasoning at the other vertices shows that the partial rotations in M and in M' at the points $1, 2, \ldots, 6$ are, up to reversals, as shown in Table 7.1. Note also that these partial rotations in M and M' are isomorphic; for example the permutation (3 4) takes one to the other.

M	M'
$1:\ 23\cdots 45\cdots$	$1:\ 24\cdots 35\cdots$
$2:\ 31\cdots 64\cdots$	$2:\ 36\cdots 14\cdots$
$3:\ 12\cdots 56\cdots$	$3:\ 15\cdots 26\cdots$
$4:\ 51\cdots 62\cdots$	$4:\ 56\cdots 12\cdots$
$5:\ 14\cdots 36\cdots$	$5:\ 13\cdots 46\cdots$
$6:\ 24\cdots 35\cdots$	$6:\ 23\cdots 45\cdots$

Table 7.1: Partial Pasch surface trade.

Next consider the question of orientability. Assuming a consistent orientation of M and starting with $1:\ 23\cdots 45\cdots$, we require $2:\ 31\cdots 64\cdots$ and $4:\ 51\cdots 62\cdots$. However, these give respectively $6:\ 42\cdots$ and $6:\ 24\cdots$, contradicting orientability. Therefore a consistent orientation of M, and similarly M', is not possible. Thus a surface trade based on the combinatorial Pasch trade is necessarily between nonorientable embeddings.

We have shown the necessity of Table 7.1 for the existence of a Pasch surface trade, but we have not demonstrated that such a trade exists. In order to do this, take the rows of the partial rotation scheme for M with the undetermined sections eliminated and then determine any resulting non-triangular faces. From each such face, eliminate multiple vertices, if any, by the insertion of additional triangles involving new faces as illustrated below in Figure 7.1, where the twice repeated vertex x is eliminated from the face F by the insertion of new vertices x_1 and x_2.

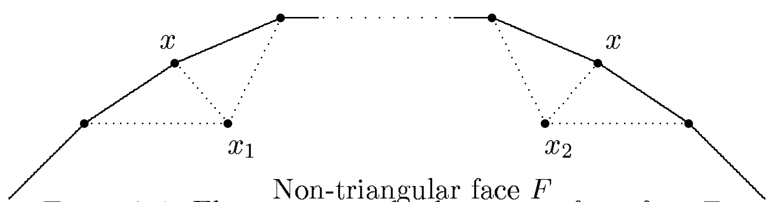

Figure 7.1: Eliminating multiple vertices from face F.

Having completed this elimination, for a non-triangular face without multiple vertices, insert a new vertex into the interior of that face and join it by non-intersecting edges to all the vertices on the boundary, thereby forming a triangular embedding of some simple connected graph.

Application of this algorithm to the case of the Pasch trade given in Table 7.1 give the rotations M and M' as shown below in Table 7.2

M	M'
$1: 23x45y$	$1: 24x35y$
$2: 31y64z$	$2: 36y14z$
$3: 12z56x$	$3: 15z26x$
$4: 51x62z$	$4: 56x12z$
$5: 14z36y$	$5: 13z46y$
$6: 24x35y$	$6: 23x45y$
$x: 1364$	$x: 1364$
$y: 1265$	$y: 1265$
$z: 2354$	$z: 2354$

Table 7.2: Example of a Pasch surface trade.

The same algorithm may be applied to produce examples of other surface trades from partial rotation schemes; it preserves orientability in the sense that if a partial rotation scheme is potentially orientable, then the resulting triangular embedding M will be orientable. This does not, however, ensure that the traded embedding M' is orientable. Also note that it is always possible to render both M and M' nonorientable by gluing on a nonorientable triangular embedding which shares a common face with M and M'.

The results of [18] for all five surface trades having volume at most 6, are summarized in Table 7.3. In the case of Trade-X, every possible geometric realization permits a surface trade. In the case of a face 2-colourable embedding M both Trade-X and Trade-Y necessarily involve both colour classes. The entry “28” against the semihead trade reduces to 19 if we allow M and M' to be exchanged. This only arises for semihead trades because the geometric realizations of the partial rotations in M and M' can be nonisomorphic in this case.

Name	Number of nonisomorphic geometric realizations	Comments
Pasch	1	M and M' are necessarily nonorientable.
6-cycle	4	In one case it is possible for one or both of M and M' to be orientable.
Semihead	28	In one case it is possible for one or both of M and M' to be orientable.
Trade-X	7	In one case it is possible for both M and M' to be orientable, but not to have one orientable and the other nonorientable.
Trade-Y	3	In one case it is possible for one or both of M and M' to be orientable.

Table 7.3: Small surface trades.

Perhaps the most compelling reason for considering surface trades is the possibility of using them to obtain lower bounds of the form 2^{an^2} on the numbers of triangular embeddings of K_n for residue classes not covered by the methods described in Section 4. Such potential use depends on constructing embeddings having a large number of independent trades, possibly using current graphs. So far, at least, we have not been able to implement this strategy.

8 Maximum genus embeddings

Whenever a biembedding of two STS(n)s exists, it represents a minimum genus face 2-colourable embedding of K_n in a surface and hence may be considered to be a minimum genus embedding of each of the two STS(n)s involved. From Euler's formula, in the orientable case the minimum genus is $(n-3)(n-4)/12$ and in the nonorientable case it is $(n-3)(n-4)/6$.

Our focus in this section lies at the opposite extreme, namely on cellular embeddings of Steiner triple systems of maximum genus. To be precise, we seek a face 2-colourable embedding of a complete graph K_n in a surface in which the black faces are triangles and so determine an STS(n), while there is just one white face and the interior of that face is homeomorphic to an open disc. This latter condition ensures that the embedding is cellular and it precludes artificial inflation of the genus by the addition of unnecessary handles or crosscaps. In the orientable case, the corresponding genus is $(n-1)(n-3)/6$, and in the nonorientable case, $(n-1)(n-3)/3$. To avoid trivialities, we shall assume that $n > 3$ and then the single white face, which has $n(n-1)/2$ edges, may be referred to unambiguously as the *large face*. In topological graph theory, graphs which are cellularly embeddable with precisely one face are called "upper-embeddable". By analogy with this usage, we use the term *upper-embedding* for embeddings of STS(n)s of the type just described, appending the qualifier "orientable" or "nonorientable" as appropriate.

By contrast with biembeddings, it is easy to prove that for $n > 3$ every STS(n) has both an orientable and a nonorientable upper-embedding. It is also possible to give detailed results about the possible automorphisms of such embeddings. We represent handles and crosscaps in diagrams as shown in Figure 8.1. The results of this section are taken from our joint paper with Širáň [47].

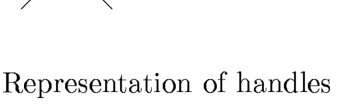

Figure 8.1: Representation of handles and crosscaps.

Theorem 8.1 *Every STS(n) has an orientable upper-embedding.*

Proof The triples of the STS(n) will be represented as black triangles of the embedding. The initial step is to take all of the black triangles containing a fixed point

∞ of the STS(n). From these, one constructs a face 2-coloured planar embedding of a connected simple graph G on n vertices, having for its faces the $(n-1)/2$ black triangles incident with ∞, and one white face. The graph G and its embedding are illustrated in Figure 8.2.

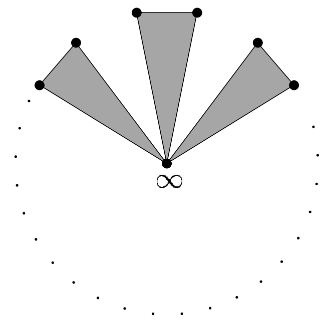

Figure 8.2: The planar embedding of G.

We proceed to add the remaining $(n-1)(n-3)/6$ triples of the STS(n), one at a time, increasing the genus by 1 at each step. Consider at any stage the boundary of the white face. We assume that every point of the STS(n) appears on this boundary at least once. This assumption is certainly true for the initial embedding illustrated in Figure 8.2. If the next triple to be added is $\{u, v, w\}$ then we locate one occurrence of each of these points on the boundary of the white face, add a handle to the white face, and paste on the triangle (u, v, w) (or (u, w, v), depending on the order of the selected points around the white face). This is illustrated in Figure 8.3 which shows the location of the triangle relative to the handle.

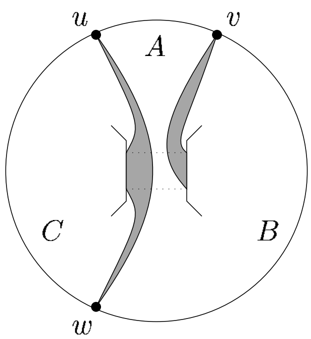

Figure 8.3: Adding a black triangle.

