MULTIPLE INTEGRALS

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Contents

1 Preamble

1.1 About this package

This package is for people who need to be able to integrate functions of two or three variables in various coordinate systems. It doesn't contain a lot of theory. It isn't really designed for pure mathematicians who require a course discussing conditions for the existence of double and triple integrals. You will find that you need some knowledge of integration of a function of one variable in order to get the most out of this package. In particular, you need to be able to integrate using the methods of substitution and parts. You will also find it helpful to have some familiarity with partial differentiation for functions of several variables. If you are a bit rusty, don't worry - but it would be sensible to do some revision either at the start or as the need arises. Reasonable revision texts are given in the bibliography (Section 10).

If you complete the whole package you should be able to

- understand what is meant by the **double integral** of a function of two variables,
- interpret the **double integral** geometrically,
- evaluate double integrals in **cartesian coordinates**,
- interchange the **order of integration** in a double integral,
- evaluate double integrals in **polar coordinates**,
- apply double integrals to finding the **mass, centre of mass and moments of inertia** of a lamina,
- understand what is meant by the **triple integral** of a function of three variables,
- evaluate triple integrals in **cartesian coordinates**,
- evaluate triple integrals in **cylindrical polar coordinates**,
- evaluate triple integrals in **spherical polar coordinates**,
- apply triple integrals to finding the **mass, centre of mass and moments of inertia** of a body,
- understand what is meant by the **Jacobian** of a transformation,
- use **Jacobians** to transform the variables in a double integral,

Depending on your own programme of study you may not need to cover everything in this package. Your tutor will advise you what, if anything, can be omitted.

1.2 How to use this package

You MUST do examples! Doing lots of examples for yourself is generally the most effective way of learning the contents of this package and covering the objectives listed above. We recommend that you

- first read the theory - make your own notes where appropriate,
- then work through the worked examples - compare your solutions with the ones in the notes,
- finally do similar examples yourself in a workbook.

The original printing of these notes leaves every other page blank. Use the spare space for your own comments, notes and solutions. You will see certain symbols appearing in the right hand margin from time to time:

◯ denotes the end of a worked example,

V denotes a reference to videos (see below for details),

EX highlights a point in the notes where you should try examples.

By the time you have reached a package like this one you will probably have realised that learning mathematics rarely goes smoothly! When you get stuck, use your accumulated wisdom and cunning to get around the problem. You might try:

- re-reading the theory/worked examples,
- putting it down and coming back to it later,
- reading ahead to see if subsequent material sheds any light,
- talking to a fellow student,
- looking in a textbook (see the bibliography),
- watching the appropriate video (see the video summaries),
- raising the problem at a tutorial.

1.3 Videos, tutorials and self-help

The videos cover the main points in the notes. The areas covered are indicated in the notes, usually at the ends of sections and subsections. To resolve a particular difficulty you may not need to watch a whole video (they are each about 30 minutes long). They are broken up into sections prefaced with titles which can be read on fast scan. In addition, a summary of the videos associated with this package appears as an appendix to these notes.

Your tutor will tell you about the arrangements for viewing the videos. Try the worked examples **before** watching the solution unfold on the screen. Make notes of any points you cannot follow so that you can explain the difficulty in a subsequent tutorial session. If you are viewing a video individually, remember the rewind button! Unlike a lecture you can get instant and 100 percent accurate replay of what was said.

Your tutor will tell you about tutorial arrangements. These may be related to assessment arrangements. If attendance at tutorials is compulsory then make sure you know the details! The tutorials provide you with individual contact with a tutor. Use this time wisely - staff time is the most expensive of all our resources.

You should come to tutorials in a prepared state. This means that you should have read the notes and the worked examples. You should have tried appropriate examples for yourself. If you have had difficulty with a particular section then you should watch the corresponding video. If your tutor finds that you haven't done these things then s/he may refuse to help you. Your tutor will find it easier to assist you if you can make any queries as specific as possible.

Your fellow students are an excellent form of self-help. Discuss problems with one another and compare solutions. Just be careful that

1. any assessed coursework submitted by you is yours alone,
2. you yourself do really understand solutions worked out jointly with colleagues.

Familiarize yourself with the layout and contents of these notes; scan them before reading them more carefully. The contents page will help you find your way about - use it. The bibliography will point you to textbooks covering the same material as these notes.

When you graduate, your future employer will be just as interested in your capacity for learning as in what you already know. If you can learn mathematics from this package and from textbooks then you will not only have learnt a particular mathematical topic. You will also (and more importantly) have learnt **how to learn** mathematics.

2 Double integrals

Previously we have extended the idea of differentiation to functions of more than one variable by introducing partial derivatives. Here we examine the possibility of extending the concept of the definite integral to functions of several variables.

Suppose we have a function of one variable, $f(x)$, which is positive for $a \leq x \leq b$. Then, geometrically, $\int_a^b f(x)dx$ represents the area between the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$.

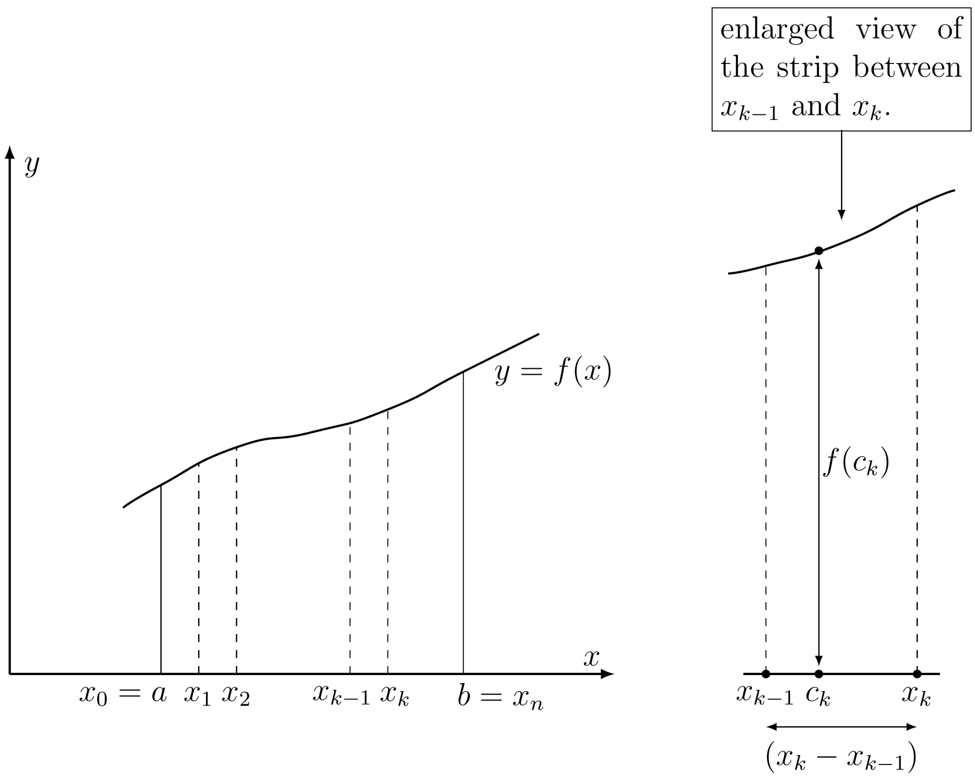

Figure 1: Area under $y = f(x)$ between $x = a$ and $x = b$.

To illustrate this we subdivide the interval $[a, b]$ by choosing $x_0, x_1, \ldots, x_n$ such that (see Figure 1)

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

Now we choose

$$c_1 \in [x_0, x_1], \quad c_2 \in [x_1, x_2], \ldots, \quad c_n \in [x_{n-1}, x_n].$$

Then $f(c_k)(x_k - x_{k-1})$ is the area of a rectangle with base $(x_k - x_{k-1})$ and height $f(c_k)$. This is an approximation to the area under the curve between $x = x_{k-1}$ and $x = x_k$. An approximation for the total area under the curve is

$$\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).$$

Suppose that

$$\triangle = \max_{k=1,\ldots,n} (x_k - x_{k-1}).$$

Then the **definite integral** of $f(x)$ between $x = a$ and $x = b$ is defined as

$$\int_a^b f(x)dx = \lim_{\triangle \to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

provided that the limit exists. This limit is exactly what we mean by the area under the curve.

We now try to extend this idea to a function of two variables, $u = f(x, y)$. As we have already seen this equation defines a surface in a three-dimensional space. We assume that $f(x, y)$ is defined for all x and y in a certain region of the xy- plane, denoted by R. The situation is illustrated in Figure 2.

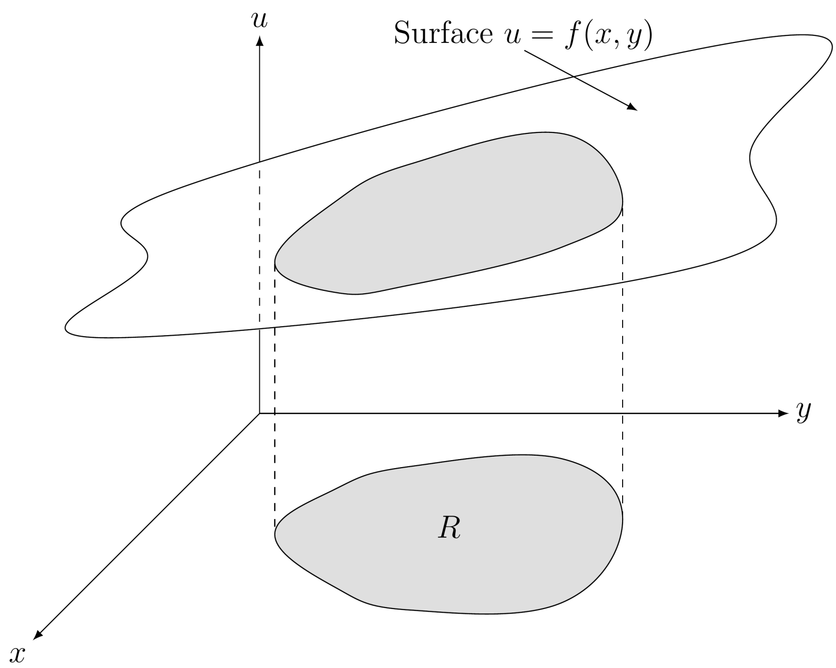

Figure 2: The surface $u = f(x, y)$.

We assume for the time being that $u > 0$ for all $(x, y) \in R$. The region R corresponds to the interval $[a, b]$ in the one variable case. We now divide R into n subregions in any manner and we let the areas of the subregions be $\delta A_k,\ k = 1, \ldots, n$ (see Figure 3).

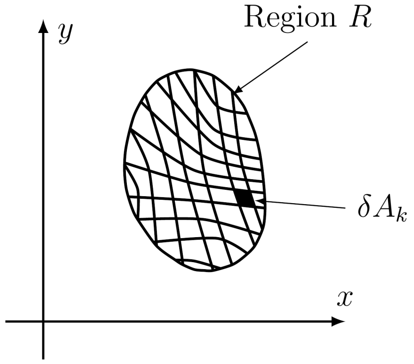

Figure 3: Subdivisions of R.

Now choose a point (x'_k, y'_k) in each subregion; i.e. we choose

$$(x'_1, y'_1) \in \delta A_1,\ (x'_2, y'_2) \in \delta A_2, \ldots,\ (x'_n, y'_n) \in \delta A_n.$$

We form the product $f(x'_k, y'_k)\delta A_k$ and this represents the volume of a column of base area δA_k and height $f(x'_k, y'_k)$. This is an approximation to **the volume under the surface** $u = f(x, y)$ over the subregion δA_k. The sum of all such products

$$\sum_{k=1}^{n} f(x'_k, y'_k)\delta A_k \tag{1}$$

is an approximation to the volume under the surface $u = f(x, y)$ over R. The equivalent geometrical problem for a function of two variables is that of finding a volume under a surface rather than an area under a curve. We now proceed to the limit in equation (1) as the size of the subregions tends to zero. Let $\triangle = \max\limits_{k=1,\ldots,n} \delta A_k$ and define the **double integral** of $f(x, y)$ over R as

$$\lim_{\triangle \to 0} \sum_{k=1}^{n} f(x'_k, y'_k)\delta A_k$$

provided this limit exists. The double integral is denoted by $\displaystyle\iint\limits_R f(x, y) dA$

and so,

$$\iint_R f(x,y)dA = \lim_{\triangle\to 0} \sum_{k=1}^{n} f(x'_k, y'_k)\delta A_k. \tag{2}$$

The limit must be independent of the way in which R is subdivided and of the particular choice of (x'_k, y'_k) within each subregion. We shall assume that these conditions hold for all the integrals with which we are dealing.

The double integral has some properties which are similar to those of the ordinary (single) definite integral. The most important of these is the **linearity property**. This states that given two functions of two variables, $f(x,y)$ and $g(x,y)$, and two constants, a and b,

$$\iint_R \{af(x,y) + bg(x,y)\}dA = a \iint_R f(x,y)dA + b \iint_R g(x,y)dA.$$

The double integral is linear because the summation in equation (2) has the linearity property.

The second property states that if R can be divided into two mutually disjoint subregions, R_1 and R_2; i.e. such that $R_1 \cup R_2 = R$ and R_1 and R_2 do not overlap, then

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA.$$

All that this really says is that the volume above the region R is the sum of the volumes above R_1 and R_2.

Note that if $f(x,y) = 1$ for all $(x,y) \in R$, then

$$\iint_R f(x,y)dA = \iint_R dA = \text{ area of } R.$$

Geometrically, we are finding the volume of a cylinder whose cross-section is R and whose height is unity. Numerically this is the same as the area of the cross- section.

Finally also note that although we assumed that $u > 0$ for all $(x,y) \in R$, the definition of the double integral is equally valid if this assumption is relaxed. Parts of the surface for which $u < 0$ will give a negative contribution to the double integral and also to the volume under the surface.

(The video revises the definite integral, defines the double integral of a function of two variables and discusses its properties) V

3 Evaluation of double integrals

3.1 Cartesian coordinates

In section 2 we have introduced the definition and geometrical interpretation of double integrals. In this section we examine some techniques of evaluating them.

Since the subdivision of the region of integration, R, may be carried out in any manner, we choose ways of subdividing it that are the most convenient. The most important way is to employ a set of cartesian axes and use rectangles bounded by straight lines parallel to the x- and y-axes. The region R is then covered by a rectangular grid as shown in Figure 4

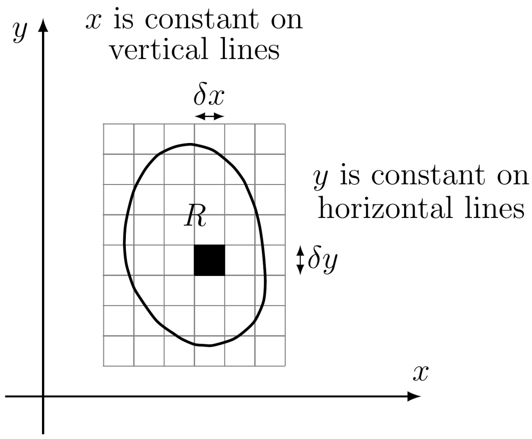

Figure 4: Subdivisions of R in cartesian coordinates.

The lines parallel to the x-axis have equation $y = constant$ for different values of the constant, whereas the lines parallel to the y-axis have equation $x = constant$. Let a particular rectangle in R have sides δx and δy so that $\delta A = \delta x \delta y$. Now the boundary of R is a curve in the xy-plane. We let the lower boundary of R be given by $y = y_1(x)$ and the upper boundary of R by $y = y_2(x)$. Then these two boundaries will intersect at $x = a$ and $x = b$, where $a < b$. Thus a and b are the minimum and maximum values of x within R. The situation is shown in Figure 5.

We are thus making some simplifying assumptions about the nature of the region, R, and its bounding curve. Regions which look like those in Figure 6 are excluded.

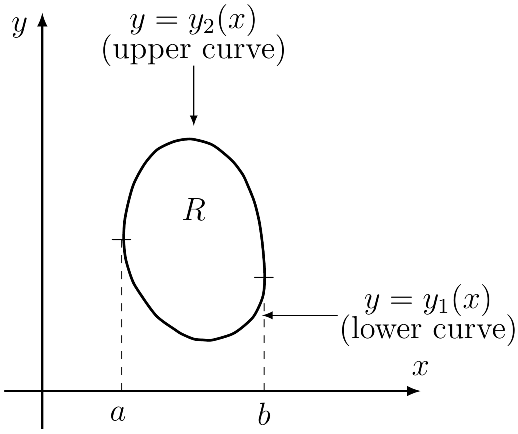

Figure 5: Boundaries of R.

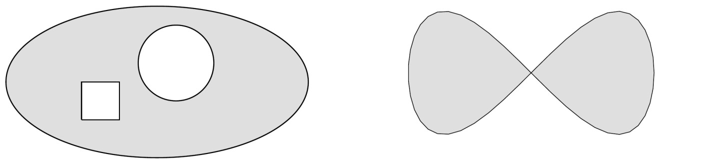

Figure 6: Examples of regions excluded from definition.

Suppose we now have a function, $u = f(x, y)$, defined for all $(x, y) \in R$ and we wish to evaluate

$$\iint_R f(x, y) dA.$$

We do this in two stages. Firstly, we hold x constant and integrate with respect to y between $y = y_1(x)$ and $y = y_2(x)$. This results in a function of x alone. This function of x is then integrated with respect to x between $x = a$ and $x = b$. We can express this process mathematically as

$$\iint_R f(x, y) dA = \int_a^b \left\{ \int_{y_1(x)}^{y_2(x)} f(x, y) dy \right\} dx. \tag{3}$$

The integral on the right hand side is called a **repeated integral** or **iterated**

integral, and is usually written without the brackets:

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)dydx.$$

The order in which the differentials (dy and dx) is written is very important because it shows the order in which the integration must be carried out. The inner integral sign is associated with the first differential, dy and the outer integral sign is associated with the second differential, dx.

Geometrically, equation (3) can be interpreted as follows. For convenience we imagine $f(x,y)$ is positive throughout R so that the surface $u = f(x,y)$ lies above the region R. The left-hand side of (3) represents the total volume under the surface. It is the sum of the volumes of lots of small "matchsticks". (see Figure 7.)

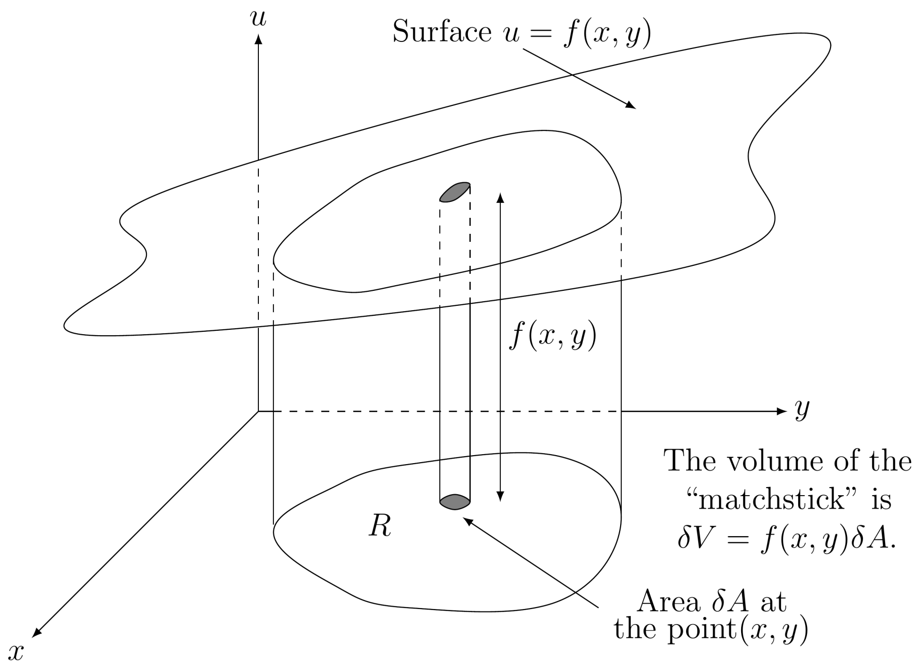

Figure 7: A typical "matchstick".

The volume of the "matchstick" is $\delta V = f(x,y)\delta A$. Working in Cartesian coordinates we interpret δA as $\delta y \delta x$ (see Figure 8).

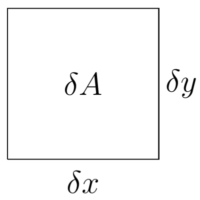

Figure 8: $\delta A = \delta y \delta x$.

Our "matchsticks" therefore have a rectangular cross-section δy by δx. This ensure that they will fit together nicely. The inner integral in equation (3) multiplied by δx, namely

$$\left(\int_{y_1(x)}^{y_2(x)} f(x, y) dy \right) \delta x$$

adds up these "matchsticks" in the y-direction to form a "slab" (see Figure 9).

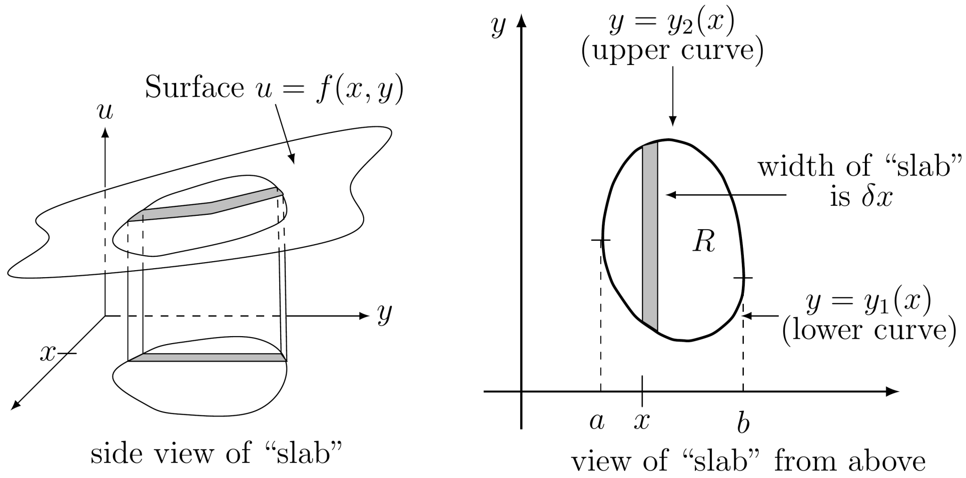

Figure 9: "Matchsticks" assembled to form a "slab".

In this inner integral the value of x is held fixed (see the view of the "slab" from above). Also the value of y varies from the bottom curve $y = y_1(x)$ (of course this value depends on x) to the upper curve $y = y_2(x)$ (again the value will depend on x).

Having assembled the "matchsticks" into "slabs" we glue the slabs together (in the x-direction) to obtain the total volume. Here x varies from the lowest value a to the greatest value b (see Figure 9 again). This finally produces

$$\int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)dy \right) dx.$$

Example 3.1 Evaluate

$$\int_1^3 \int_2^3 (x^2 - 2xy)dydx$$

and sketch the region over which the integral is taken.

Solution The order of the differentials in this repeated integral indicates that we must integrate first with respect to y between $y = 2$ and $y = 3$ and then with respect to x between $x = 1$ and $x = 3$. So

$$\begin{aligned}
\int_1^3 \int_2^3 (x^2 - 2xy)dydx &= \int_1^3 \left[x^2y - xy^2\right]_2^3 dx \\
&= \int_1^3 (\{3x^2 - 9x\} - \{2x^2 - 4x\})dx \\
&= \int_1^3 (x^2 - 5x)dx \\
&= \left[\frac{x^3}{3} - \frac{5x^2}{2}\right]_1^3 \\
&= \left(9 - \frac{45}{2}\right) - \left(\frac{1}{3} - \frac{5}{2}\right) \\
&= -11\frac{1}{3}
\end{aligned}$$

Note that once the y integration has been performed and the limits put in, the resulting integral is a definite integral of one variable, x.

The region of integration is defined by $y = 2$ and $y = 3$ and $x = 1$ and $x = 3$. This region is the rectangle illustrated in Figure 10.

Since the answer is negative we can deduce that for the region R the major part of the surface $u = x^2 - 2xy$ lies below the plane $u = 0$. In fact since $u = x(x - 2y)$ and $x - 2y < 0$ at all points of R, the whole surface $u = x^2 - 2xy$ lies below the plane $u = 0$.

In this example, because the limits of integration are all constant the region of integration is necessarily a rectangle. If the integrand can be written as a product of a function of x alone and a function of y alone, the double

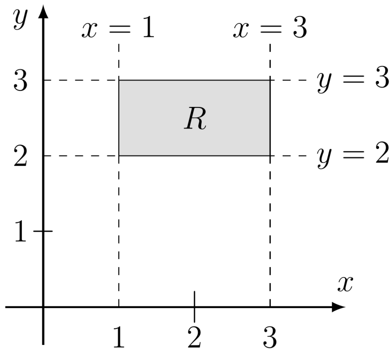

Figure 10: Region of integration for $\int_1^3 \int_2^3 (x^2 - 2xy)dydx$.

integral over a rectangle can be expressed as a product of single integrals. For

$$\int_a^b \int_c^d f(x)g(y)dydx = \int_a^b f(x) \int_c^d g(y)dydx$$

since $f(x)$ is held constant for the y-integration. Now $\int_c^d g(y)dy$ is a constant and so

$$\int_a^b \int_c^d f(x)g(y)dydx = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right).$$

Example 3.2 Evaluate

$$\int_0^1 \int_x^{2x} xe^y dydx$$

and sketch the region over which the integral is taken.

Solution We have

$$\begin{aligned} \int_0^1 \int_x^{2x} xe^y dydx &= \int_0^1 \left[xe^y\right]_x^{2x} dx \\ &= \int_0^1 (xe^{2x} - xe^x)dx \\ &= \int_0^1 x(e^{2x} - e^x)dx. \end{aligned}$$

This is now a definite integral of one variable and may be integrated by parts.

$$\begin{aligned}
\int_0^1 \int_x^{2x} xe^y dydx &= \left[x \left(\frac{e^{2x}}{2} - e^x \right) \right]_0^1 - \int_0^1 \left(\frac{e^{2x}}{2} - e^x \right) dx \\
&= \frac{e^2}{2} - e - \left[\frac{e^{2x}}{4} - e^x \right]_0^1 \\
&= \frac{e^2}{2} - e - \left\{ \frac{e^2}{4} - e - \left(\frac{1}{4} - 1 \right) \right\} \\
&= \frac{e^2}{4} - \frac{3}{4} = \frac{1}{4}(e^2 - 3)
\end{aligned}$$

The region of integration is the area between the lines $y = x$ and $y = 2x$, for which $0 \le x \le 1$. The region is the triangle illustrated in Figure 11.

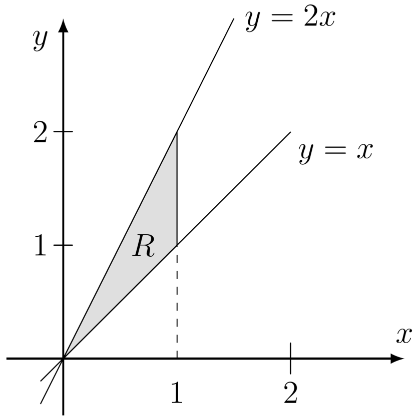

Figure 11: Region of integration for $\displaystyle\int_0^1 \int_x^{2x} xe^y dydx$.

In some problems the region over which the double integral is taken is described by the equations of its boundaries. From these equations appropriate limits for the repeated integral must be chosen. To do this a sketch of the region of integration should be drawn.

Example 3.3 Evaluate

$$\iint\limits_R xydA$$

where R is the region bounded by $y = 0$, $x = 2$ and $x^2 = 4y$.

Solution We first sketch the region R, by drawing the curves $y = 0$, $x = 2$ and $x^2 = 4y$. (see Figure 12).

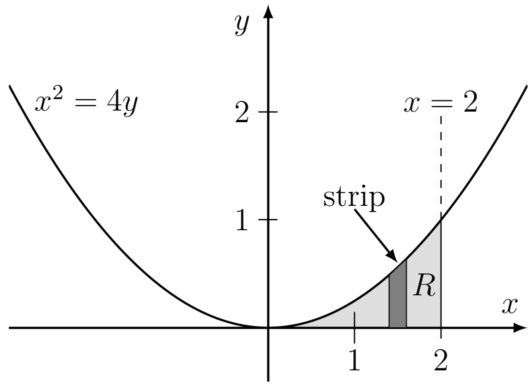

Figure 12: Region of integration bounded by $y = 0$, $x = 2$ and $x^2 = 4y$.

Clearly the only region that is bounded (i.e. entirely enclosed) by the three curves is the region denoted by R.

To select the limits for the integration consider an arbitrary, but fixed value of x in $0 \leq x \leq 2$. At this value of x draw a narrow strip parallel to the y-axis. The lower limit for the y-integration will be the value of y in terms of x where the strip meets the lower boundary of R; here it is $y = 0$. Similarly the upper limit for the y-integration is the value of y in terms of x where the strip meets the upper boundary of R; here this is $y = \dfrac{x^2}{4}$.

The limits for the second integral are the extreme values of x in R; i.e. $x = 0$ and $x = 2$. So

$$\iint\limits_R xy dA = \int_0^2 \int_0^{x^2/4} xy dy dx.$$

The repeated integral is now easy to evaluate in the usual way.

$$\begin{aligned}
\iint\limits_R xy dA &= \int_0^2 \left[\frac{xy^2}{2}\right]_0^{x^2/4} dx \\
&= \int_0^2 \frac{x}{2}\left(\frac{x^2}{4}\right)^2 dx \\
&= \int_0^2 \frac{x^5}{32} dx \\
&= \left[\frac{x^6}{6\times 32}\right]_0^2 = \frac{1}{3}
\end{aligned}$$

Example 3.4 Evaluate $\iint\limits_R x dA$ over the smaller segment of the circle $x^2+y^2=4$, cut off by the line $x=1$.

Solution We first sketch the curve $x^2+y^2=4$ and the line $x=1$.

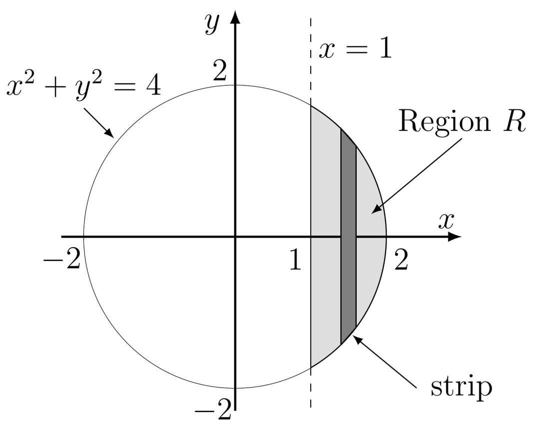

Figure 13: Region of integration bounded by $x^2+y^2=4$ and $x=1$.