If the points u, v, w originally divided the boundary of the white face into three sections A, B and C, then it is easy to see that after the addition of the black triangle (u, v, w) as shown in Figure 8.3, there still remains just one white face with boundary $A(vw)C(uv)B(wu)$. This face has three more edges than at the previous stage and every point of the STS(n) still appears on the boundary. It is also clear that if the interior of the white face was homeomorphic to an open disc prior to the addition of the black triangle, then it remains so after this addition. $\square$

We remark that it is not necessary to start with the planar embedding specified in the proof. All that is required is a planar embedding of some graph G containing only black triangles from the STS(n) and a single white face, the interior of which is homeomorphic to an open disc, incident with all the points of the STS(n).

Theorem 8.2 *Every STS(n) with $n > 3$ has a nonorientable upper-embedding.*

Proof The proof is identical with that of Theorem 8.1 up to the addition of the final black triangle. This is added to the white face using two crosscaps rather than one handle. Figure 8.4 illustrates this step. For clarity, the edges uv, vw and wu are labelled a, b and c respectively.

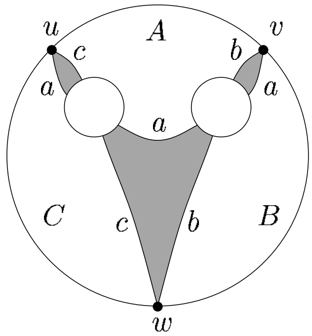

Figure 8.4: Adding the final black triangle.

Using the same notation as in the proof of Theorem 8.1, the boundary of the white face after the addition of the black triangle (u, v, w) is $A(vw)B(vu)\ C(wu)$. The resulting surface has $((n-1)(n-3)/6) - 1$ handles and 2 crosscaps, giving nonorientable genus $(n-1)(n-3)/3$. □

It follows from Theorems 8.1 and 8.2 that for each admissible n, the number of orientable (or nonorientable) upper-embeddings of STS(n)s is at least as great as the number of STS(n)s.

We next give some results about the possible automorphisms of upper-embeddings of STS(n)s. We repeat the assumption that $n > 3$.

Theorem 8.3 *If ϕ is an automorphism of an orientable or nonorientable upper-embedding of an STS(n) then ϕ, represented as a permutation of the points, has one of two forms:*
(a) *ϕ comprises a product of disjoint cycles of equal length, or*
(b) *ϕ comprises a single fixed point together with a product of disjoint cycles of equal length.*
Furthermore, ϕ preserves the direction around the large face, and the common cycle length is odd.

Proof Suppose that ϕ has two fixed points, a and b. Since ϕ must preserve the large face and the edge ab appears somewhere on the boundary of this face, it must fix the points adjacent to the edge ab on this boundary. By repetition of this argument, ϕ fixes every point of the STS(n). Thus ϕ is the identity mapping and so is both of type (a) and type (b). It follows that if ϕ is not the identity mapping then it can have at most one fixed point.

Next suppose that ϕ contains two disjoint cycles of lengths p and q, where $1 < p < q$. Then ϕ^p is an automorphism with p fixed points and a cycle of length at least 2. By the previous paragraph, this is not possible. Hence ϕ must take one of the forms (a) or (b) defined in the statement of the theorem.

Now assume that ϕ has the form (a) and that it reverses the direction around the large face. Clearly ϕ is not the identity. Consider any edge ab which is mapped by ϕ to an edge $a'b'$ appearing on the boundary of the large face as shown in Figure 8.5.

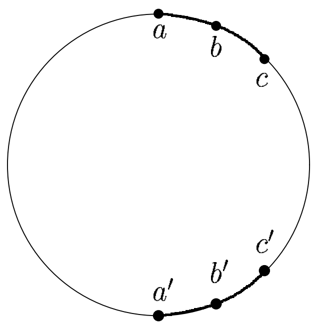

Figure 8.5: Points around the large face.

If c is adjacent to b on this boundary then it must be mapped to c' adjacent to b' as shown in Figure 8.5. Proceeding in this fashion we deduce that $\phi(a') = a$ and, further, that $\phi^2(x) = x$ for every point x of the STS(n). Since ϕ is not the identity and has the form (a), we see that ϕ must be the product of disjoint transpositions, contradicting the fact that n is odd.

Next, assume that ϕ has the form (b) and that it reverses the direction around the large face. Again, ϕ is clearly not the identity. Suppose that ϕ fixes the point ∞ (and no other point). Arguing as before we see that ϕ fixes ∞ and contains $(n-1)/2$ disjoint transpositions. Suppose that three of these are $(a_1\ b_1), (a_2\ b_2)$ and $(a_3\ b_3)$. Consider the edge a_1b_1. Since this edge is stabilized by ϕ, it must appear midway between two successive occurrences of ∞ on the boundary of the large face. But the edge a_2b_2 must also appear midway between the same two successive occurrences of ∞, and the same is true of the edge a_3b_3. Since there are only two midway positions, we have a contradiction. We conclude that ϕ preserves the direction around the large face.

Finally, consider the cycle length. If ϕ has the form (a), then the cycle length is necessarily odd. If ϕ has the form (b) and the cycle length is k, suppose that k is even. Then $\psi = \phi^{k/2}$ is an automorphism which comprises a fixed point and $(n-1)/2$ transpositions. If $(a_1\ b_1)$ is one of these transpositions, then ψ will reverse the direction of the edge a_1b_1 and so fails to preserve the direction around the large face, a contradiction. Thus k must be odd. $\square$

By using arguments based on voltage graphs, more can be said in case (a) of Theorem 8.3. The following result is given in [47].

Theorem 8.4 *If ϕ is an automorphism of an orientable upper-embedding of an STS(n), and if ϕ comprises a product of disjoint cycles of equal length k, then either $k = 1$ (in which case ϕ is the identity permutation) or $k = 3$ (in which case $n \equiv 3 \pmod 6$).*

Direct constructions using voltage graphs are then used in [47] to show that the restrictions described in Theorems 8.3 and 8.4 are, in a sense, best possible. For automorphisms without a fixed point, the following results are obtained.

Theorem 8.5 *If $n \equiv 3 \pmod 6$, then there exists an orientable upper-embedding of an STS(n) having an automorphism that is a product of disjoint 3-cycles.*

Theorem 8.6 *If $n \equiv 1$ or $3 \pmod 6$ and $n > 3$, then every cyclic STS(n) has a nonorientable upper-embedding with a cyclic automorphism. Consequently, if $k|n$, then every cyclic STS(n) has a nonorientable upper-embedding having an automorphism which is the product of disjoint k-cycles.*

For automorphisms with a single fixed point, i.e. case (b) of Theorem 8.3, constructions given in [47] yield the following result.

Theorem 8.7 *Let S be an STS(n) with an automorphism ϕ having a single fixed point and l cycles each of length k, where k is odd and $n = kl + 1$. Then there exist both an orientable and a nonorientable upper-embedding of S having ϕ as an automorphism.*

9 Hamiltonian embeddings

A *Hamiltonian embedding* of K_n is an embedding of K_n in a surface, which may be orientable or nonorientable, in such a way that each face is a Hamiltonian cycle. In a triangular embedding of a complete graph, each face is as small as possible. At the opposite extreme, for every n there exists an embedding of K_n having a single face [32]. Around this single face every vertex appears $n - 1$ times. The problem of constructing Hamiltonian embeddings of K_n is intermediate between the two extremes: the face lengths are as large as possible subject to the restriction that no vertex is repeated on the boundary of any face. In design theory terminology, if the embedding is face 2-colourable then the faces in each colour class form an n-cycle system, in other words a decomposition of the edge set of K_n into Hamiltonian cycles. Whether or not the embedding is face 2-colourable, the complete set of faces forms a twofold n-cycle system.

In a Hamiltonian embedding of K_n the number of faces is $n - 1$. In the nonorientable case Euler's formula gives the genus as $\gamma = (n-2)(n-3)/2$. In the orientable case the genus is $g = (n - 2)(n - 3)/4$, which implies that $n \equiv 2$ or $3 \pmod 4$ is a necessary condition for the embedding. Face 2-colourability requires n to be odd, so that $n \equiv 1$ or $3 \pmod 4$. The recent paper by Ellingham and Stephens [33] established the existence of Hamiltonian embeddings in nonorientable surfaces for $n = 4$ and $n \geq 6$. We summarize their results in sufficient detail to give the flavour, giving a somewhat simpler construction in the case $n \equiv 3 \pmod 4$. We also present a novel construction given by Širáň and ourselves in [48] which produces Hamiltonian embeddings of K_n from triangular embeddings of K_n.

Theorem 9.1 (Ellingham and Stephens) *For $n = 4$ or $n \geq 6$, K_n has a Hamiltonian embedding in a nonorientable surface. Moreover, when n is odd, there is such an embedding that is face 2-colourable. There is no orientable or nonorientable Hamiltonian embedding of K_5.*

Proof First consider the case n even and write $n = 2k + 2$. Take K_n to have vertex set $\mathbb{Z}_{2k+1} \cup \{\infty\}$. Let C_i be the Hamiltonian cycle $(\infty, i, i+1, i-1, i+2, i-2, \dots, i+k, i-k)$. The cycle C_0 is illustrated in Figure 9.1 and C_i is obtained from it by rotating i places clockwise.