Selecting a fixed value of x in $1 \le x \le 2$, we draw a narrow strip in the y- direction. We can see from Figure 13 that the limits for the y integration are both determined by the equation of the circle.

Since

$$x^2+y^2=4,$$

$$y^2 = 4 - x^2$$
$$\text{and } y = \pm\sqrt{4 - x^2}$$

Thus the lower limit for the y-integration is $y = -\sqrt{4-x^2}$ and the upper limit is $y = \sqrt{4-x^2}$. The limits for the x-integration are $x = 1$ and $x = 2$.

So

$$\begin{aligned}
\iint\limits_R x\,dA &= \int_1^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx \\
&= \int_1^2 [xy]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\, dx \\
&= \int_1^2 \{x\sqrt{4-x^2} - (-x\sqrt{4-x^2})\}dx \\
&= 2\int_1^2 x\sqrt{4-x^2}dx
\end{aligned}$$

Use the substitution $u = 4 - x^2$, so that $\dfrac{du}{dx} = -2x$. Then, when $x = 1$, $u = 3$ and when $x = 2$, $u = 0$. So

$$\begin{aligned}
\iint\limits_R x\,dA &= 2\int_3^0 u^{1/2}\left(\frac{du}{-2}\right) \\
&= \int_0^3 u^{1/2}du \\
&= \left[\frac{2}{3}u^{3/2}\right]_0^3 = 2\sqrt{3}.
\end{aligned}$$

○

(The video discusses how to evaluate double integrals in cartesian coordinates and explains several examples. V

At this point you should try several examples of evaluating double integrals in cartesian coordinates. Note that in the evaluation of $\iint\limits_R f(x,y)dydx$, *the y-integration is performed first. Thus the limits on the inner integral may involve x [but* **cannot** *involve y]. The limits on the outer integral are constants [i.e. they* **cannot** *involve x* **or** *y].)* EX

3.2 Change of order of integration

In all the previous examples the repeated integral has been evaluated by doing the y-integration first followed by the x-integration. However, we can also express double integrals over suitable regions, R, as repeated integrals involving an x- integration followed by a y-integration. To see how to do this we examine Figure 14 which is a redrawing of Figure 5.

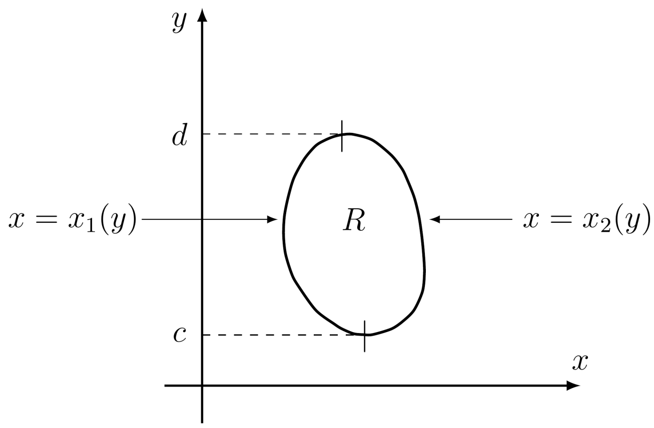

Figure 14: Region of integration R.

This time the boundary of R is divided into a left hand boundary, $x = x_1(y)$ and a right hand boundary, $x = x_2(y)$. These two boundaries intersect at $y = c$ and $y = d$ where $c < d$.

To evaluate the double integral we first hold y constant and integrate with respect to x between $x = x_1(y)$ and $x = x_2(y)$. This results in a function of y alone, which is then integrated between $y = c$ and $y = d$. Thus

$$\iint\limits_R f(x,y)dA = \int_c^d \left\{ \int_{x_1(y)}^{x_2(y)} f(x,y)dx \right\} dy.$$

(Note that the limits on the inner integral may now involve y [but **cannot** *involve x]. The limits on the outer integral are again constants [i.e. they don't involve x or y].)* If you look at Examples 3.3 and 3.4, instead of drawing strips in the y-direction all we are really doing here is drawing strips in the x-direction.

Since the value of a double integral is independent of the way in which it is evaluated, it follows that

$$\int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)dxdy = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)dydx.$$

We can verify this in a particular case by reworking Example 3.3 by carrying out the x-integration first.

Example 3.3 (Again!) Evaluate

$$\iint\limits_R xydA$$

where R is the region bounded by $y = 0$, $x = 2$ and $x^2 = 4y$.

Solution Referring to Figure 12, to determine the limits for the x-integration we consider a fixed value of y in R, and draw a strip parallel to the x-axis. The left hand boundary of the strip is given by the parabola, $x^2 = 4y$ and so $x = \sqrt{4y}$. The right hand boundary is $x = 2$. The limits for the y-integration are the extreme values of y in R. These are $y = 0$ and $y = 1$ (since when $x = 2$, $y = x^2/4 = 1$). So

$$\begin{aligned} \iint\limits_R xydA &= \int_0^1 \int_{\sqrt{4y}}^2 xydxdy \\ &= \int_0^1 \left[\frac{x^2y}{2}\right]_{\sqrt{4y}}^2 dy \\ &= \int_0^1 \{2y - 2y^2\}dy \\ &= \left[y^2 - \frac{2}{3}y^3\right]_0^1 = \frac{1}{3} \end{aligned}$$

○

Clearly there is no point in evaluating double integrals by both routes. However, for some integrals one route may be much easier than the other. Indeed, for some integrals it may not be possible to find an indefinite integral (in terms of the usual elementary functions) by one of the routes. For example if we try to evaluate

$$\int_0^1 \int_x^1 x^2 e^{-y^2} dydx$$

we would not be able to find the integral of e^{-y^2}. By swapping the order of integration and integrating x^2 first we can do this integral. When changing the order of integration we have to be **extremely careful** with the limits. **ALWAYS draw a diagram!**

Example 3.5 By interchanging the order of integration evaluate

$$\int_0^1 \int_x^1 x^2 e^{-y^2} dy dx.$$

Solution We start by drawing a sketch of the region of integration, R. From the limits R is defined by

$$x \leq y \leq 1 \text{ and } 0 \leq x \leq 1.$$

So we draw the lines $y = x$ and $y = 1$ in Figure 15.

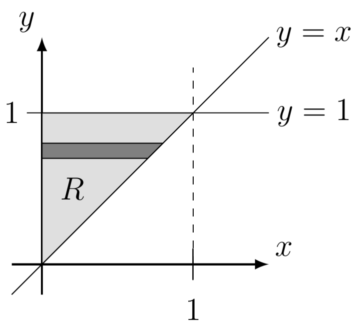

Figure 15: Region of integration R.

R is between $y = x$ and $y = 1$, with $0 \leq x \leq 1$; so it is the triangular region shown on Figure 15. To select the limits for the x-integration first, we pick an arbitrary y in R and consider the strip parallel to the x-axis. The end points are $x = 0$ and $x = y$. The extreme values of y are $y = 0$ and $y = 1$. So

$$\begin{aligned} \int_0^1 \int_x^1 x^2 e^{-y^2} dy dx &= \int_0^1 \int_0^y x^2 e^{-y^2} dx dy \\ &= \int_0^1 \left[\frac{x^3}{3} e^{-y^2} \right]_0^y dy \\ &= \int_0^1 \frac{y^3}{3} e^{-y^2} dy. \end{aligned}$$

Although this integral may not look easy, it is much easier than trying to integrate e^{-y^2}. If we put $u = y^2$ we obtain $\dfrac{du}{dy} = 2y$, together with $u = 0$ when $y = 0$ and $u = 1$ when $y = 1$. Then

$$\begin{aligned}
\int_0^1 \frac{y^3}{3} e^{-y^2} dy &= \int_0^1 \frac{u}{6} e^{-u} du \\
&= \frac{1}{6}\left\{[-ue^{-u}]_0^1 + \int_0^1 e^{-u} du\right\} \\
&= \frac{1}{6}\left\{-e^{-1} + [-e^{-u}]_0^1\right\} \\
&= \frac{1}{6}\left\{-e^{-1} - e^{-1} + 1\right\} \\
&= \frac{1}{6}\left(1 - \frac{2}{e}\right)
\end{aligned}$$

○

Example 3.6 Interchange the order of integration for

$$\int_0^1 \int_y^{2-y} f(x, y) dx dy.$$

Solution Here the region R is defined by

$$y \le x \le 2 - y \text{ and } 0 \le x \le 1.$$

So we draw the lines $x = y$ and $x = 2 - y$.

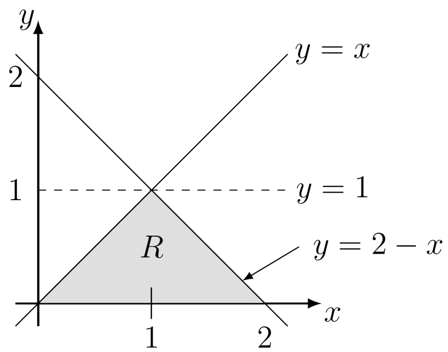

Figure 16: Region of integration R.

R is between $x = y$ and $x = 2 - y$, with $0 \le x \le 1$; so it is the triangular shaded region shown in Figure 16. To interchange the order of integration we need to draw strips parallel to the y-axis. The difficulty that arises here is that the upper bounding curve of R has a different equation depending whether $0 \le x \le 1$ or $1 \le x \le 2$. To overcome this difficulty we divide R into two subregions R_1 and R_2 as shown in Figure 17.

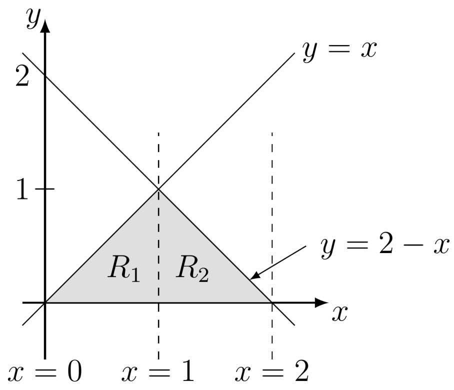

Figure 17: Regions of integration, R_1 and R_2.

Then using one of the properties of double integrals from Section 2

$$\iint\limits_{R} f(x,y)dA = \iint\limits_{R_1} f(x,y)dA + \iint\limits_{R_2} f(x,y)dA.$$

The limits for R_1 are from $y = 0$ to $y = x$ and from $x = 0$ to $x = 1$. The limits for R_2 are from $y = 0$ to $y = 2 - x$ and from $x = 1$ to $x = 2$. Thus

$$\int_0^1 \int_y^{2-y} f(x,y)dxdy = \int_0^1 \int_0^x f(x,y)dydx + \int_1^2 \int_0^{2-x} f(x,y)dydx.$$

(The video discusses how to change the order of integration in a double integral and solves the examples of this subsection. V

Now you should try some examples involving changing the order of integration.) EX

3.3 Polar coordinates

For some problems, particularly where the region of integration is partially bounded by arcs of circles, it is more convenient to use polar coordinates than cartesian coordinates.

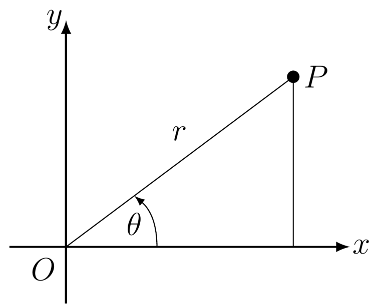

Figure 18: Polar coordinates.

In Figure 18 the polar coordinates of P are the distance of P from the origin, r, and the angle that OP makes with the x-axis, θ. These are related to cartesian coordinates by

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}$$

where $r \geq 0$ and $-\pi < \theta \leq \pi$. Care must be taken when selecting θ since the principal value of the inverse tan function is not always the correct value. e.g. for the point with cartesian coordinates $(-1, 1)$, $\tan\theta = -1$, but the correct value of θ is $\dfrac{3\pi}{4}$ and not the prinicipal value, $-\dfrac{\pi}{4}$. The point $(1, -1)$ also gives $\tan\theta = -1$ but here $\theta = -\dfrac{\pi}{4}$; θ must be selected so that the point lies in the correct quadrant. To change from polar coordinates to cartesians we use

$$x = r\cos\theta, \; y = r\sin\theta.$$

In polar coordinates we subdivide the region R in a similar manner to the way we did it in cartesian coordinates. We draw the lines $r = constant$ and $\theta = constant$ for different values of the constant. The lines $r = constant$ are circles centred on the origin, whereas the lines $\theta = constant$ are radii from the origin. We obtain the grid illustrated in Figure 19.

The grid divides R into subregions of which a typical one is $PQRS$ (see Figure 20). Let it have area δA and let P have polar coordinates (r, θ). Then since Q lies on the same $\theta = constant$ line as does P, Q will have polar coordinates $(r + \delta r, \theta)$ where $PQ = \delta r$. Similarly since S lies on the same $r = constant$ curve as does P, S will have polar coordinates $(r, \theta + \delta\theta)$ where $S\hat{O}P = \delta\theta$. Now in the subregion $PQRS$, PQ is of length δr and PS is the arc of a circle of radius r which subtends and angle $\delta\theta$ at its centre. Hence the length of PS is $r\delta\theta$. (Note that $\delta\theta$ must be in radians!) Hence the area of $PQRS$ is approximately

$$\delta A \approx (\delta r)(r\delta\theta) = r\delta\theta\delta r.$$

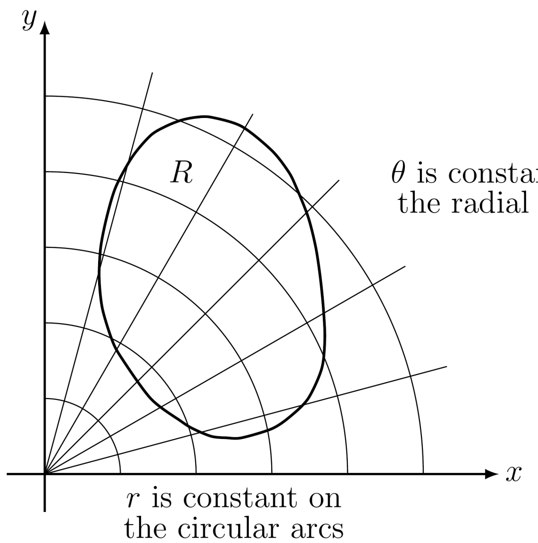

Figure 19: Subdivisions of R in polar coordinates.

Although $PQRS$ is not, in fact, a rectangle, it is near enough for our purposes here, as long as δr and $\delta\theta$ are sufficiently small.

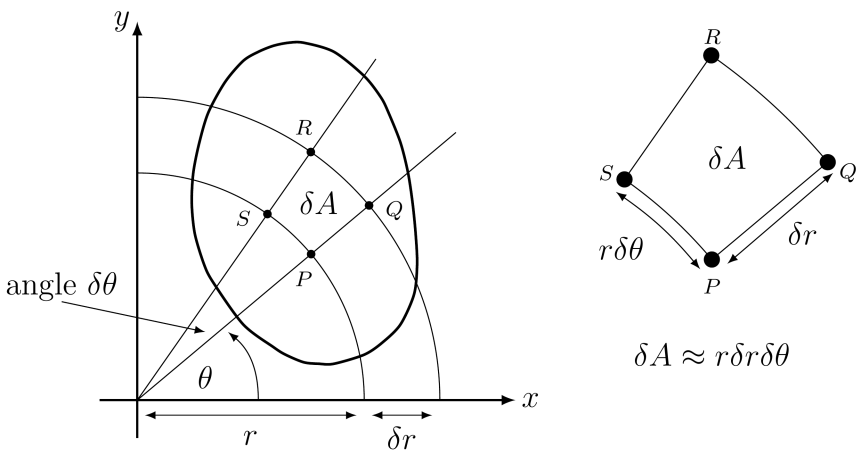

Figure 20: An element of area in polar coordinates.

Thus when we are evaluating double integrals in polar coordinates we use

$$dA = r d\theta dr.$$

The approximation involved can be justified rigorously as δr and $\delta\theta \to 0$.

The limits for the integration are determined in a similar way to that used for cartesian coordinates. If we are doing the r-integration first the limits will be from $r = r_1(\theta)$ to $r = r_2(\theta)$ where $r = r_1(\theta)$ is the equation of curve C_1, and $r = r_2(\theta)$ is the equation of curve C_2. (see Figure 21). The limits for θ will be θ_1 and θ_2, the minimum and maximum values of θ in R and the intersections of the curves C_1 and C_2. Again R is assumed to be a suitable region such that these points may be defined.

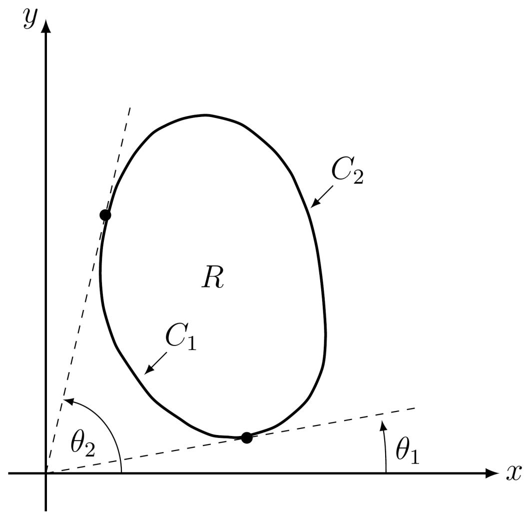

Figure 21: Boundaries of R in polar coordinates.

Then

$$\iint\limits_R f(x,y)dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta.$$

Alternatively if we do the θ-integration first the limits are $\theta = \theta_1(r)$ to $\theta = \theta_2(r)$ where $\theta = \theta_1(r)$ is the equation of curve C_3 and $\theta = \theta_2(r)$ is the equation of curve C_4 (see Figure 22).

The limits for r are r_1 and r_2, the minimum and maximum values of r in R and the intersections of the curves C_3 and C_4.

In this case

$$\iint\limits_R f(x,y)dA = \int_{r_1}^{r_2} \int_{\theta_1(r)}^{\theta_2(r)} f(r\cos\theta, r\sin\theta) r d\theta dr.$$

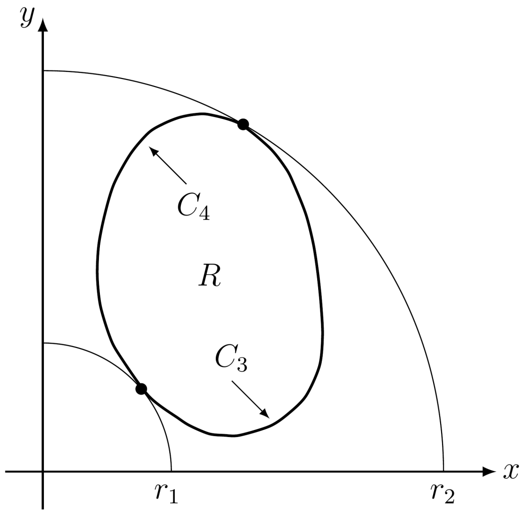

Figure 22: Boundaries of R in polar coordinates.

Example 3.7 By transforming to polar coordinates evaluate the double integral

$$\iint_R \sqrt{x^2 + y^2}\, dA$$

where R is the region in the first quadrant bounded by the curves $x^2+y^2 = a^2$, $y = x$ and $x = 0$, and a is a constant.

Solution We first sketch the region R.

This problem is particularly suited to polar coordinates since the region of integration is a sector of a circle (see Figure 23). To determine the limits of integration we draw a small radial strip across R. The ends of the strip are $r = 0$ and $r = a$, so these are limits for the r-integration. The extreme values of θ in R are $\frac{\pi}{4}$ and $\frac{\pi}{2}$ (remembering to use radians).

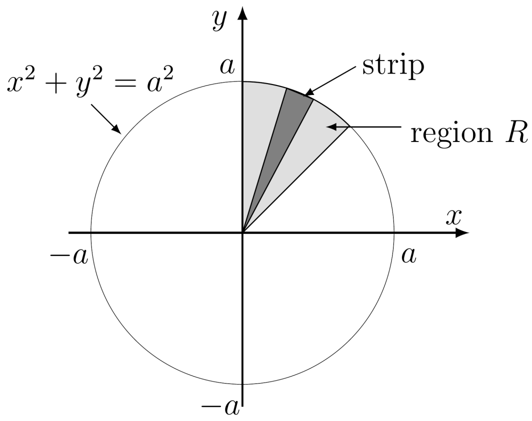

Figure 23: Sketch of the region R.

Also, converting the integrand to polar coordinates, $\sqrt{x^2 + y^2} = r$. So

$$
\begin{aligned}
\iint\limits_R \sqrt{x^2+y^2}\,dA &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^a r(rdrd\theta) \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^a r^2 drd\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{1}{3}r^3\right]_0^a d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}a^3 d\theta \\
&= \left[\frac{1}{3}a^3\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{\pi a^3}{12}.
\end{aligned}
$$

○

Example 3.8 Show, using double integration, that the volume of a sphere really is $\frac{4}{3}\pi a^3$.

Solution Let the sphere be $x^2 + y^2 + z^2 = a^2$. One eighth of the volume of the sphere will lie in the first quadrant.(See Figure 24).

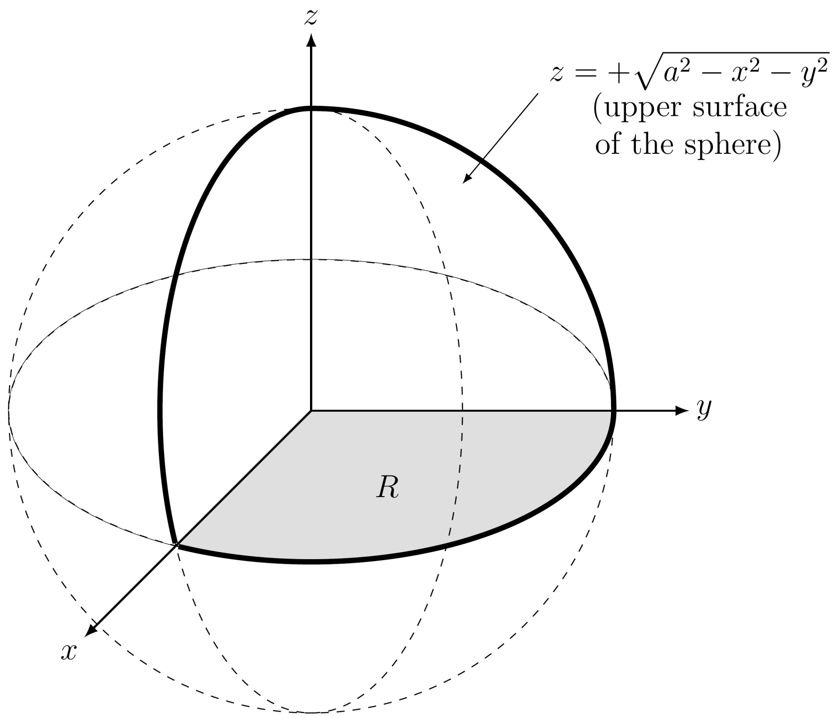

Figure 24: One eighth of the sphere $x^2 + y^2 + z^2 = a^2$.

To find the volume of the sphere we need to find eight times the volume under the surface $z = \sqrt{a^2 - y^2 - x^2}$ above the plane $z = 0$ and for which $x,\ y \geq 0$. Therefore

$$V = 8 \iint_R \sqrt{a^2 - y^2 - x^2}dA$$

where R is the region shaded in Figure 24. In polar coordinates the limits for R are $r = 0$ to $r = a$ and $\theta = 0$ to $\theta = \dfrac{\pi}{2}$. So

$$V = 8 \int_0^{\frac{\pi}{2}} \int_0^a (a^2 - r^2)^{1/2} r dr d\theta.$$

To evaluate the inner integral, let $u = a^2 - r^2$. Then

$$\frac{du}{dr} = -2r, \text{ so } rdr = \left(-\frac{du}{2}\right)$$

Also $u = a^2$ when $r = 0$ and $u = 0$ when $r = a^2$. So

$$\begin{aligned} V &= 8\int_0^{\frac{\pi}{2}} \int_{a^2}^{0} u^{1/2}\left(-\frac{du}{2}\right) d\theta \\ &= 4\int_0^{\frac{\pi}{2}} \left[\frac{2}{3}u^{3/2}\right]_0^{a^2} d\theta \\ &= \frac{8}{3}\int_0^{\frac{\pi}{2}} a^3 d\theta \\ &= \frac{8a^3}{3}[\theta]_0^{\frac{\pi}{2}} \\ &= \frac{4}{3}\pi a^3 \end{aligned}$$

○

Example 3.9 By converting to polar coordinates evaluate

$$\iint\limits_R e^{-x^2-y^2} dA$$

where R is the whole of the first quadrant. Hence deduce that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Solution Note in this example that it is not possible to obtain an indefinite integral of either e^{-x^2} or e^{-y^2} (at any rate, not in terms of the usual elementary functions). The region of integration here is not bounded by a closed curve. Nevertheless we can describe the region in terms of polar coordinates and evaluate the integral. The limits are $r = 0$ to $r = \infty$ and $\theta = 0$ to $\theta = \frac{\pi}{2}$. (see Figure 25).

So

$$\iint\limits_R e^{-x^2-y^2} dA = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta.$$

Let $u = r^2$, and then $\dfrac{du}{dr} = 2r$, and so $rdr = \dfrac{du}{2}$. When $r = 0$, $u = 0$ and

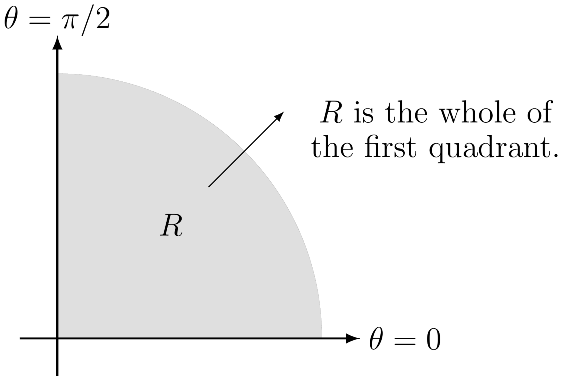

Figure 25: The region of integration R.

when $r \to \infty$, $u \to \infty$. Hence we have

$$
\begin{aligned}
\iint\limits_R e^{-x^2-y^2} dA &= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-u} \frac{du}{2} d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[-e^{-u}\right]_0^{\infty} d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \{0-(-1)\} d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{\pi}{4}. \qquad (4)
\end{aligned}
$$

If we express the integral in Cartesian coordinates we have

$$
\begin{aligned}
\iint\limits_R e^{-x^2-y^2} dA &= \int_0^{\infty} \int_0^{\infty} \left(e^{-x^2}\right) \left(e^{-y^2}\right) dx dy \\
&= \left(\int_0^{\infty} e^{-x^2} dx\right) \left(\int_0^{\infty} e^{-y^2} dy\right) \\
&= \frac{\pi}{4} \text{ (from equation 4).}
\end{aligned}
$$

But $\displaystyle\int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-y^2} dy$, since x and y are just dummy variables.

Hence

$$\left(\int_0^\infty e^{-x^2} dx\right)^2 = \frac{\pi}{4}.$$

i.e.

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

But e^{-x^2} is an even function. Hence

$$\int_{-\infty}^\infty e^{-x^2} dx = 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$$

The integrand in this question is related to the normal distribution function in statistics:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}\right\}$$

where m and σ are the mean and standard deviation of the distribution. We can now prove that

$$\int_{-\infty}^\infty p(x)dx = 1.$$

Let $z = \dfrac{x-m}{\sqrt{2}\sigma}$ and then $\dfrac{dz}{dx} = \dfrac{1}{\sqrt{2}\sigma}$. When $x \to \pm\infty$, $z \to \pm\infty$. So

$$\begin{aligned}\int_{-\infty}^\infty p(x)dx &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty \exp(-z^2)(\sqrt{2}\sigma)dz \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-z^2} dz = 1.\end{aligned}$$

Example 3.10 Sketch the cardioid $r = a(1+\sin\theta)$. Use a double integral to find the area which is inside the cardioid and outside the circle $r = a$.

Solution We tabulate the function $r = a(1+\sin\theta)$ for θ in $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	a	$a\left(1+\frac{\sqrt{2}}{2}\right)$	$2a$	$a\left(1+\frac{\sqrt{2}}{2}\right)$	a

θ	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	$a\left(1-\frac{\sqrt{2}}{2}\right)$	0	$a\left(1-\frac{\sqrt{2}}{2}\right)$	a

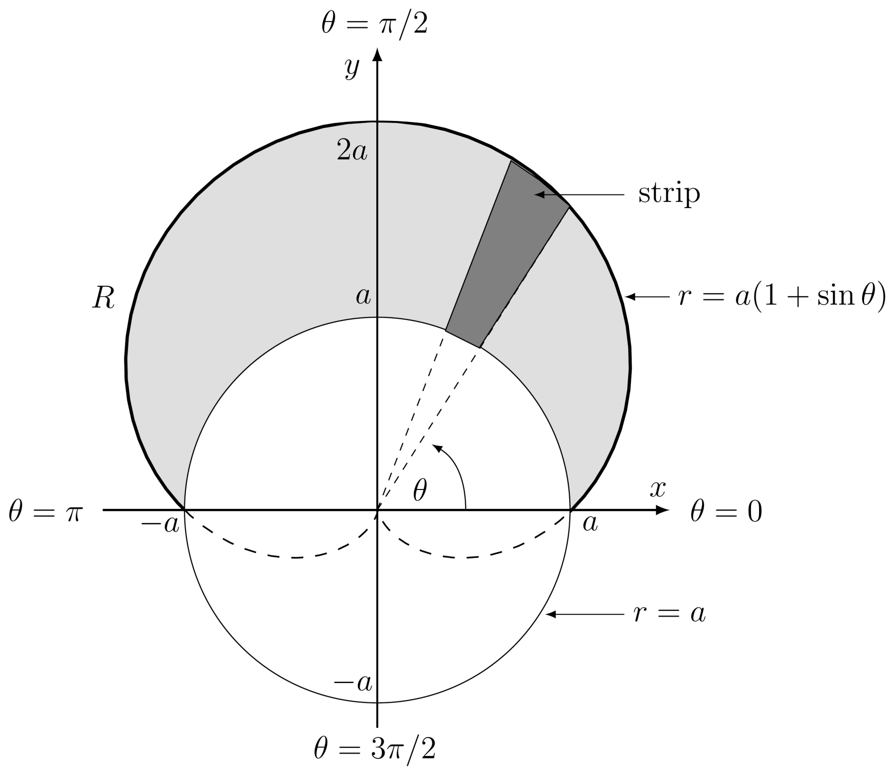

Figure 26: The cardioid $r = a(1 + \sin\theta)$.