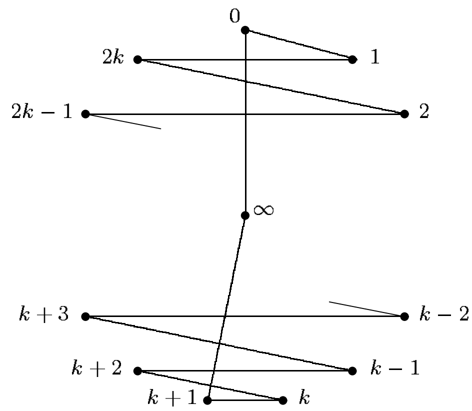

Figure 9.1: The cycle C_0.

The set of $2k + 1$ Hamiltonian cycles $\{C_i : i = 0, 1, \dots, 2k\}$ may be sewn together along common edges to produce a Hamiltonian embedding of K_{2k+2}. To verify this, we compute the rotations at ∞ and i. These are as follows.

$$\begin{array}{ccccccccccccc}
\infty & : & 0 & k & 2k & k-1 & 2k-1 & k-2 & \dots & 2 & k+2 & 1 & k+1 \\
i & : & \infty & i+1 & i+2 & i+3 & i+4 & i+5 & \dots & i-4 & i-3 & i-2 & i-1
\end{array}$$

Since each of these is a single cycle of length $2k + 1$, it follows that the construction produces a Hamiltonian embedding of K_{2k+2}. To see that the embedding is nonorientable for $k \geq 1$, delete the point ∞ and the edges incident with ∞, and examine the boundary of the resulting single face embedding of K_{2k+1}. This has the form

$$(\overbrace{0,1,2k,2,2k-1,\dots,k-1,k+2,k,k+1}^{C_0},\underbrace{k+2,k,k+3,\dots,0,\overbrace{1\ \cdots\cdots}^{C_1}}_{C_{k+1}}\ \cdots\cdots),$$

where the bracings indicate the Hamiltonian cycles from which the sections are derived. Since the edge 01 is encountered twice in the same direction, the embedding cannot be orientable.

Next consider the case $n = 4s + 3$, $s \geq 1$. Take K_{4s+3} to have vertex set $\{\infty, a_0, a_1, \ldots, a_{2s}, b_0, b_1, \ldots, b_{2s}\}$. With subscript arithmetic modulo $2s+1$, let H_i be the Hamiltonian cycle

$$\begin{aligned} H_i &= (\infty a_i b_i b_{2s+i} a_{1+i} a_{2s+i} b_{1+i} b_{2s-1+i} a_{2+i} a_{2s-1+i} b_{2+i} b_{2s-2+i} \cdots \\ &\qquad \cdots a_{s+2+i} b_{s-1+i} b_{s+1+i} a_{s+i} a_{s+1+i} b_{s+i}). \end{aligned}$$

The cycle H_0 is illustrated in Figure 9.2 and H_i is obtained from it by rotating $2i$ places clockwise.

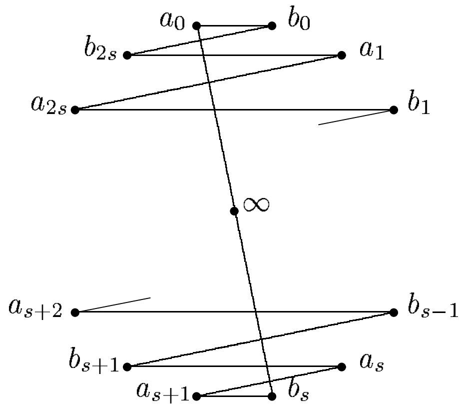

Figure 9.2: The cycle H_0.

A second Hamiltonian decomposition may be formed from this one by applying the mapping $a_j \to a_{j+1}$ $(j = 0, 1, \ldots, 2s)$. This produces Hamiltonian cycles G_i which may be written most conveniently with the cyclic order reversed as

$$\begin{aligned} G_i &= (\infty b_{s+i} a_{s+2+i} a_{s+1+i} b_{s+1+i} b_{s-1+i} a_{s+3+i} \cdots \\ &\qquad \cdots b_{1+i} a_i a_{2+i} b_{2s+i} b_i a_{1+i}). \end{aligned}$$

The set of $4s+2$ Hamiltonian cycles $\{H_i, G_i : i = 0, 1, \ldots, 2s\}$ described above may be sewn together along common edges to produce a Hamiltonian embedding of K_{4s+3}. To verify this, we compute the rotations at ∞, a_i and b_i. These are as follows.

∞	:	a_0	b_s	a_1	b_{s+1}	a_2	b_{s+2}	$\ldots$	a_{2s}	b_{s-1}
a_i	:	∞	b_i	a_{1+i}	b_{1+i}	a_{2+i}	b_{2+i}	$\ldots$	a_{2s+i}	b_{2s+i}
b_i	:	∞	a_{1+i}	b_{2s+i}	a_i	b_{2s-1+i}	a_{2s+i}	$\ldots$	b_{1+i}	a_{2+i}

Since each of these is a single cycle of length $4s+2$, it follows that the construction produces a Hamiltonian embedding of K_{4s+3}. Because each of $\{H_i : i = 0, 1, \ldots, 2s\}$ and $\{G_i : i = 0, 1, \ldots, 2s\}$ is a Hamiltonian decomposition of K_{4s+3}, it also follows that the Hamiltonian embedding is face 2-colourable. To see that the embedding is nonorientable for $s \geq 1$, delete the point ∞ and the edges incident with ∞, and examine the boundary of the resulting single face embedding of K_{4s+2}. This has the form

$$(\overbrace{a_0 b_0 \ldots b_s}^{H_0} \underbrace{\,b_s\, a_{s+2} a_{s+1} b_{s+1} \ldots a_1}_{G_0} \overbrace{a_1}^{H_1} \ldots\ldots\ldots\ldots \overbrace{a_s b_s \ldots b_{2s}}^{H_s} \underbrace{b_{2s}\, a_1 a_0 b_0 \ldots a_{s+1}}_{G_s} \overbrace{a_{s+1}}^{H_{s+1}} \ldots\ldots),$$

where the bracings again indicate the Hamiltonian cycles from which the sections are derived. Since the edge a_0b_0 is encountered twice in the same direction, the embedding cannot be orientable.

For $n = 4s + 1$, Ellingham and Stephens take a similar Hamiltonian decomposition of K_n into cycles H_i, again apply a permutation of the vertices to give a second Hamiltonian decomposition into cycles G_i, and then combine the two decompositions to produce the embedding. The permutation required is somewhat more complicated than that given above for $n = 4s + 3$. By this method, the embedding is certainly face 2-colourable, and it is again easily shown to be nonorientable. For the details, we refer the reader to the original paper [33].

To see that K_5 does not have a Hamiltonian embedding, suppose the contrary. Take the vertices as $0, 1, 2, 3, 4$, and delete the vertex 0 together with edges incident with it to obtain a single-face embedding of K_4 whose face boundary may be taken, without loss of generality, as $(1, a, b, 2, c, d, 3, e, f, 4, g, h)$, where $\{a, b\} = \{3, 4\}, \{c, d\} = \{1, 4\}, \{e, f\} = \{1, 2\}$ and $\{g, h\} = \{2, 3\}$. Since every edge of K_4 must appear twice, it is easy to check that there are precisely four possibilities, all of which lie in one isomorphism class. One of the possibilities for the face boundary is $(1, 3, 4, 2, 4, 1, 3, 2, 1, 4, 3, 2)$. Consideration of the rotation at the vertex 2 shows that this does not produce a surface embedding. □

We next show how Hamiltonian embeddings of K_n may be derived by surface surgery from triangular embeddings of K_n. Such triangular embeddings exist for $n \equiv 0$ or 1 (mod 3); whether the triangular embedding is in an orientable or nonorientable surface is immaterial. To avoid trivial cases we assume that $n \geq 4$. This work comes from our joint paper with Širáň [48].

Construction 9.1

Take a triangular embedding of K_n on the vertex set $\{\infty, a_1, a_2, \ldots, a_{n-1}\}$ and, without loss of generality, take the rotation scheme to have the following form.

$$\begin{array}{ccccccccc}
\infty & : & a_1 & a_2 & a_3 & a_4 & \ldots & a_{n-2} & a_{n-1} \\
a_1 & : & \infty & a_2 & b_{1,1} & b_{1,2} & \ldots & b_{1,n-4} & a_{n-1} \\
a_2 & : & \infty & a_3 & b_{2,1} & b_{2,2} & \ldots & b_{2,n-4} & a_1 \\
\vdots & & & & & & \vdots & & \vdots \\
a_i & : & \infty & a_{i+1} & b_{i,1} & b_{i,2} & \ldots & b_{i,n-4} & a_{i-1} \\
\vdots & & & & & & \vdots & & \vdots \\
a_{n-1} & : & \infty & a_1 & b_{n-1,1} & b_{n-1,2} & \ldots & b_{n-1,n-4} & a_{n-2}
\end{array}$$

where, for each $i = 1, 2, \ldots, n-1$, $(b_{i,1}\ b_{i,2}\ \ldots\ b_{i,n-4})$ is some permutation of $\{a_1, a_2, \ldots, a_{n-1}\} \setminus \{a_{i-1}, a_i, a_{i+1}\}$, with subscript arithmetic modulo $n-1$.

From the n lines of the rotation scheme, create $n-1$ Hamiltonian cycles by discarding the first line and, for each i, replacing the line corresponding to a_i by the cycle $A_i = (\infty a_i a_{i+1} b_{i,1} b_{i,2} \ldots b_{i,n-4} a_{i-1})$. It is easy to see that these cycles form a Hamiltonian decomposition of $2K_n$. The Hamiltonian face corresponding to A_i is formed from the triangular faces that comprise the rotation at a_i in the original triangular embedding, with the triangle $(\infty\ a_i\ a_{i+1})$ removed. It remains to show that these Hamiltonian faces may be sewn together along common edges to produce a Hamiltonian embedding of K_n. In order to prove this, it is only necessary to prove that the resulting rotation about any vertex comprises a single cycle of length $n-1$, rather than a set of shorter cycles with total length $n-1$. This may be done as in the proof of Theorem 9.1, and the details are given in [48]. To consider the question of orientability, delete the point ∞ and the edges incident with ∞ from the embedding to obtain a single face embedding of K_{n-1}. It is then easy to show that an orientable triangular embedding of K_n will, by this construction, produce a nonorientable Hamiltonian embedding of K_n. Although it appears conceivable that a nonorientable triangular embedding might produce an orientable Hamiltonian embedding of K_n for $n \equiv 3, 6, 7$ or 10 (mod 12), we have no examples of this and examination of the boundary of the single face suggests that such situations are likely to be rare. □

An advantage of Construction 9.1 is that it produces a large number of nonisomorphic Hamiltonian embeddings. The following result is proved in [48].

Theorem 9.2 *If there exist M nonisomorphic triangular embeddings of K_n, $n \equiv 0$ or 1 (mod 3), then there exist at least $M/4n^2(n-1)$ nonisomorphic Hamiltonian embeddings of K_n.*

Some easy consequences that follow from this and the results given in Section 4 are as follows.

Corollary 9.3 *For $n \equiv 0$ or 1 (mod 3) there are at least $2^{n/6-o(n)}$ nonisomorphic Hamiltonian embeddings of K_n.*

Proof For $n \equiv 0$ or 1 (mod 3), Korzhik and Voss [67] established that there are at least $2^{n/6-o(n)}$ nonisomorphic triangular embeddings of K_n. The result follows immediately from this and Theorem 9.2. □

Corollary 9.4 *For $n \equiv 1, 3, 7$ or 9 (mod 18) there are at least $2^{n^2/54-o(n^2)}$ nonisomorphic Hamiltonian embeddings of K_n.*

Corollary 9.5 *The constant $1/54$ that appears in the exponent in the preceding corollary may be improved to $2/81$ for $n \equiv 1, 3, 7, 9, 19, 21, 25$ or 27 (mod 54).*

Finally in this section, we mention a further result of Ellingham and Stephens [34] that gives a recursive construction for orientable Hamiltonian embeddings of K_n.

Theorem 9.6 *Suppose that $s \geq 1$ and that K_{4s+2} has an orientable Hamiltonian embedding. Then K_{8s+2} also has an orientable Hamiltonian embedding.*

With the aid of an orientable Hamiltonian embedding of K_{10} found by a computer search, this facilitates the construction of an infinite family of such embeddings. Apart from rumours of an orientable Hamiltonian embedding of K_{30}, and the resulting infinite series, we know of no other orientable cases.

10 Latin squares

The constructions of Section 4 and their generalizations rely on face 2-colourable triangular embeddings of complete tripartite graphs $K_{n,n,n}$. It is therefore of interest to investigate these. Note that the faces in each colour class form a decomposition of $K_{n,n,n}$ into triples and hence a TD$(3,n)$ transversal design or, equivalently, a Latin square of side n. If we adopt a similar definition of biembeddability for Latin squares to that given for Steiner triple systems in Section 2, then a face 2-colourable triangular embedding of $K_{n,n,n}$ may be regarded as a biembedding of two Latin squares of side n. We may reasonably enquire about existence of these for each n, the number of biembeddings for each n, whether every Latin square is biembeddable, and whether every pair of Latin squares of the same size is biembeddable. Much of the material in this Section is taken from our joint papers with Knor and Širáň [46, 39, 40, 42].

The first result, taken from [40], is the equivalence of face 2-colourability and orientability.

Theorem 10.1 *A triangular embedding of $K_{n,n,n}$ is orientable if and only if it is face 2-colourable.*

Proof Suppose that $K_{n,n,n}$ has tripartition $\{A, B, C\}$. If an orientable embedding is given, then triangles with clockwise orientation (A, B, C) may be coloured black and those with clockwise orientation (A, C, B) may be coloured white. Conversely, suppose that a face 2-colourable triangular embedding is given. If a black triangle of the embedding with vertices $a \in A$, $b \in B$, $c \in C$ is oriented, say clockwise, as (A, B, C), then all black triangles incident with a also have clockwise orientation (A, B, C), while the white triangles incident with a have orientation (A, C, B). Since the vertices of these triangles span $B \cup C$, all remaining black triangles have clockwise orientation (A, B, C) and all remaining white triangles have clockwise orientation (A, C, B). It follows that the rotation scheme for the embedding satisfies Ringel's Rule Δ^* (see Section 2) and therefore represents an orientable embedding. □

The existence of orientable triangular embeddings of $K_{n,n,n}$ for every n was established by Ringel and Youngs in [79], and a proof using a voltage graph based on a dipole with n parallel edges embedded in a sphere is indicated by Stahl and White [80]. Generalizing this voltage graph slightly to the one shown in Figure 10.1 gives Construction 10.1.

Construction 10.1

Suppose that $\{a_0, a_1, \ldots, a_{n-1}\} = \{0, 1, \ldots, n-1\}$ and that for $0 \le i \le n-1$, the differences $a_i - a_{i-1}$ are coprime with n, where subscripts are taken modulo n. Then the lift of the embedding M shown in Figure 10.1, with voltages as shown in the group $\mathbb{Z}_n$, gives an embedding of the complete bipartite graph $K_{n,n}$ in an orientable surface in which every face is bounded by a Hamiltonian cycle. If, for each i, a new vertex w_i is placed into that face obtained by lifting the 2-gon with voltages a_i and $-a_{i-1}$, and this new vertex is joined by non-intersecting edges to all the vertices lying on the boundary of that cycle, then a triangular embedding of $K_{n,n,n}$ in an orientable surface is formed. □

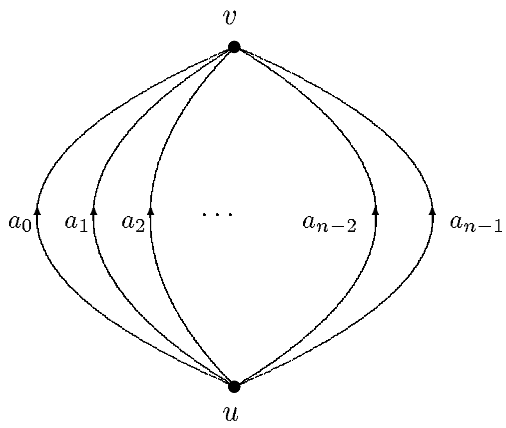

Figure 10.1: Dipole embedded in a sphere.

A careful analysis of possible isomorphisms between embeddings obtained from this construction yields the following growth estimate.

Theorem 10.2 *If n is prime then there are at least $\frac{(n-2)!}{6n}$ nonisomorphic orientable triangular embeddings of the complete tripartite graph $K_{n,n,n}$.*

For a proof see [42] where results are also given for the case when n is not prime.

The particular case of Construction 10.1 when $a_i = i$ for $0 \leq i \leq n-1$ results in one colour class of triangular faces containing all triangles of the form $(u_j v_{j+k} w_k)$ and the other containing all triangles of the form $(u_j v_{j-k+1} w_k)$ for $0 \leq j, k \leq n-1$. The corresponding Latin squares are both copies of the cyclic square

$$C_n = \begin{array}{|ccccc|} \hline 0 & 1 & 2 & \dots & n-1 \\ 1 & 2 & 3 & \dots & 0 \\ 2 & 3 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & 0 & 1 & \dots & n-2 \\ \hline \end{array}$$

Thus, Construction 10.1 asserts, *inter alia*, that for each n the cyclic Latin square C_n is biembeddable with a copy of itself. In fact, as is shown in [42], this embedding is the unique regular triangular embedding of $K_{n,n,n}$ in an orientable surface. By saying that an orientable embedding M of a graph G is *regular*, we mean that for every two *flags*, that is ordered triples (v_1, e_1, f_1) and (v_2, e_2, f_2), where e_i is an edge incident to vertex v_i and face f_i, $1 \leq i \leq 2$, there exists an automorphism of M which maps v_1 to v_2, e_1 to e_2, and f_1 to f_2. Note that this definition requires automorphisms which reverse the global orientation of the surface. A regular embedding has the greatest possible number of automorphisms because the image of any one flag under an automorphism is sufficient to determine the automorphism completely. Thus the total number of automorphisms in a regular orientable triangular embedding of $K_{n,n,n}$ is just the number of flags, which is easily seen to be $12n^2$. Conversely, an orientable triangular embedding M of $K_{n,n,n}$ having $12n^2$ automorphisms must be regular.