We can now sketch the curve. (see Figure 26)

The limits for the region R in polar coordinates are $r = a$ to $r = a(1 + \sin\theta)$ and from $\theta = 0$ to $\theta = \pi$. To express an area as a double integral we need to choose the integrand to be 1 (see section 2). If A denotes the area required then

$$
\begin{aligned}
A &= \int_0^\pi \int_a^{a(1+\sin\theta)} r dr d\theta \\
&= \int_0^\pi \left[\frac{r^2}{2}\right]_a^{a(1+\sin\theta)} d\theta \\
&= \frac{a^2}{2}\int_0^\pi \{(1+\sin\theta)^2 - 1\} d\theta \\
&= \frac{a^2}{2}\int_0^\pi (2\sin\theta + \sin^2\theta) d\theta \\
&= \frac{a^2}{2}\int_0^\pi (2\sin\theta + \frac{1}{2}(1-\cos 2\theta)) d\theta \\
&= \frac{a^2}{2}\left[-2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^\pi \\
&= \frac{a^2}{2}\left\{\left(-2\cos\pi + \frac{\pi}{2}\right) - (-2\cos 0)\right\} \\
&= \frac{a^2}{2}\left(4 + \frac{\pi}{2}\right) \\
&= a^2(2 + \pi/4).
\end{aligned}
$$

○

(The video explains how to use polar coordinates to evaluate appropriate double integrals and solves some of the examples of this subsection. V

At this point you should try some examples which involve the use of polar coordinates.) EX

4 Applications of double integrals

(This section is not covered in the videos).

We have already seen that we can use double integrals to find areas of regions in the xy-plane and volumes under the surface $u = f(x, y)$ over a given region in the xy-plane. We can also apply double integrals to finding various physical properties of thin plates or laminae.

Consider a plate of negligible thickness which occupies a region R of the xy- plane. Such a thin plate is called a **planar lamina**. Let the density of the plate be $\rho(x, y)$ units of mass per unit area. Note that this density is a mass per **unit area**, since we are dealing with a plate of negligible thickness. Also $\rho(x, y)$ is not assumed to be constant. It is a function of position within the sheet.

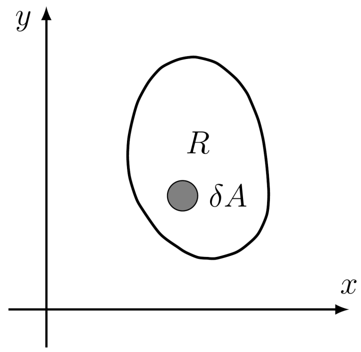

Figure 27: Planar lamina.

A small element of area, δA (see Figure 27) will have an approximate mass of $\rho(x', y')\delta A$ where (x', y') is a point in δA. The total mass, M, of the lamina is obtained by summing the individual contributions $\rho(x', y')\delta A$ and then taking the limit as $\delta A \to 0$. Hence

$$M = \iint_R \rho(x, y) dA.$$

If $\rho(x, y)$ is a constant, say ρ_0, then this reduces to

$$M = \rho_0 \iint_R dA = \rho_0 \times \text{area of } R$$

as expected. **The first moment** of the element of area about an axis is the product of its mass and its perpendicular distance from that axis. Thus the first moment of δA about the x-axis is approximately $(\rho(x', y')\delta A)y'$ where (x', y') is a point in δA. The **first moment of the lamina** about the x-axis is

$$M_x = \iint\limits_R y\rho(x, y)dA.$$

Similarly the first moment of the lamina about the y-axis is

$$M_y = \iint\limits_R x\rho(x, y)dA.$$

The **centre of mass** of the lamina is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{M} \text{ and } \bar{y} = \frac{M_x}{M}$$

The centre of mass is a very important point in the lamina since in many dynamical applications the lamina can be treated as though its mass were concentrated at this point. The centre of mass can be thought of as the point of balance of the lamina.

In a similar way, the **second moments** or **moments of inertia** of the lamina about the x-axis and the y-axis may be defined:-

$$I_x = \iint\limits_R y^2\rho(x, y)dA,$$

$$I_y = \iint\limits_R x^2\rho(x, y)dA.$$

The moment of inertia is a measure of the resistance of the lamina to rotational motion. We can also define the moment of inertia about an axis through the origin perpendicular to both x- and y-axes. This is the **polar moment of inertia**,

$$I_0 = \iint\limits_R (x^2 + y^2)\rho(x, y)dA = I_x + I_y.$$

Example 4.1 Find the centre of mass and moment of inertia about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density, $\rho(x, y) = y + 1$.

Solution The curves $x = y^2$ and $x = 2y - y^2$ are parabolas. They intersect when $y^2 = 2y - y^2$. i.e. when

$$2y^2 - 2y = 0$$

$$y(y - 1) = 0$$

$$y = 0 \text{ or } y = 1.$$

Thus they intersect at $(0, 0)$ and $(1, 1)$. We next sketch a diagram of the thin plate. This is shown in Figure 28.

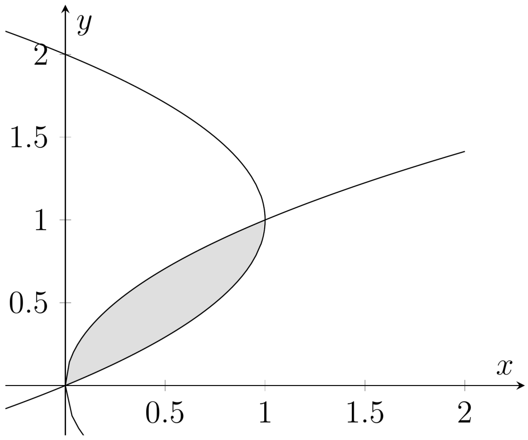

Figure 28: Thin plate bounded by $x = y^2$ and $x = 2y - y^2$.

If the mass of the plate is M, then

$$
\begin{aligned}
M &= \int_0^1 \int_{y^2}^{2y-y^2} \rho(x,y)dxdy \\
&= \int_0^1 \int_{y^2}^{2y-y^2} (y+1)dxdy \\
&= \int_0^1 [(y+1)x]_{y^2}^{2y-y^2} dy \\
&= \int_0^1 (y+1)(2y-2y^2)dy \\
&= 2\int_0^1 (y-y^3)dy \\
&= 2\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1 \\
&= \frac{1}{2}
\end{aligned}
$$

Also

$$
\begin{aligned}
M_x &= \iint\limits_R y\rho(x,y)dA \\
&= \int_0^1 \int_{y^2}^{2y-y^2} (y+1)ydxdy \\
&= 2\int_0^1 (y^2-y^4)dy \\
&= 2\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1 \\
&= \frac{4}{15}.
\end{aligned}
$$

So

$$\bar{y} = \frac{M_x}{M} = \frac{8}{15}.$$

Similarly

$$\begin{aligned}
M_y &= \iint\limits_R x\rho(x,y)dA \\
&= \int_0^1 \int_{y^2}^{2y-y^2} (y+1)x dx dy \\
&= \int_0^1 \left[\frac{(y+1)x^2}{2}\right]_{y^2}^{2y-y^2} dy \\
&= \frac{1}{2}\int_0^1 (y+1)\{(2y-y^2)^2 - y^4\}dy \\
&= 2\int_0^1 (y+1)(y^2-y^3)dy \\
&= \frac{4}{15}.
\end{aligned}$$

So

$$\bar{x} = \frac{M_y}{M} = \frac{8}{15}.$$

Hence the centre of mass is $\left(\frac{8}{15}, \frac{8}{15}\right)$.

Now

$$\begin{aligned}
I_x &= \iint\limits_R y^2\rho(x,y)dA \\
&= \int_0^1 \int_{y^2}^{2y-y^2} (y+1)y^2 dx dy \\
&= \int_0^1 (y+1)y^2(2y-2y^2)dy \\
&= 2\int_0^1 (y^3-y^5)dy \\
&= 2\left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1 \\
&= \frac{1}{6}.
\end{aligned}$$

○

(Now you should try some examples involving applications of double integrals to thin plates.) EX

5 Triple integrals

A **triple** or **volume integral** may be defined in a similar way to double integrals although a direct geometrical interpretation is not possible. When the integrand is positive we can however regard a triple integral as determining the mass of a volume of variable density (the density being given by the integrand). This is described in more details in section 7

Let $f(x, y, z)$ be a function of 3 independent variables, x, y and z which is defined for all x, y and z within a volume V. Clearly we cannot represent the function $u = f(x, y, z)$ graphically since we would require 4 mutually perpendicular axes to do so. However, we can construct the triple integral mathematically by the process described in section 2 of this package.

We divide V into n subregions in any manner and let the volumes of the subregions be $\delta V_k, \quad k = 1, 2, \ldots, n$. Choose a point (x'_k, y'_k, z'_k) in each subregion. Then the **triple** or **volume integral** of $f(x, y, z)$ over V is defined as

$$\iiint\limits_V f(x, y, z)dV = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x'_k, y'_k, z'_k)\delta V_k$$

(where $\Delta = \max\{\delta V_k, \quad k = 1, 2, \ldots, n\}$), provided the limit exists.

Again the limit must be independent of the way in which V is subdivided and the particular choice of (x'_k, y'_k, z'_k) within each subregion. We assume these conditions to be true for all the integrals with which we shall be dealing.

As mentioned above, the triple integral can be interpreted as giving the mass of a material of varying density, $f(x, y, z)$, occupying the volume V. In the particular case when $f(x, y, z) = 1$ for all (x, y, z) in V,

$$\iiint\limits_V f(x, y, z)dV = \iiint\limits_V dV = \text{the volume of } V.$$

The volume integral also satisfies the linearity property:

$$\iiint\limits_V \{af(x, y, z) + bg(x, y, z)\}dV = a \iiint\limits_V f(x, y, z)dV + b \iiint\limits_V g(x, y, z)dV$$

and if V is divided into two mutually disjoint subregions, V_1 and V_2 then

$$\iiint\limits_{V} f(x,y,z)dV = \iiint\limits_{V_1} f(x,y,z)dV + \iiint\limits_{V_2} f(x,y,z)dV$$

(The video introduces triple integrals and discusses the properties associated with them). V

6 Evaluation of triple integrals

6.1 Cartesian coordinates

By drawing the planes $x = constant,\ y = constant,\ z = constant$, V is subdivided into cuboids with edges parallel to the coordinates axis (Figure 29).

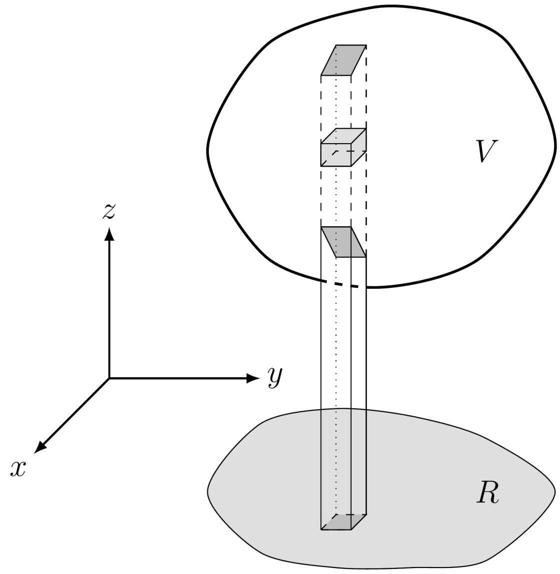

Figure 29: V subdivided into cuboids.

From Figure 29, $\delta V = \delta z \delta y \delta x$. Consider performing the z-integration first. Fix x and y and consider a rectangular column extending through the volume (a lift-shaft?). The integration must be carried out between the lower surface of V (say $z = z_1(x, y)$) and the upper surface of V (say $z = z_2(x, y)$). These two surfaces will meet in a curve around V; this curve is not necessarily in a plane parallel to the xy-plane. We now need to consider the contributions of all the columns. Let R be the projection of V in the xy-plane; i.e. R is the set of all points $(x, y, 0)$ for which (x, y, z) is in V. Then each column in V is projected onto a rectangle in R. We need to sum the contributions of each column by performing a double integral over R.

$$\text{i.e.} \qquad \iiint_V f(x,y,z)dV = \iint_R \left\{ \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \right\} dA$$

$$= \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dzdydx$$

where the limits $y = y_1(x)$ to $y = y_2(x)$ and $x = a$ to $x = b$ are chosen to define R.

Example 6.1 Evaluate $\displaystyle\iiint\limits_V xz dV$ where V is the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$ and $z = 4$, which lies in the first octant.

Solution The first octant is defined by $x \geq 0$, $y \geq 0$, $z \geq 0$. We first sketch a diagram of V (Figure 30). The equation $z = x^2$ defines a parabola at any fixed value of y.

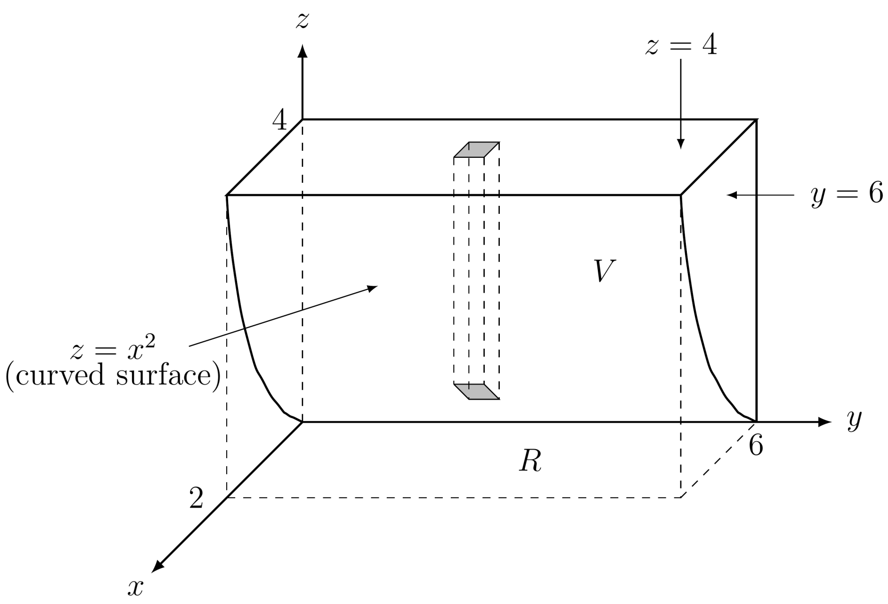

Figure 30: The volume V.

From Figure 30 the limits for z are $z = x^2$ to $z = 4$. The projection of V on the xy-plane is the rectangle bounded by $x = 0$, $y = 0$, $x = 2$ and $y = 6$. The limits for y and x are therefore $y = 0$ to $y = 6$ and $x = 0$ to $x = 2$. So

$$
\begin{aligned}
\iiint\limits_V xz dV &= \int_0^2 \int_0^6 \int_{x^2}^4 xz dz dy dx \\
&= \int_0^2 \int_0^6 \left[\frac{xz^2}{2}\right]_{x^2}^4 dy dx \\
&= \frac{1}{2}\int_0^2 \int_0^6 (16x - x^5) dy dx \\
&= \frac{1}{2}\int_0^2 \left[(16x - x^5)y\right]_0^6 dx \\
&= 3\left[8x^2 - \frac{x^6}{6}\right]_0^2 \\
&= 3 \times 2^5 \left(1 - \frac{1}{3}\right) = 2^6 = 64
\end{aligned}
$$

○

Example 6.2 Evaluate $\iiint\limits_V 45x^2 y dV$ where V is the closed region bounded by $4x + 2y + z = 8$, $x = 0$, $y = 0$ and $z = 0$.