This regular embedding may be constructed directly from the Latin square C_n and an isomorphic copy C'_n. Index rows and columns of these squares by the group $\mathbb{Z}_n$ so that the entry in row i, column j of C_n is $C_n(i,j) = i + j$, and then define C'_n by $C'_n(i,j) = i + j - 1$.

To see how these squares are combined to produce the embedding, consider the case $n = 3$, so that

$$C_n = \begin{array}{|ccc|} \hline 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ \hline \end{array} \qquad C'_n = \begin{array}{|ccc|} \hline 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \\ \hline \end{array}$$

Then take the nine points of $K_{3,3,3}$ to be $0_r, 1_r, 2_r, 0_c, 1_c, 2_c, 0_e, 1_e, 2_e$. Black triangles with clockwise orientation (r, c, e), are read from the first square so that, for example, the $(0, 2)$ entry 2 gives the triangle $(0_r 2_c 2_e)$. White triangles with clockwise orientation (r, e, c) are read from the second. The resulting rotation scheme is

$$\begin{array}{llllllll}
0_r: & 0_c & 0_e & 1_c & 1_e & 2_c & 2_e \\
1_r: & 0_c & 1_e & 1_c & 2_e & 2_c & 0_e \\
2_r: & 0_c & 2_e & 1_c & 0_e & 2_c & 1_e \\
0_c: & 0_e & 0_r & 2_e & 2_r & 1_e & 1_r \\
1_c: & 0_e & 2_r & 2_e & 1_r & 1_e & 0_r \\
2_c: & 0_e & 1_r & 2_e & 0_r & 1_e & 2_r \\
0_e: & 0_r & 0_c & 1_r & 2_c & 2_r & 1_c \\
1_e: & 0_r & 1_c & 1_r & 0_c & 2_r & 2_c \\
2_e: & 0_r & 2_c & 1_r & 1_c & 2_r & 0_c
\end{array}$$

Returning to the general case, this biembedding has n^2 automorphisms of the form $\phi_{\alpha,\beta} : (i_r, j_c, k_e) \to ((i + \alpha)_r, (j + \beta)_c, (k + \alpha + \beta)_e)$, and these all preserve the colour classes, the orientation, and the rows, columns and entries. In addition, the mapping $\chi : (i_r, j_c, k_e) \to (i_c, -j_e, -k_r)$ gives an automorphism of order 3 which permutes rows, columns and entries, but preserves the colour classes and the orientation. The mapping $\mu : (i_r, j_c, k_e) \to (i_c, j_r, k_e)$ gives an automorphism of order 2 which preserves the colour classes but reverses orientation, and the mapping $\nu : (i_r, j_c, k_e) \to (-i_c, -j_r, (-k - 1)_e)$ gives an automorphism of order 2 which reverses the colour classes but preserves the orientation. It follows that the group of automorphisms generated by these mappings has order at least $12n^2$. Since this is the maximum possible order, we deduce that this group is the full automorphism group of the biembedding and that the biembedding is regular.

A useful feature of the cyclic Latin square is that for odd values of n it contains a transversal and hence any associated biembedding contains a parallel class of triangles in the corresponding colour class. In particular, for odd n, the regular biembedding has a parallel class of triangles in each colour. A parallel class in one colour is required for the $K_{3,3,3}$ bridges used in the recursive constructions for biembeddings of Steiner triple systems in Section 4, and for the $K_{n,n,n}$ bridges used in generalizations of these constructions. There is a similar recursive construction for Latin squares first given in [46] which we now present and which enables us to give lower bounds on the numbers of biembeddings of Latin squares in certain cases.

Construction 10.2

Take any biembedding of two Latin squares of side n in a (necessarily orientable) surface S. Next take m copies of the given biembedding on m disjoint surfaces $S^0, S^1, \ldots, S^{m-1}$. We use superscripts in a similar way to identify corresponding points on these surfaces. We attempt to join these surfaces together to produce a biembedding of Latin squares of side mn. To do this we will use as bridges biembeddings of Latin squares of side m. So let T denote the bridging surface supporting such an embedding, say M, and assume that the graph $K_{m,m,m}$ embedded in T has vertex parts $\{a^i\}, \{b^i\}$ and $\{c^i\}$ and that the embedding has black faces $(a^i c^i b^i)$ for $i = 0, 1, \ldots, m-1$. *Note this requires M to have a parallel class of black triangles.*

For each white triangular face (uvw) in S we bridge $S^0, S^1, \ldots, S^{m-1}$ using a copy of M, obtained by renaming a^i, b^i and c^i as u^i, v^i and w^i respectively. The black face $(u^i w^i v^i)$ from the copy of M is glued to the white face $(u^i v^i w^i)$ in S^i.

It is now routine to check that the resulting embedding represents a biembedding of two Latin squares of side mn, that is a triangular embedding of $K_{mn,mn,mn}$ in an orientable surface. □

As with the constructions of Section 4, certain generalizations are possible. We may use alternative bridges provided they all have a common parallel class of black triangles having the same orientation. Likewise, we may vary the embeddings in the surfaces S^i provided that they all have the same white triangles with the same orientations. Reapplication of the construction may also be possible in certain circumstances. For reasons of space we cannot present all the ramifications here. However the following points are worthy of remark as they produce large lower bounds for the number of biembeddings in many cases. For further details see [46].

Remark Take Construction 10.2 with $m = 3$, and use as bridges the two differently labelled $K_{3,3,3}$ embeddings given in Section 4. Since a face 2-colourable triangular embedding of $K_{n,n,n}$ has n^2 white faces, varying the bridges gives 2^{n^2} differently labelled embeddings of $K_{3n,3n,3n}$. Replacing $3n$ by n, we may express this by saying that there are at least $2^{n^2/9}$ differently labelled orientable triangular embeddings of $K_{n,n,n}$ for $n \equiv 0 \pmod 3$. Since the maximum possible size of an isomorphism class is $6(n!)^3$, this gives a lower bound of $2^{n^2/9-o(n^2)}$ for the number of nonisomorphic biembeddings of Latin squares when $n \equiv 0 \pmod 3$.

Remark In view of the previous remark, it is clearly useful to have a large supply of differently labelled orientable triangular embeddings of $K_{m,m,m}$, all having a common oriented parallel class of triangular faces in one of the two colour classes. So, on the assumption that one such embedding, say M, exists, apply to it all permutations which fix this parallel class, including its orientation, and which preserve the tripartition. There are $3(m!)$ such permutations. Suppose that π is one of these permutations and that π is also an automorphism of M. Since π preserves the orientation, the parallel class and the tripartition, π is determined by the image of any single vertex. Consequently, there are at most $3m$ such permutations π. It follows that, provided one such embedding exists, there are at least $3(m!)/3m = (m-1)!$

differently labelled orientable triangular embeddings of $K_{m,m,m}$ all having a common oriented parallel class of triangular faces in one of the two colour classes. Hence for m odd there are at least $((m-1)!)^{n^2}$ differently labelled orientable triangular embeddings of $K_{mn,mn,mn}$.

The same bound also holds for those even values of m for which there exists a biembedding of two Latin squares of side m, at least one of which has a transversal. Such biembeddings do not exist for $m = 2$ and $m = 4$, but they do exist for $m = 6$ and $m = 8$ and, in the light of the computational results described below, it would be surprising if they did not exist for all even $m \geq 10$.

The failure of the construction method for $m = 2$ and $m = 4$ is not quite the end of the story. We have one more construction which is new but similar to Construction 10.2. It takes a biembedding of Latin squares of side n and produces a biembedding of Latin squares of side $2n$. The notation is similar to the previous case.

Construction 10.3

Take any biembedding of two Latin squares of side n in a surface S. Next take two copies of the given biembedding on disjoint surfaces S^0 and S^1 with the colour classes on S^1 reversed so that a white triangle $(u^0v^0w^0)$ in S^0 corresponds to a black triangle $(u^1v^1w^1)$ in S^1. The bridges are formed from copies of a face 2-colourable embedding M of $K_{2,2,2}$ in a sphere having vertex parts $\{a^0, a^1\}, \{b^0, b^1\}, \{c^0, c^1\}$, a black face $(a^0c^0b^0)$ and a white face $(a^1c^1b^1)$. For each white triangular face (uvw) in S we bridge S^0 and S^1 using a copy of M, obtained by renaming a^i, b^i and c^i as u^i, v^i and w^i respectively. The black (respectively white) face $(u^iw^iv^i)$ from the copy of M is glued to the white (respectively black) face $(u^iv^iw^i)$ in S^i.

Again it is now routine to check that the resulting embedding represents a biembedding of two Latin squares of side $2n$. □

We next turn our attention to some computational results. Again for reasons of space, we must merely summarize these, pointing out what appear to be interesting features. Fuller details are given in [40]. When we speak of the number of Latin squares of side n, we refer to the number of *main classes*, that is the number of nonisomorphic TD$(3, n)$ designs. A representative of each main class for $n = 4, 5, 6$ and 7 is given in [24].

Firstly, for each of $n = 1, 2$ and 3 there is only one Latin square of side n and one biembedding. For $n = 4$ there are two Latin squares of side n, but only one biembedding which, from above, is the regular biembedding of the cyclic square with a copy of itself. The other Latin square of side 4 is the Cayley table of the Klein 4-group. This is not biembeddable, either with itself or the cyclic square as can be easily shown. Let the Latin square be given by

$L_1 =$

	4	5	6	7
0	8	9	X	Y
1	9	8	Y	X
2	X	Y	8	9
3	Y	X	9	8

For clarity we represent the rows, columns and entries by different symbols. Without loss of generality it can be assumed that the rotation about the point 8 is

$$8: \ 0\ 4\ 1\ 5\ 2\ 6\ 3\ 7$$

This determines the coordinates of the entry 8 in the Latin square, say L_2, with which we are attempting to biembed L_1, namely (row, column) = (0, 7), (1, 4), (2, 5) and (3, 6). Now the only way of completing row 0 and column 4 of the Latin square L_2 without the rotation about either the point 0 or the point 4 not being a complete cycle is as follows.

$L_2 =$

	4	5	6	7
0	X	Y	9	8
1	8			
2	Y	8		
3	9		8	

But now it is impossible to place any entry in the (3, 5) position.

There are two Latin squares of side 5 and three biembeddings, but these biembeddings all involve two copies of the cyclic square, and the other square is not biembeddable. For $n = 6$ there are 12 Latin squares and 29 biembeddings. The Latin squares of side 6 numbered 3, 4, 7 and 10 in the listing of [24] do not feature in any of the 29 biembeddings, but the remaining eight squares each have a biembedding with a copy of themselves. For $n = 7$ there are 147 Latin squares and 23,664 biembeddings of which 4,761 are biembeddings of a Latin square with itself. However, although every Latin square of side 7 features in some biembedding, several do not biembed with themselves. But perhaps the most interesting feature of these biembeddings is that it is possible to partition the set of 147 squares into 16 subsets, of cardinalities 1, 1, 1, 2, 3, 3, 3, 6, 6, 8, 8, 9, 18, 19, 26 and 33 respectively, so that within each subset most squares biembed with most squares, and no two squares from different subsets biembed. More details of this partition appear in [40]. This bizarre partitioning, which also occurs for the Latin squares of side 6 although in not such a startling fashion, is wholly unexplained. It may just be a feature for small values of n but it may be more general and have a deeper significance. It also suggests that some form of surface trade may be involved.

From the previous paragraph it will be seen that there are six Latin squares, one each of sides 4 and 5, and four of side 6, that do not feature in any biembeddings. These include, as well as the Cayley table of the Klein 4-group, that of the non-Abelian group of order 6, #7 in the listing of [24]. It is an interesting question whether these squares are the only ones with this property. In an attempt to answer this question, with Martin Knor we have looked at those Latin squares of side 8 that come from the Cayley tables of the five groups of order 8. One of these groups is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and another is $\mathbb{Z}_4 \times \mathbb{Z}_2$, both of which might be considered as close relatives of the Klein 4-group ($= \mathbb{Z}_2 \times \mathbb{Z}_2$). The two non-Abelian groups of order 8, namely the dihedral group $\mathcal{D}_4$ and the quaternion group $\mathcal{Q}$ are also of interest. However, we have found that each of the resulting five Latin squares biembeds and we know of no further cases of non-biembeddable Latin squares. In examining the biembeddings of these five squares of side 8, we find that, apart from the cyclic square, these biembeddings never contain two copies of the same square.

11 Symmetric configurations

The term "configuration" is nowadays used rather loosely; it has come to refer to any fixed small number of blocks which form part of a design. In this section we revert to the original meaning and define an (n_r, b_k) *configuration* to be an incidence structure of n points and b lines such that

1. each line contains k points,
2. each point lies on r lines,
3. two different points are connected by at most one line.

If $b = n$, and therefore $r = k$, the configuration is said to be *symmetric* and denoted by n_k. Our interest, in the case where $k = 3$, is in the problem of biembedding a pair of symmetric configurations of triples in a closed surface. The embedded graph is the incidence graph of each of the two configurations, where two vertices are joined by an edge if they occur together in some triple. This graph is 6-regular and, by Euler's formula, the supporting surface must be either the torus or the Klein bottle. Examples of symmetric configurations are the Fano plane or STS(7), which is the unique 7_3 configuration, and the Pappus and Desargues configurations which are 9_3 and 10_3 configurations respectively. Already in the nineteenth century enumeration results of n_3 configurations were available for small values of n. Kantor [59] showed that there is one 8_3, three 9_3 and ten 10_3 configurations and Martinetti [71] extended this catalogue by enumerating all 31 11_3 configurations.

We now have a sequence of questions concerning biembeddings of n_3 configurations which are analogous to those asked at the end of Section 2 in relation to Steiner triple systems.

1. Given an n_3 configuration, does it have a biembedding with some other n_3 configuration in the torus, the Klein bottle or both? In particular for each $n \geq 7$, is there an n_3 configuration which has such a biembedding in one or the other or both of the surfaces?
2. Given a pair of n_3 configurations do they have a biembedding in the torus, the Klein bottle or both?
3. If such biembeddings exist, how many are there?

An answer to these questions in the case of the torus was provided by Altshuler [4] and then for both the torus and the Klein bottle by Negami [74, 75]. But all three papers are written from a different viewpoint; the term "configuration" is not mentioned at all. In each case, the problem of biembedding symmetric configurations is related to the classification of which 6-regular graphs have a triangulation in the torus or the Klein bottle (or both). Negami refers to these as 6-regular *toroidal graphs* and 6-regular *Klein bottlal graphs* respectively, but for the latter we use the term *Klein bottleable graphs*. The simpler terms "torus graph" and "Klein bottle graph" might be thought preferable, but these are used by Negami to describe embeddings rather than graphs and it would be confusing for us to use them for a different purpose. In the main, our account and notation follows that given in [74].

Considering first triangulations of the torus, we define *the standard 6-regular triangulation* $T(p,q,r)$ of the torus. To do this consider the triangulation, shown in Figure 11.1, of the domain

$$\{(x,y) \in R^2 : \ 0 \le x \le r, \quad 0 \le y \le p\},$$

where p and r are positive integers.

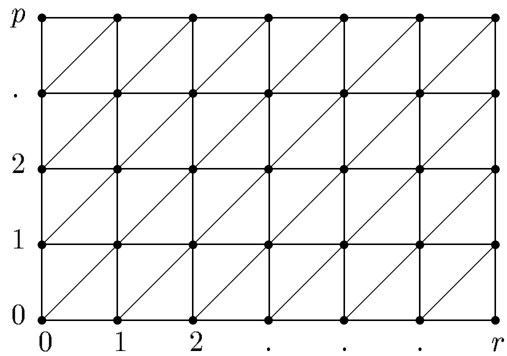

Figure 11.1: Triangulation of $\{(x,y) \in R^2 : \ 0 \le x \le r, \quad 0 \le y \le p\}$.

In order to convert this into a triangulation of the torus, first identify the upper and lower sides of the rectangle in the usual way to form an open-ended cylinder. The embedded graph of this triangulation we denote by H_r^p and we make use of this again when considering embeddability in the Klein bottle. Now glue one of the boundaries of the cylinder to the other so that the point $(0,y)$, $0 \le y \le p$ coincides with the point (r,y'), $0 \le y' \le p$ if $y - y' \equiv q \pmod p$, where q is an integer satisfying $0 \le q < p$. Informally we make a "twist" in the cylinder before gluing the two boundaries. This procedure defines the standard triangulation $T(p,q,r)$. Note that $T(p,q,r)$ is face 2-colourable and that a rotation of the diagram by π gives an isomorphism between the face sets of the two colour classes. For our purposes, the main result in both [4] and [74] is the following theorem.

Theorem 11.1 *If G is a 6-regular toroidal graph and M is an embedding of G in the torus, then M is isomorphic to some standard triangulation $T(p,q,r)$.*

We remark that different ordered triples (p,q,r) and (p',q',r') can lead to isomorphic triangulations. For example, as shown in [74], $T(p,q,r)$ is isomorphic to $T(p,q',r)$ if $q' \equiv -(q+r) \pmod p$. Also, the embedded graph of $T(p,q,r)$ need not be simple, although Negami identifies those which are not. He also goes on to prove that if G is a simple 6-regular toroidal graph, then the embedding is unique up to isomorphism.

To determine if an n_3 configuration has a biembedding in the torus, it therefore suffices to decide if its incidence graph is isomorphic to the embedded graph of some

$T(p,q,r)$. If this is the case, then the biembedding exists. If it is not the case, then the configuration has no biembedding in the torus. When the biembedding exists, it is unique and the two biembedded configurations are isomorphic.