Solution The equation $4x + 2y + z = 8$ defines a plane which intersects the axes at (2,0,0), (0,4,0) and (0,0,8) (see Figure 31).

The limits for the z-integration are $z = 0$ and $z = 8 - 4x - 2y$. The projection of V in the xy-plane is the triangular region R, bounded by $x = 0$, $y = 0$ and $4x + 2y = 8$. The limits are $y = 0$ to $y = 4 - 2x$ and $x = 0$ to $x = 2$.

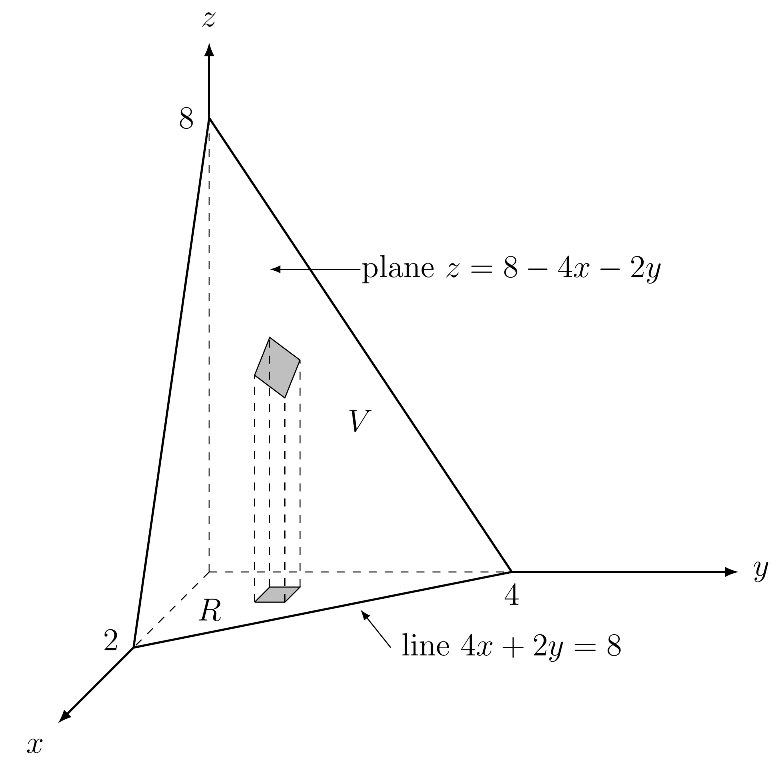

Figure 31: The volume V.

So

$$
\begin{aligned}
\iiint\limits_V 45x^2 y dV &= \int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y} 45x^2 y dz dy dx \\
&= 45 \int_0^2 \int_0^{4-2x} x^2 y(8-4x-2y) dy dx \\
&= 90 \int_0^2 \int_0^{4-2x} \{x^2(4-2x)y - x^2y^2\} dy dx \\
&= 90 \int_0^2 \left[\frac{x^2(4-2x)y^2}{2} - \frac{x^2y^3}{3} \right]_0^{4-2x} dx \\
&= 90 \int_0^2 \left\{ \frac{x^2(4-2x)^3}{2} - \frac{x^2(4-2x)^3}{3} \right\} dx.
\end{aligned}
$$

The integrand may be combined into a single term since $x^2(4-2x)^3$ is a common factor.

$$\text{Thus} \qquad \iiint_V 45x^2 y\,dV = 15\int_0^2 x^2(4-2x)^3\,dx$$

Rather than multiply this out, we can integrate by parts twice to give

$$\begin{aligned}
\iiint_V 45x^2 y\,dV &= 15\left\{\left[\frac{x^2(4-2x)^4}{4\times(-2)}\right]_0^2 - \int_0^2 \frac{2x(4-2x)^4}{4\times(-2)}dx\right\} \\
&= \frac{15}{4}\int_0^2 x(4-2x)^4 dx \\
&= \frac{15}{4}\left\{\left[\frac{x(4-2x)^5}{5\times(-2)}\right]_0^2 - \int_0^2 \frac{(4-2x)^5}{5\times(-2)}dx\right\} \\
&= \frac{3}{8}\int_0^2 (4-2x)^5 dx \\
&= \frac{3}{8}\left[\frac{(4-2x)^6}{6\times(-2)}\right]_0^2 \\
&= \frac{3\times 4^6}{16\times 6} = 128.
\end{aligned}$$

○

(The video shows how to evaluate triple integrals in cartesian coordinates and covers the examples in this subsection. V

At this point you should try some examples of evaluating triple integrals in cartesian coordinates. You will find drawing the diagrams difficult. Most of us have to draw several rough sketches before we produce a useful picture. Practice is the answer!) EX

6.2 Cylindrical polar coordinates (w, ϕ, z)

The **cylindrical polar coordinates** (w, ϕ, z) of a point in a 3-dimensional space are related to its cartersian coordinates (x, y, z) by the equations

$$x = w\cos\phi, \qquad y = w\sin\phi, \qquad z = z.$$

Thus z is the same coordinate in both systems; w and ϕ are defined in a similar way to the plane polar coordinates (r, θ) defined in section 3.3. We use a different notation because r and θ are reserved for spherical polar coordinates (see section 6.3). The cylindrical polar coordinates (w, ϕ, z) are given by

$$w = \sqrt{x^2 + y^2}, \qquad \tan\phi = \frac{y}{x}, \qquad z = z$$

where $w \geq 0$ and $-\pi < \phi \leq \pi$. The same careful considerations for choosing the appropriate value of ϕ need to be employed here as for θ in plane polars. Note that for points on the z-axis, ϕ is undefined. This does not usually cause any difficulty in practice. Figure 32 illustrates cylindrical polar coordinates.

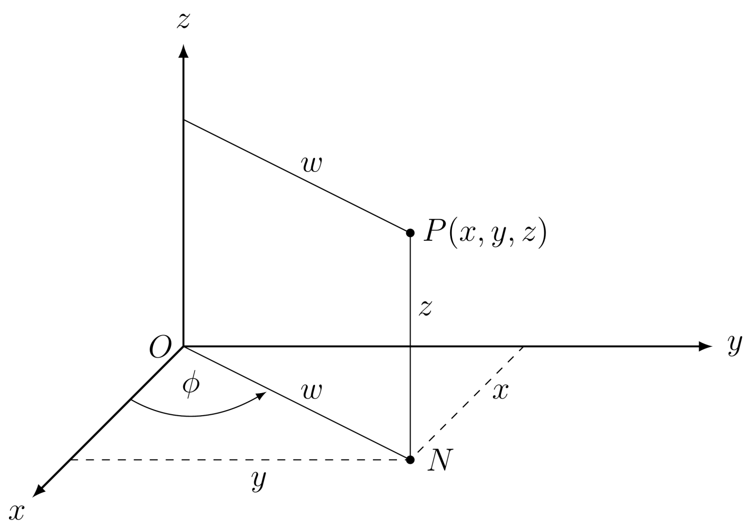

Figure 32: Cylindrical polar coordinates.

In cylindrical polar coordinates we subdivide a volume V by drawing the surfaces, $w = constant$, $\phi = constant$, and $z = constant$. This will produce elements of volume whose shape is shown in Figure 33.

Consider the point P with cylindrical polar coordinates (w, ϕ, z). Then the point Q will have coordinates $(w+\delta w, \phi, z)$ where $PQ = \delta w$. The point R will have coordinates $(w, \phi+\delta\phi, z)$ and S will have coordinates $(w, \phi, z+\delta z)$. Thus the volume of the element is approximately $\delta V = PQ \times PR \times PS$. But $PQ = \delta w$, $PS = \delta z$ and since PR is an arc of a circle of radius w, $PR = w\delta\phi$. Therefore

$$\delta V \approx \delta w(w\delta\phi)(\delta z) = w\delta w\delta\phi\delta z$$

Note there will also be higher order terms as with plane polar coordinates, but these can be neglected as $\delta V \to 0$. Thus in cylindrical polar coordinates:

$$dV = wdwd\phi dz$$

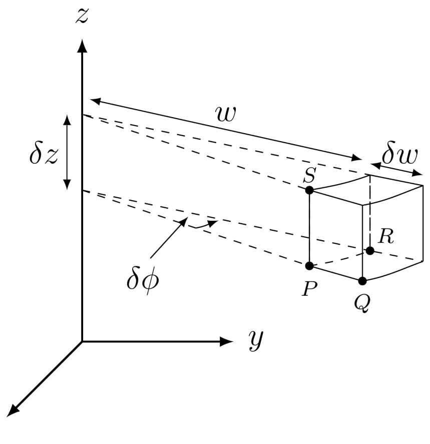

Figure 33: A volume element in cylindrical polars.

We carry out triple integrals in cylindrical polar coordinates in a similar manner to cartesians by doing the z-integration first between the lower and upper surfaces of V (say from $z = z_1(w, \phi)$ to $z = z_2(w, \phi)$). We then integrate over the projection of V in the xy-plane, treating w, ϕ as plane polar coordinates. So

$$\iiint\limits_V f(x,y,z)dV = \int_{\phi_1}^{\phi_2} \int_{w_1(\phi)}^{w_2(\phi)} \int_{z_1(w,\phi)}^{z_2(w,\phi)} F(w,\phi,z)w dz dw d\phi$$

where $F(w, \phi, z) = f(x, y, z)$. In other words $F(w, \phi, z)$ is the function of w, ϕ and z obtained when x and y are replaced by $w \cos\phi$ and $w \sin\phi$ respectively.
Example 6.3 Use cylindrical polar coordinates to evaluate

$$\iiint\limits_V \sqrt{x^2 + y^2} dV$$

where V is the region bounded by the surfaces $z = x^2+y^2$ and $z = 8-(x^2+y^2)$.

Solution The surface $z = x^2 + y^2$ is a paraboloid. Cross-sections at $x =$ *constant* or $y =$ *constant* are parabolas, whereas cross-sections at $z =$ *constant* (> 0) are circles. Note that $z \geq 0$ for all x, y. Similarly $z = 8 - (x^2 + y^2)$ is also a paraboloid, but $z \leq 8$. Figure 34 shows the two paraboloids and V is the volume contained between them.

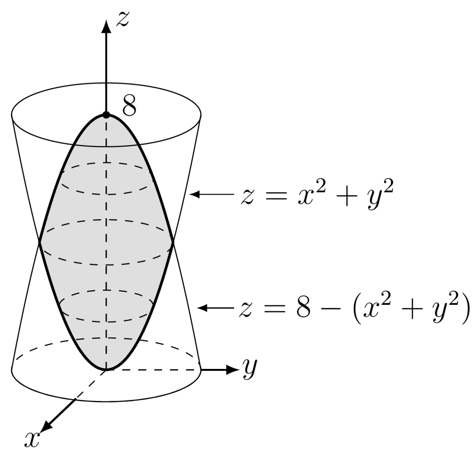

Figure 34: The volume V.

The paraboloids intersect when $8 - (x^2 + y^2) = x^2 + y^2$, i.e. $x^2 + y^2 = 4$, and this gives $z = 4$. The projection of V on the xy-plane is thus the interior of the circle $x^2 + y^2 = 4$.

In cylindrical polars, the integration limits are

$$\begin{aligned} z = w^2 &\quad \text{to} \quad z = 8 - w^2 \\ w = 0 &\quad \text{to} \quad w = 2 \\ \phi = -\pi &\quad \text{to} \quad \phi = \pi \end{aligned}$$

Therefore

$$\begin{aligned} \iiint\limits_V \sqrt{x^2 + y^2} &= \int_{-\pi}^{\pi} \int_0^2 \int_{w^2}^{8-w^2} w.w dz dw d\phi \\ &= \int_{-\pi}^{\pi} \int_0^2 \left[w^2 z\right]_{w^2}^{8-w^2} dw d\phi \\ &= \int_{-\pi}^{\pi} \int_0^2 \{w^2(8 - w^2) - w^4\} dw d\phi \\ &= \int_{-\pi}^{\pi} \left[\frac{8w^3}{3} - \frac{2w^5}{5}\right]_0^2 d\phi \\ &= \int_{-\pi}^{\pi} \frac{2^7}{15} d\phi = \frac{2^8 \pi}{15} = \frac{256\pi}{15}. \end{aligned}$$

(The video shows how to use cylindrical polar coordinates to evaluate appropriate triple integrals and covers the example in this subsection. V

You should now try to evaluate some triple integrals using cylindrical polars). EX

6.3 Spherical polar coordinates (r, θ, ϕ)

The **spherical polar coordinates,** (r, θ, ϕ) of a point in a 3-dimensional space are related to its cartesian coordinates (x, y, z) by

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta.$$

The coordinate r is the radial distance of a point, P, from the origin, O; θ is the angle that OP makes with the positive z-axis and ϕ is the angle that ON makes with the positive x-axis, where N is the foot of the perpendicular from P to the xy-plane. See Figure 35 for an illustration.

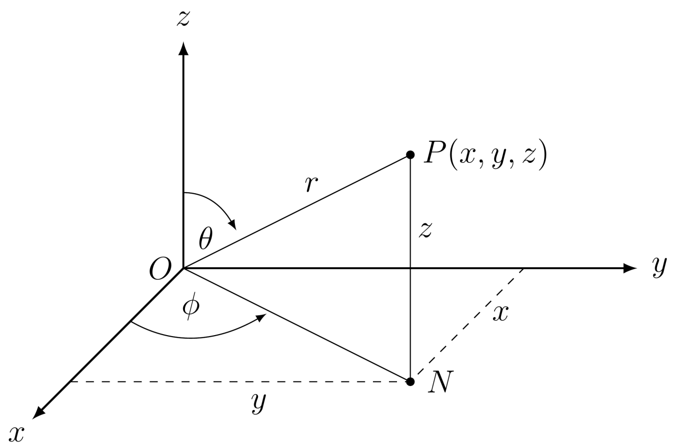

Figure 35: Spherical polar coordinates.

Note that $r \geq 0$, $0 \leq \theta \leq \pi$ and $-\pi < \phi \leq \pi$. The angle θ is called the **polar angle** and ϕ is called the **azimuthal angle**. The azimuthal angle ϕ is the same one that occurs in cylindrical polars.

Expressions giving r, θ and ϕ in terms of x, y and z are

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \cos^{-1}\left\{\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right\}, \qquad \tan\phi = \frac{y}{x}$$

where the appropriate value of ϕ must be chosen.

In spherical polar coordinates a volume V is subdivided by drawing the surfaces $r = \textit{constant}$, $\theta = \textit{constant}$ and $\phi = \textit{constant}$. The surfaces $r = \textit{constant}$ are concentric spheres centred on the origin; $\theta = \textit{constant}$ generates conical surfaces with the z-axis as the axis of symmetry and $\phi = \textit{constant}$ generates planes perpendicular to the xy-plane. This will produce elements of volume whose shape is shown in Figure 36.

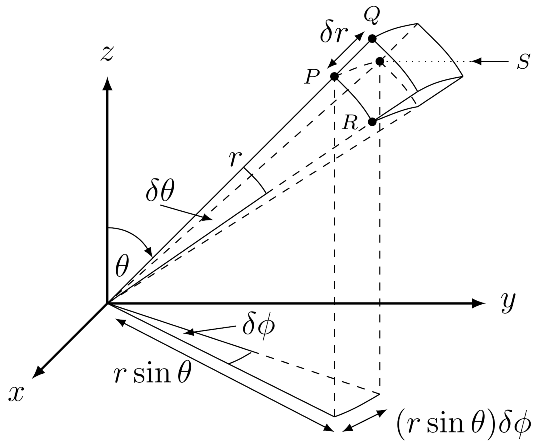

Figure 36: A volume element in spherical polars.

Consider the point P with spherical polar coordinates (r, θ, ϕ). Then the point Q will have coordinates $(r + \delta r, \theta, \phi)$, R will have coordinates $(r, \theta + \delta\theta, \phi)$ and S will have coordinates $(r, \theta, \phi + \delta\phi)$. Note that $PQ = \delta r$, $PR = r\delta\theta$ and, because PS is an arc of a circle of radius $r\sin\theta$, $PS = r\sin\theta\delta\phi$. Thus the volume of the element is approximately

$$\begin{aligned}\delta V &\approx PQ \times PR \times PS \\ &= \delta r(r\delta\theta)(r\sin\theta\delta\phi) \\ &= r^2\sin\theta\delta r\delta\theta\delta\phi\end{aligned}$$

There will also be higher order terms in an exact expression for δV, but these may be neglected as $\delta V \to 0$. Thus in spherical polar coordinates

$$dV = r^2\sin\theta dr d\theta d\phi$$

Triple integrals in spherical polars are normally carried out by integrating firstly with respect to r between the inner surface, $r = r_1(\theta, \phi)$ and the outer surface $r = r_2(\theta, \phi)$. (See Figure 37). We then integrate with respect to θ from $\theta = \theta_1(\phi)$ to $\theta = \theta_2(\phi)$ and finally we integrate with respect to ϕ from $\phi = \phi_1$ to $\phi = \phi_2$.

So $$\iiint\limits_V f(x, y, z)dV = \int_{\phi_1}^{\phi_2}\int_{\theta_1(\phi)}^{\theta_2(\phi)}\int_{r_1(\theta,\phi)}^{r_2(\theta,\phi)} F(r, \theta, \phi)r^2\sin\theta dr d\theta d\phi$$

where $F(r, \theta, \phi) = f(x, y, z)$. Spherical polar coordinates are most useful for integrating over spheres or parts of spheres. The limits are then usually much easier than the very general ones given above.

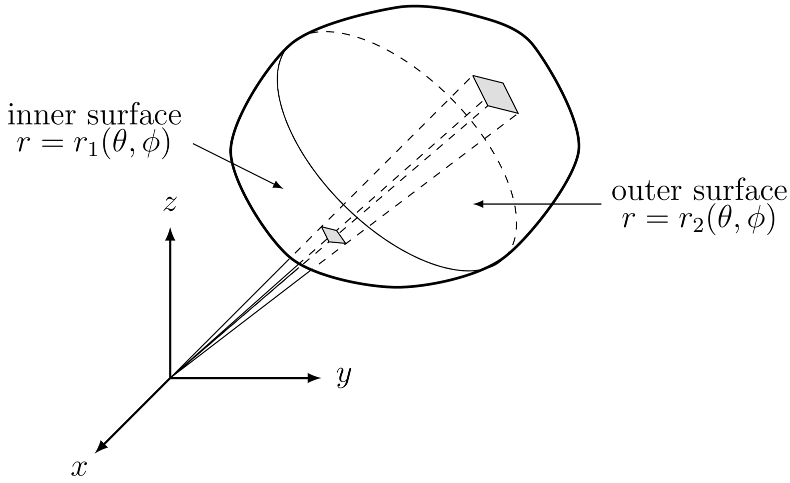

Figure 37: Integration in spherical polars.

Example 6.4 Integrate xyz over the volume in the first octant bounded by the coordinate planes and the sphere $x^2 + y^2 + z^2 = 1$.

Solution The volume here is just one eighth of the volume of the unit sphere. It is illustrated in Figure 38.

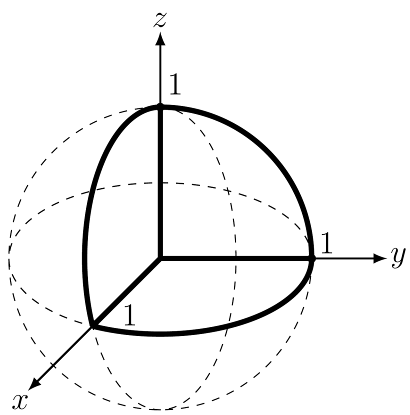

Figure 38: The volume V.

The limits in spherical polar coordinates are easy. They are

$$\begin{aligned} r = 0 &\quad \text{to} \quad r = 1 \\ \theta = 0 &\quad \text{to} \quad \theta = \frac{\pi}{2} \\ \phi = 0 &\quad \text{to} \quad \phi = \frac{\pi}{2}. \end{aligned}$$

So $$\iiint_V xyzdV$$

$$\begin{aligned} &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r\cos\theta)\times r^2\sin\theta dr d\theta d\phi \\ &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^5\sin^3\theta\cos\theta\sin\phi\cos\phi dr d\theta d\phi \\ &= \left(\int_0^{\pi/2}\sin\phi\cos\phi d\phi\right)\left(\int_0^{\pi/2}\sin^3\theta\cos\theta d\theta\right)\left(\int_0^1 r^5 dr\right) \\ &= \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^1 \\ &= \frac{1}{2}\times\frac{1}{4}\times\frac{1}{6} = \frac{1}{48}. \end{aligned}$$

○

Example 6.5 Find the volume of the region, V, bounded by the surface $r = 1 + \cos\theta$ and the sphere $r = 2$.

Solution For any fixed value of ϕ, the equation $r = 1 + \cos\theta$, $(0 \le \theta \le \pi)$, generates half of a cardioid. Rotating this around the z-axis produces the solid shown in Figure 39.

This shape lies wholly inside the sphere $r = 2$ (just touching it at the top where $\theta = 0$). The limits in spherical polar coordinates are therefore:

$$\begin{aligned} r = 1 + \cos\theta &\quad \text{to} \quad r = 2 \\ \theta = 0 &\quad \text{to} \quad \theta = \pi \\ \phi = -\pi &\quad \text{to} \quad \phi = \pi \end{aligned}$$

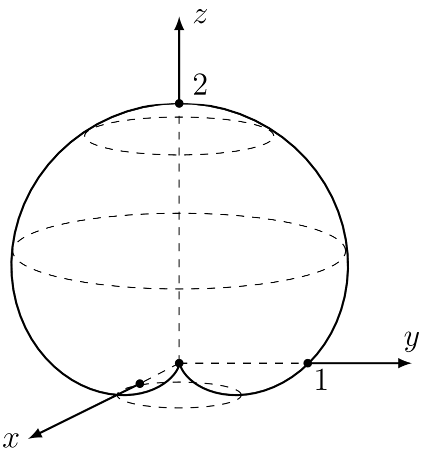

Figure 39: The surface $r = 1 + \cos\theta$.

$$
\begin{aligned}
\text{So} \qquad V &= \iiint\limits_V dV \\
&= \int_{-\pi}^{\pi}\int_0^{\pi}\int_{1+\cos\theta}^{2} r^2 \sin\theta dr d\theta d\phi \\
&= \frac{1}{3}\int_{-\pi}^{\pi}\int_0^{\pi} \left[r^3 \sin\theta\right]_{1+\cos\theta}^{2} d\theta d\phi \\
&= \frac{1}{3}\int_{-\pi}^{\pi}\int_0^{\pi} \{8-(1+\cos\theta)^3\} \sin\theta d\theta d\phi \\
&= \left(\frac{1}{3}\int_{-\pi}^{\pi} d\phi\right)\left(\int_0^{\pi} \{8-(1+\cos\theta)^3\} \sin\theta d\theta\right) \\
&= \frac{2\pi}{3}\left[-8\cos\theta + \frac{(1+\cos\theta)^4}{4}\right]_0^{\pi} \\
&= \frac{2\pi}{3}\left\{8 - \{-8 + \frac{2^4}{4}\}\right\} \\
&= \frac{2\pi}{3}\{16-4\} = 8\pi.
\end{aligned}
$$

○

(The video explains how to use spherical polar coordinates to evaluate certain triple integrals and covers the first example in this subsection. V

You should now try some examples using spherical polars to evaluate triple integrals.) EX

7 Applications of triple integrals

We have already seen that we can use triple integrals to find the volumes of regions in a 3-dimensional space. Just as with double integrals we can use triple integrals to find physical properties of materials. Consider a material whose density is $\rho(x, y, z)$, occupying a volume V in a 3-dimensional space. The density here is a mass per unit volume. We do not assume that $\rho(x, y, z)$ is constant, but that it is a function of position in space. Physically we might have a gas occupying V. A small element of volume will have a mass $\rho(x', y', z')\delta V$ where (x', y', z') is a point in δV. The total mass within V is therefore

$$M = \iiint\limits_V \rho(x, y, z) dV$$

If $\rho(x, y, z)$ is a constant, say ρ_0, then this reduces to

$$M = \rho_0 \iiint\limits_V dV = \rho_0 \times \text{(the volume of V)}$$

i.e. mass = density × volume, as expected.

The **centre of mass** of the material within V is the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M} \iiint\limits_V x\rho(x, y, z) dV, \qquad \bar{y} = \frac{1}{M} \iiint\limits_V y\rho(x, y, z) dV,$$

$$\text{and} \qquad \bar{z} = \frac{1}{M} \iiint\limits_V z\rho(x, y, z) dV.$$

The **moment of inertia** of the material within V about a given axis is

$$I = \iiint\limits_V \rho(x, y, z)\{d(x, y, z)\}^2 dV$$

where $d(x, y, z)$ is the perpendicular distance of a point (x, y, z) of the material from the given axis (see Figure 40).

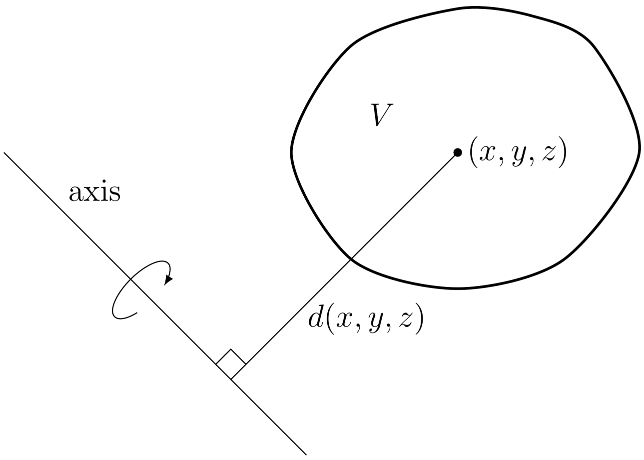

Figure 40: Calculating the moment of inertia about an axis.

Example 7.1 The density of a right circular cone of height h, and base radius r, at a point at distance w from its axis of symmetry is $\rho = \dfrac{\rho_0 w}{a}$. Calculate

(i) its mass,

(ii) the position of its centre of mass, and

(iii) its moment of inertia about the axis of symmetry.

Solution

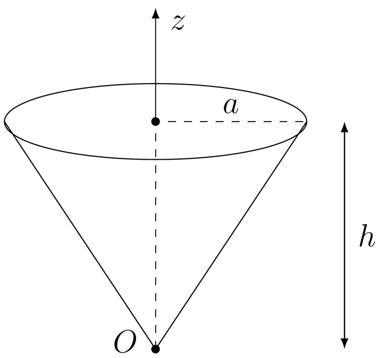

Figure 41: The cone positioned with vertex at the origin and axis vertical.

(i) Because of the axial symmetry it is natural to use cylindrical polar coordinates for this problem (see figure 41). We need to determine the correct limits to describe the cone. If we perform the z-integration first, the lower limit will be determined by the curved surface of the cone. Figure 42 shows how to calculate this lower limit for z in terms of the distance, w from the axis, of a point on the surface of the cone.

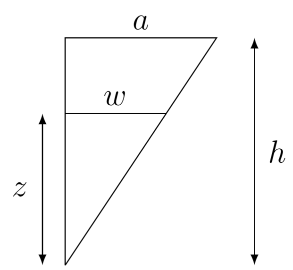

Figure 42: Calculation of the lower limit for z in terms of w.

By similar triangles in Figure 42

$$\frac{z}{w} = \frac{h}{a}$$

so $z = hw/a$. The limits are therefore:

$$\begin{aligned} z = \frac{hw}{a} &\quad \text{to} \quad z = h \\ w = 0 &\quad \text{to} \quad w = a \\ \phi = -\pi &\quad \text{to} \quad \phi = \pi \end{aligned}$$

Hence

$$\begin{aligned} M &= \int_{-\pi}^{\pi}\int_{0}^{a}\int_{hw/a}^{h} \frac{\rho_0 w}{a}.wdzdwd\phi \\ &= \frac{\rho_0}{a}\int_{-\pi}^{\pi}\int_{0}^{a} \left[w^2 z\right]_{hw/a}^{h} dwd\phi \\ &= \frac{\rho_0 h}{a}\int_{-\pi}^{\pi}\int_{0}^{a} \left\{w^2 - \frac{w^3}{a}\right\} dwd\phi \\ &= \frac{\rho_0 h}{a}\int_{-\pi}^{\pi} \left[\frac{w^3}{3} - \frac{w^4}{4a}\right]_0^a d\phi \\ &= \frac{\rho_0 h}{a}\int_{-\pi}^{\pi} \frac{a^3}{12} d\phi = \frac{\pi\rho_0 h a^2}{6}. \end{aligned}$$

(ii) By symmetry the centre of mass will be located on the axis of the cone, i.e. at $(0, 0, \bar{z})$ where

$$\begin{aligned}
\bar{z} &= \frac{1}{M}\iiint\limits_V z\rho(x,y,z)dV \\
&= \frac{1}{M}\int_{-\pi}^{\pi}\int_0^a\int_{hw/a}^h z.\frac{\rho_0 w}{a}.wdzdwd\phi \\
&= \frac{\rho_0}{Ma}\int_{-\pi}^{\pi}\int_0^a\left[\frac{w^2z^2}{2}\right]_{hw/a}^h dwd\phi \\
&= \frac{\rho_0 h^2}{2Ma}\int_{-\pi}^{\pi}\int_0^a\left\{w^2-\frac{w^4}{a^2}\right\}dwd\phi \\
&= \frac{\rho_0 h^2}{2Ma}\int_{-\pi}^{\pi}\left[\frac{w^3}{3}-\frac{w^5}{5a^2}\right]_0^a d\phi \\
&= \frac{\rho_0 h^2}{2Ma}\int_{-\pi}^{\pi}\frac{2a^3}{15}d\phi \\
&= \frac{2\pi\rho_0 h^2 a^2}{15M} \\
&= \frac{2\pi\rho_0 h^2 a^2}{15}.\frac{6}{\pi\rho_0 ha^2} = \frac{4h}{5}.
\end{aligned}$$

Hence the centre of mass is at $(0,0,4h/5)$.
(iii) The perpendicular distance of a point inside the cone from its axis of symmetry is the cylindrical polar coordinate w. So the required moment of inertia is

$$\begin{aligned}
I &= \int_{-\pi}^{\pi}\int_0^a\int_{hw/a}^h \frac{\rho_0 w}{a}.w^2.wdzdwd\phi \\
&= \frac{\rho_0}{a}\int_{-\pi}^{\pi}\int_0^a\left[w^4z\right]_{hw/a}^h dwd\phi \\
&= \frac{\rho_0 h}{a}\int_{-\pi}^{\pi}\int_0^a\left\{w^4-\frac{w^5}{a}\right\}dwd\phi \\
&= \frac{\rho_0 h}{a}\int_{-\pi}^{\pi}\left[\frac{w^5}{5}-\frac{w^6}{6a}\right]_0^a d\phi \\
&= \frac{\rho_0 h}{a}\int_{-\pi}^{\pi}\frac{a^5}{30}d\phi \\
&= \frac{\pi\rho_0 ha^4}{15} \qquad \left(=\frac{6Ma^2}{15}=\frac{2}{5}Ma^2\right)
\end{aligned}$$

○

(The video shows the application of triple integrals to mass, centre of mass and moments of inertia and covers the example of this section. V

At this point you should try some examples involving the applications of triple integrals.) EX

8 Jacobians

(This section is not covered in the videos).