Also in [74], Negami lists the isomorphism classes for standard triangulations $T(p,q,r)$ on fewer than 15 vertices. For 11 vertices or less, those with simple embedded graphs comprise $T(n,2,1)$, $7 \le n \le 11$ together with $T(3,0,3)$. In general, $T(n,2,1)$ is the biembedding of the cyclic symmetric configuration on the base set $\{0,1,2,\ldots,n-1\}$ generated from the triple $\{0,2,3\}$ under the action of the mapping $z \mapsto z+1 \pmod{n}$, and the two colour classes that result are isomorphic under $z \mapsto -z \pmod{n}$. The particular case $T(3,0,3)$ is the biembedding of the Pappus configuration with a copy of itself. It follows that the unique 7_3 and 8_3 configurations, two of the three 9_3 configurations and one of each of the ten 10_3 and 31 11_3 configurations are biembeddable in the torus, and that the remaining configurations on 11 vertices or less are not. Further analysis shows that for the 12_3, 13_3 and 14_3 configurations respectively, four of 229, two of 2,036 and two of 21,399 are biembeddable in the torus, and the remainder are not. The classification also implies that any connected cyclic symmetric configuration n_3 has a unique biembedding with an isomorphic copy of itself in the torus. (Here "connected" means that the incidence graph is connected.) This is because the incidence graph of such a configuration is isomorphic to the embedded graph of $T(p,q,r)$ for some values of p,q,r. An alternative and purely combinatorial proof of this result appears in [41].

Turning now to biembeddings of symmetric configurations n_3 in the Klein bottle, the classification of which 6-regular graphs have triangulations in this surface is given in [75]. This paper is a preprint and seems not to have been published in a journal. But the results are both important and interesting and deserve to be better known. We describe the relevant graphs beginning with H_r^p defined above. This has $p(r+1)$ vertices, those vertices with coordinates $(0,j)$ or (r,j) for $0 \le j \le p-1$ have degree 4, but all other vertices have degree 6. From the graph H_r^p and its cylindrical embedding, two families of triangulations of the Klein bottle may be constructed.

The first of these is achieved by identifying, for each y, $0 \le y \le p$, the points with coordinates $(0,y)$ and $(r,p-y)$. These embeddings are called *Klein bottle triangulations of handle type* and denoted by $Kh(p,r)$.

The construction of the second family of triangulations depends on the parity of p. Again referring to H_r^p, if $p=2m$ is even, identify the point $(0,y)$ with $(0,y+m)$ and the point (r,y) with $(r,y+m)$, $0 \le y \le m$. If $p=2m+1$ is odd, use the graph H_{r-1}^p and join the point $(0,y)$ to $(0,y+m)$ and the point $(r-1,y)$ to $(r-1,y+m)$, $0 \le y \le p$, with arithmetic on the second coordinate modulo p. In this second case, when p is odd, the Klein bottle is formed by placing the additional joins across two crosscaps. The resulting triangulations, for p even or odd, are called *Klein bottle triangulations of crosscap type* and denoted by $Kc(p,r)$.

In both $Kh(p,r)$ and $Kc(p,r)$ the number of vertices is pr and the two families of triangulations are distinct. The classification theorem given in [75] is now as follows.

Theorem 11.2 *If G is a 6-regular Klein bottleable graph and M is an embedding of G in the Klein bottle, then M is isomorphic to precisely one of $Kh(p,r)$, $p \ge 3, r \ge 3$ or $Kc(p,r)$, $p \ge 5, r \ge 2$.*

As with the toroidal graphs, Negami proves that the triangular embedding of any 6-regular Klein bottleable graph is unique. In fact, the triangulations $Kh(p,r)$ are face 2-colourable while the triangulations $Kc(p,r)$ are not. So, in seeking the answer to the question of biembeddability of an n_3 configuration in the Klein bottle, it is only necessary to determine whether or not its incidence graph is isomorphic to the embedded graph of some $Kh(p,r)$. As in the toroidal case, there is an isomorphism between the face sets of the two colour classes of $Kh(p,r)$. It remains to consider the question of whether any symmetric configuration can be biembedded in both the torus and the Klein bottle. This is not so and follows from the fact that none of the embedded graphs of $T(p,q,r)$ triangulations are isomorphic to any of those of $Kh(p,r)$ triangulations. An alternative and perhaps simpler proof, which does not rely on the above classification, is given in [69].

Combining the results for the torus and the Klein bottle, we have the following theorem.

Theorem 11.3 *A symmetric configuration n_3 is biembeddable in the torus if and only if its incidence graph is isomorphic to the embedded graph of some $T(p,q,r)$. It is biembeddable in the Klein bottle if and only if its incidence graph is isomorphic to the embedded graph of some $Kh(p,r)$, $p \geq 3, r \geq 3$. Any such biembedding is unique and the two n_3 configurations that appear in the biembedding are isomorphic. No n_3 configuration has a biembedding in both the torus and the Klein bottle.*

The third 9_3 configuration which is not biembeddable in the torus corresponds to $Kh(3,3)$ and is therefore biembeddable in the Klein bottle.

Perhaps some readers may feel it is somewhat unsatisfactory that the answer to the question of the biembeddability of symmetric configurations is given in terms of whether their incidence graphs are isomorphic to any of the embedded graphs from $T(p,q,r)$ or $Kh(p,r)$. But this is a situation in which a design-theoretic problem can be successfully attacked by methods of topological graph theory. This is in contrast to Section 3, where the existence of an orientable triangulation of the complete graph K_n, $n \equiv 3 \pmod{12}$, was determined by exclusively design-theoretic methods and shows the interplay between the two areas.

Finally in this section we mention the work of White and in particular the papers [37, 82]. As the titles imply the emphasis here is on finding topological models of configurations on appropriate surfaces. The biembedding of the Pappus configuration with itself in the torus appears explicitly in these papers as well as an embedding of the Desargues configuration in the double torus.

12 Concluding remarks

In this final section, we review some open problems and briefly discuss other work in this area. We begin with the questions 2 to 4 posed at the end of Section 2, which we consider in reverse order.

The results given in Section 6 show that not every pair of Steiner triple systems of order $n = 15$ has an orientable biembedding, and it seems possible that similar nonexistence results may apply to all $n \equiv 3$ or $7 \pmod{12}$ with $n \geq 15$. However, for $n = 15$, the situation regarding nonorientable biembeddings is, as we described, quite different, with every pair of STS(15)s having at least one biembedding. This led us to make Conjecture 6.1 that every pair of STS(n)s, $n \equiv 1$ or 3 (mod 6) and $n \geq 9$, has at least one nonorientable biembedding. A proof of this conjecture would represent a major step forward.

Confining our attention to the orientable case, we know that the STS(7) and all 80 STS(15)s have minimum genus embeddings. Does every STS(n), $n \equiv 3$ or 7 (mod 12) have such an embedding, necessarily a biembedding? We think that the answer is likely to be in the affirmative though it may be a very difficult result to prove. But we did show in Section 8 that every STS(n) has a maximum genus embedding in which the black faces are triangles corresponding to the triples of the STS(n) and there is just one white face. An intermediate result where the black faces are triangles and there are $(n-1)/2$ white faces, all of which are Hamiltonian cycles, might be of interest.

The theorems of Section 4 give, for n lying in certain residue classes, a lower bound of the form 2^{an^2} for the number of biembeddings of STS(n)s in both orientable and nonorientable surfaces. What is the true order of magnitude of this number? We can obtain a crude upper estimate by using the known upper bound for the number of labelled Steiner triple systems of order n, namely $(e^{-1/2}n)^{n^2/6}$ [83]. It follows easily from this fact that, in both the orientable and nonorientable cases, the number of nonisomorphic biembeddings is less than $n^{n^2/3}$.

If it were the case that each pair of STS(n)s has a biembedding, then we could obtain a lower bound for the number of nonisomorphic biembeddings in a similar fashion, since the number of such pairs is at least $n^{n^2/3-o(n^2)}$. So, if the rate of growth of the number of nonisomorphic biembeddings were really of the order 2^{an^2} then this would imply that almost all STS(n)s are <u>not</u> biembeddable either orientably or nonorientably. Conjecture 6.1, based on the STS(15) data, therefore constrains us to the view that the correct rate of growth in the number of biembeddings is $n^{n^2/3-o(n^2)}$, at least in the nonorientable case.

Whatever the true rate of growth for biembeddings (that is, face 2-colourable triangulations of K_n), one would expect to see similar and related growth estimates for the number of minimum genus embeddings of K_n for all residue classes.

Turning now to other problems associated with biembeddings of pairs of STS(n)s, we showed in Section 3 how certain Steiner triple systems obtained from the Bose construction can be biembedded. Specifically, the groups used are cyclic. In the orientable (respectively nonorientable) cases can the result be generalized to any Abelian group of order $4s+1$ (respectively $2s+1$)? The Bose construction itself has a number of generalizations. In the version given in Section 3, the group G is used to construct a commutative idempotent quasigroup with operation $*$ defined by $i * j = (i+j)/2$. But there are many other such quasigroups. Some of these generalizations may have topological implications.

With regard to the cyclic biembeddings described in Section 5, it seems likely that infinitely many pairs of cyclic STS($12s+7$)s do not biembed cyclically in an orientable surface. Indeed, there may be infinitely many cyclic STS($12s+7$)s that do not appear in any orientable cyclic biembedding. It seems somewhat more likely that, possibly with finitely many exceptions, each such pair biembeds cyclically in a nonorientable surface.

Most of the work surveyed in this paper has been concerned with embeddings of various kinds of triple system. An exception is Section 9 where embeddings of the complete graph K_n in which each face is a Hamiltonian cycle are considered. Theorem 9.1 gives a complete solution to the existence of such embeddings in the case of a nonorientable surface. However, the existence question for orientable surfaces is far from settled. But more generally, one could consider embeddings of K_n in which all the faces are cycles of any constant length. The logical place to begin would be with quadrangulations. The necessary condition for a quadrangulation of the complete graph K_n in a nonorientable surface is $n \equiv 0$ or 1 (mod 4) and in an orientable surface is $n \equiv 0$ or 5 (mod 8). In two papers [55, 56], Hartsfield and Ringel construct such embeddings for $n \equiv 1$ (mod 4) in the former case and $n \equiv 5$ (mod 8) in the latter. The necessary and sufficient condition for a 4-cycle system, that is a decomposition of K_n into 4-cycles, is $n \equiv 1$ (mod 8). Thus any biembedding of a pair of 4-cycle systems would necessarily be in a nonorientable surface.

In the case of Latin square biembeddings, face 2-colourability is equivalent to orientability. The results given in Section 10 show that not every pair has a biembedding, and it seems likely that there are infinitely many such pairs. However, it may be the case that all but a finite number of Latin squares appear in some biembedding. In fact, we may already have identified all the exceptional non-biembeddable Latin squares; one each of side 4 and side 5 and four of side 6. But again it may be difficult to prove that every Latin square, apart from these six exceptions, has a biembedding. However we do know that every Latin square which is the Cayley table of a cyclic group is biembeddable. Does this result extend to the Cayley table of any group, apart from $\mathcal{K}_4$ and $\mathcal{D}_3$? Our computational results concerning groups of order 8 suggest that it might, though these Latin squares do not have biembeddings with isomorphic copies of themselves, unlike the situation with the cyclic groups. Other classes of Latin square which would be of particular interest are the composition tables of Steiner quasigroups and Steiner loops, defined respectively as follows. Let $(V, \mathcal{B})$ be an STS(n). Define on V an operation $*$ by $x * x = x,\ x \in V$ and $x * y = z$ if $\{x, y, z\} \in \mathcal{B}$. Then $(V, *)$ is a *Steiner quasigroup* or *squag*. Alternatively define on $V \cup \{e\}$ an operation $\circ$ by $x \circ x = e,\ e \circ x = x \circ e = x,\ x \in V \cup \{e\}$ and $x \circ y = z$ if $\{x, y, z\} \in \mathcal{B}$. Then $(V, \circ)$ is a *Steiner loop* or *sloop*. Does the Latin square composition table of every squag or sloop have a biembedding? Finally one can make estimates and conjectures concerning the growth rate for the number of biembeddings of Latin squares and these have similar forms to those described above for Steiner triple systems.

Concerning symmetric configurations, we know that an n_3 configuration can only biembed with itself and that if it does then the biembedding is unique. But relatively few symmetric configurations seem to have such minimum genus embeddings in the torus or the Klein bottle. Possibly other higher genus embeddings such as the one mentioned of the Desargues configuration in the double torus would be interesting.

Our survey has been concerned with embeddings, usually triangulations, of graphs in surfaces. But some of the ideas can be extended to pseudosurfaces. We follow [81] in making the definitions. A *pseudosurface* is the topological space which results when finitely many identifications of finitely many points each, are made on a given surface. More precisely, distinct points $\{p_{i,j} : i = 1, 2, \ldots, k,\ j = 0, 1, \ldots, m_i\}$ on a given surface are identified to form points $p_i = \{p_{i,j} : j = 0, 1, \ldots, m_i\},\ i = 1, 2, \ldots, k$ called *singular points* or *pinch points*. The number m_i is the *multiplicity* of the pinch point p_i. It is at these pinch points that a pseudosurface fails to be a

2-manifold. A *generalized pseudosurface* is the connected topological space which results when finitely many identifications of finitely many points each, are made on a topological space of finitely many components each of which is a pseudosurface. The points subject to such identifications are also called pinch points and their multiplicities are defined in the obvious way.

The relationship between twofold triple systems and generalized pseudosurfaces is given in [3]; there is a one-to-one correspondence between TTS(n)s and triangular embeddings of the complete graph K_n in generalized pseudosurfaces. The correspondence is explored in greater depth in [68], where details of the generalized pseudosurfaces associated with twofold triple systems on 10 or less points can be found. Many of the generalized pseudosurfaces have an irregular structure but certain twofold triple systems correspond to more regular generalized pseudosurfaces. The simplest of these, for $n \equiv 1$ or 3 (mod 6), is a TTS(n) obtained by combining the block sets of two identical STS(n)s. Each pair of repeated blocks gives a triangle embedded in a sphere. By identifying points which have the same label, a generalized pseudosurface is obtained which is the union of $s = n(n-1)/6$ spheres and has n pinch points all of the same multiplicity $m = (n-1)/2$. Other generalized pseudosurfaces having a similar structure are obtained as follows. A Steiner system S(2, 4, n) is a pair $(V, \mathcal{B})$ where V is an n-element and $\mathcal{B}$ is a collection of 4-element subsets (the *blocks*) of V such that each 2-element subset of V is contained in exactly one block of $\mathcal{B}$. Such systems exist if and only if $n \equiv 1$ or 4 (mod 12) [54]. Each block corresponds to an embedding of a tetrahedron in the sphere. Again by identifying points which have the same label, a generalized pseudosurface is obtained which is the union of $s = n(n-1)/12$ spheres and has n pinch points all of multiplicity $m = (n-1)/3$. A generalized pseudosurface which is the union of $s = n(n-1)/24$ (respectively $n(n-1)/60$) spheres and has n pinch points all of multiplicity $m = (n-1)/4$ (respectively $(n-1)/5$) arises from the decomposition of the complete graph K_n into octahedra (respectively icosahedra). The former problem is solved, the spectrum is $n \equiv 1$ or 9 (mod 24) [50, 1] and is equivalent to the exact decomposition of the blocks of an STS(n) into Pasch configurations, see Section 7. The necessary condition for the latter problem is $n \equiv 1$, 16, 21 or 36 (mod 60) but only the case $n \equiv 1$ (mod 60) is resolved [2].

But probably a more interesting problem concerning pseudosurfaces is the following. The necessary and sufficient condition for the biembedding of two STS(n)s in an orientable surface is $n \equiv 3$ or 7 (mod 12). But as Emch's example given below shows, there does exist a face 2-colourable triangular embedding of the complete graph K_9 in a pseudosurface formed from an orientable surface, in fact the torus, with three pinch points of multiplicity 1.

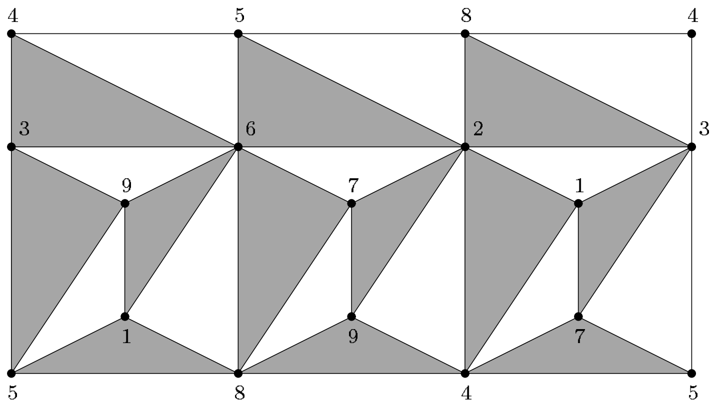

Figure 12.1 Pseudosurface biembedding of STS(9)s

In [78], a rotation scheme is given for an embedding of the complete graph K_8 in the double torus having 16 triangular faces and 2 quadrangular faces, the vertices of the quadrangular faces comprising all 8 points of the embedding. By placing two new points, say x and y, one in each quadrangle, inserting edges joining each point to the vertices of the corresponding quadrangle, and then identifying the two points x and y, we obtain a triangular embedding of the complete graph K_9 in a pseudosurface having just one pinch point of multiplicity 1. But this embedding is not face 2-colourable. These two examples naturally lead to the question of determining the pseudosurface having the least number of pinch points and/or pinch points having the least multiplicities obtained from an orientable surface for a biembedding of two STS(n)s when $n \equiv 1$ or 9 (mod 12).

More research work is needed!

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