So far we have evaluated double and triple integrals in both cartesian and polar coordinate systems. We have examined the elementary area or volume to determine δA and δV in each case. However, there are rules which will automatically generate δA and δV in general coordinate systems.

Suppose we wish to transform a double integral from (x, y) coordinates to (u, v) coordinates. We consider the elementary area formed by drawing the curves $u = constant$ and $v = constant$ (Figure 43).

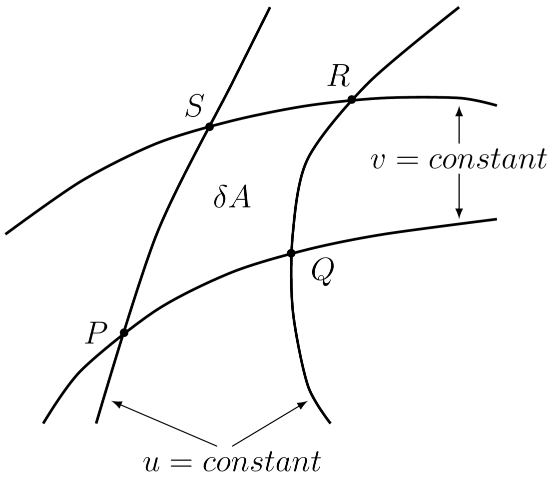

Figure 43: An element of area in uv-coordinates.

If P has (cartesian) coordinates (x, y), then Q will have coordinates $(x + \delta x, y + \delta y) \approx (x + \dfrac{\partial x}{\partial u}\delta u, y + \dfrac{\partial y}{\partial u}\delta u)$, since v is a constant along PQ. Similarly S will have coordinates $(x + \dfrac{\partial x}{\partial v}\delta v, y + \dfrac{\partial y}{\partial v}\delta v)$ approximately, since u is a constant along PS. The element of area, $PQRS$, is approximately a parallelogram with sides PQ and PS. We require an expression for the area of a parallelogram and make use of the following result.

Consider a parallelogram with vertices at $(0, 0)$, (a_1, b_1), (a_2, b_2), and $(a_1 + a_2, b_1 + b_2)$. (See Figure 44.)

The area of the parallelogram is $|a_1 b_2 - a_2 b_1|$. The modulus signs are required since we attach no meaning to the sign of the area. This result can be proved using coordinate geometry, but the proof is not included here.

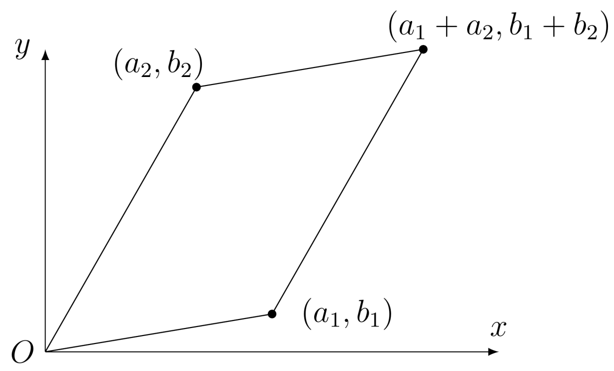

Figure 44: The area of a parallelogram.

To apply this result to $PQRS$ we need to translate P to the origin. Then Q is shifted to the point with coordinates $(\frac{\partial x}{\partial u}\delta u, \frac{\partial y}{\partial u}\delta u)$ and S to the point with coordinates $(\frac{\partial x}{\partial v}\delta v, \frac{\partial y}{\partial v}\delta v)$. Thus

$$\begin{aligned}
\delta A &= \left|\left(\frac{\partial x}{\partial u}\delta u\right)\left(\frac{\partial y}{\partial v}\delta v\right) - \left(\frac{\partial y}{\partial u}\delta u\right)\left(\frac{\partial x}{\partial v}\delta v\right)\right| \\
&= \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|\delta u \delta v \\
&= |J|\delta u \delta v
\end{aligned}$$

where J is the determinant,

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

J is called the **Jacobian** of the transformation and is alternatively written

$$J = \frac{\partial(x, y)}{\partial(u, v)}$$

to emphasise the variables involved.

Note that for the transformation from cartesians to polars,

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\text{so} \qquad J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r, \qquad \text{as expected.}$$

Thus the rule for changing from (x, y) to (u, v) coordinates is to replace dA by $|J|dudv$ where J is the Jacobian of the transformation.

A similar result applies to triple integrals. To change from (x, y, z) to (u, v, w) coordinates replace dV by $|J|dudvdw$ where

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Example 8.1 By the change of variables

$$u = \frac{x-y}{x+y} \tag{5}$$
$$v = x+y \tag{6}$$

evaluate $\iint_R e^{-(x-y)/(x+y)} dA$ where R is the region in the first quadrant bounded by the lines $x + y = 3$, $x + y = 2$, $y = 0$ and $y = x/3$.

Solution First we need to work out the Jacobian of the transformation from (x, y) to (u, v).
Since $u = \dfrac{x-y}{x+y}$ and $v = x + y$ we have $x - y = uv$ and $x + y = v$ giving

$$x = \frac{1}{2}v(u+1) \qquad \text{and} \qquad y = \frac{1}{2}v(1-u)$$

$$\text{So} \qquad \frac{\partial x}{\partial u} = \frac{1}{2}v, \qquad \frac{\partial x}{\partial v} = \frac{1}{2}(u+1), \qquad \frac{\partial y}{\partial u} = -\frac{1}{2}v, \qquad \frac{\partial y}{\partial v} = \frac{1}{2}(1-u)$$

These give

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2}v & \frac{1}{2}(u+1) \\ -\frac{1}{2}v & \frac{1}{2}(1-u) \end{vmatrix}$$
$$= \frac{1}{4}v(1-u) + \frac{1}{4}v(u+1) = \frac{1}{2}v.$$

We now need to describe the region R in terms of u and v. The region is as shown in Figure 45.

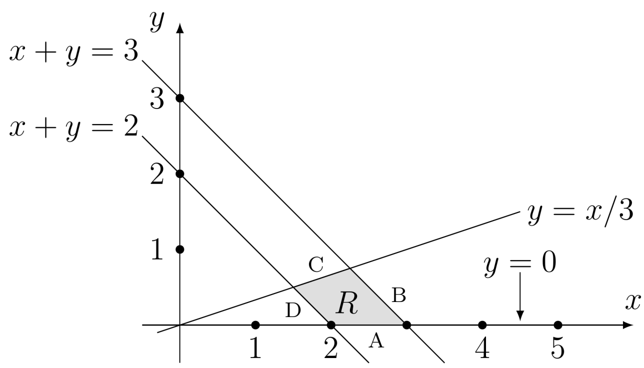

Figure 45: The region R.

Consider each of the boundaries of R in turn.
On boundary (A), $y = 0$ and $2 \le x \le 3$. Using $y = 0$, equation (5) gives $u = (x-0)/(x+0) = 1$ and equation (6) gives $v = x$. Thus boundary (A) is described by $u = 1$ and $2 \le v \le 3$.
On boundary (B), $x+y = 3$ and x increases from the intersection of $x+y = 3$ with $y = x/3$ up to $x = 3$. The point of intersection is given by $x+(x/3) = 3$, i.e. $x = 9/4$. So on boundary (B), $x + y = 3$ and $(9/4) \le x \le 3$. Using equations (5) and (6) these give $v = 3$ and $u = [x-(3-x)]/3 = (2x-3)/3$. As x varies from 9/4 to 3, u increases from 1/2 to 1, i.e. boundary (B) is described by $v = 3$ and $(1/2) \le u \le 1$.
On boundary (C), $y = x/3$ and x increases from the intersection of $x+y = 2$ with $y = x/3$ up to $x = 9/4$. The point of intersection is given by $x+(x/3) = 2$, i.e. $x = 3/2$. So on boundary (C), $y = x/3$ and $(3/2) \le x \le (9/4)$. Using equations (5) and (6) these give $u = (x-(x/3))/(x+(x/3)) = 1/2$ and $v = 4x/3$. As x increases from 3/2 to 9/4, v increases from 2 to 3. Thus boundary (C) is described by $u = 1/2$ and $2 \le v \le 3$.

Finally on boundary (D), $x + y = 2$ and $(3/2) \leq x \leq 2$. Equations (5) and (6) then give $v = 2$ and $u = [x - (2 - x)]/2 = x - 1$. As x increases from 3/2 to 2, u increases from 1/2 to 1. Hence boundary (D) is described by $v = 2$ and $(1/2) \leq u \leq 1$.
Thus the new limits when the integral is transformed to variables u and v become

$$\begin{aligned} u = \frac{1}{2} \quad &\text{to} \quad u = 1 \\ v = 2 \quad &\text{to} \quad v = 3. \end{aligned}$$

Figure 46 shows the region R in terms of u and v.

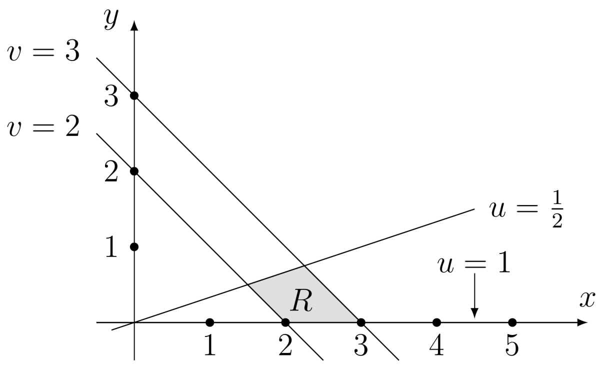

Figure 46: The region R.

We have now done all the hard work for this problem and are in a position to evaluate the integral.

$$\begin{aligned} \iint\limits_R e^{-(x-y)/(x+y)} dA &= \int_2^3 \int_{1/2}^1 e^{-u} \left(\frac{1}{2} v\right) du dv \\ &= \frac{1}{2} \left(\int_2^3 v dv\right) \left(\int_{1/2}^1 e^{-u} du\right) \\ &= \frac{1}{2} \left[\frac{1}{2} v^2\right]_2^3 \left[-e^{-u}\right]_{1/2}^1 \\ &= \frac{1}{4} (3^2 - 2^2)(-e^{-1} + e^{-1/2}) \\ &= \frac{5}{4} \left(\frac{1}{\sqrt{e}} - \frac{1}{e}\right). \end{aligned}$$

○

(Now you should try some double integrals that involve changing the variables by means of Jacobians). EX

9 Summary

When you have completed this package you should be able to do the things listed below:

1. understand what is meant by the **double integral** of a function of two variables,
2. interpret the **double integral** geometrically,
3. evaluate double integrals in **cartesian coordinates**,
4. interchange the **order of integration** in a double integral,
5. evaluate double integrals in **polar coordinates**,
6. apply double integrals to finding the **mass, centre of mass and moments of inertia** of a lamina,
7. understand what is meant by the **triple integral** of a function of three variables,
8. evaluate triple integrals in **cartesian coordinates**,
9. evaluate triple integrals in **cylindrical polar coordinates**,
10. evaluate triple integrals in **spherical polar coordinates**,
11. apply triple integrals to finding the **mass, centre of mass and moments of inertia** of a body,
12. understand what is meant by the **Jacobian** of a transformation,
13. use **Jacobians** to transform the variables in a double integral,

10 Bibliography

For textbooks covering the basic prerequisites for this package (differentiation and integration of functions of one variable) see, for example, one of the following (although there are dozens of other suitable textbooks many of which are in the University library).

Stroud, K. A. Engineering Mathematics (third edition), Macmillan, 1992.

Jeffrey, A. Mathematics for Engineers and Scientists, Van Nostrand, 1989.

Thomas, G. B. and Finney, R. L. Calculus and Analytic Geometry, Addison-Wesley, 1988.

Larson, R. E., Hostetler, R. P. and Edwards, B. H. Calculus, D. C. Heath and Company, 1990.

Gilbert, J. Guide to Mathematical Methods, Macmillan, 1991.

Anton, H. Calculus with Analytic Geometry, Wiley, 1992.

All the above books with the exceptions of Anton and Jeffrey include an elementary treatment of multiple integrals. However, there are many other textbooks in the University Library which cover the subject in greater depth. A standard work is

Cohen, G. L. Multiple Integrals, G. Bell, 1974.

Your tutor should be able to advise you which textbooks are suitable for your own needs, but you need never be short of an alternative approach or more questions to try!

11 Appendix - Video Summaries

There are six videos associated with the topic of multiple integrals. The presenter is Dave Parker from the Department of Mathematics at the University of Central Lancashire. We recommend that you read the preamble to these notes which makes some suggestions about how you should approach viewing the videos.

Video title: Multiple Integrals (part 1). (37 minutes)

Summary

1. Revision of the definite integral as the limit of a sum.
2. **Double integrals:** Definition of the double integral and its geometric interpretation. Properties of double integrals.
3. Evaluation of double integrals in **Cartesian coordinates**. Geometric interpretation. Determination of limits.
4. The examples

$$\int_1^3 \int_2^3 (x^2 - 2xy)dydx,$$

$$\int_0^\pi \int_0^x x \sin ydydx$$

and

$$\iint\limits_R xydA$$

where R is bounded by $y = 0$, $x = 2$ and $x^2 = 4y$.

Video title: Multiple Integrals (part 2). (38 minutes)

Summary

1. **Change of order** in a double integral in cartesian coordinates.

2. The examples
$$\int_0^1 \int_x^1 x^2 e^{-y^2} dy dx$$
and
$$\int_0^1 \int_y^{2-y} f(x,y) dx dy.$$

3. Definition of **polar cordinates.** Element of area in polar coordinates. Evaluation of double integrals in polar coordinates.

4. The example
$$\iint\limits_R \sqrt{x^2+y^2} dA$$
where R is the region in the first quadrant bounded by $x^2+y^2=a^2$, $y=x$ and $x=0$.

5. The example: Use a double integral to find the area which is inside the cardioid $r=a(1+\sin\theta)$ and outside the circle $r=a$.

Video title: Multiple Integrals (part 3). (28 minutes)

Summary

1. Definition of the **triple integral.** Properties of triple integrals.

2. Evaluation of triple integrals in **cartesian coordinates.** Geometric interpretation. Determination of limits.

3. The example
$$\iiint\limits_V xz dV$$
where V is the region bounded by $x=0$, $y=0$, $y=6$, $z=x^2$ and $z=4$ which lies in the first octant.

4. The example

$$\iiint\limits_V 45x^2y dV$$

where V is the region bounded by $4x + 2y + z = 8$, $x = 0$, $y = 0$ and $z = 0$.

Video title: Multiple Integrals (part 4). (21 minutes)

Summary

1. Definition of **cylindrical polar coordinates.** Element of volume in cylindrical polar coordinates.

2. Evaluation of integrals in cylindrical polar coordinates. Determination of limits.

3. The example

$$\iiint\limits_V \sqrt{x^2 + y^2} dV,$$

where V is the region bounded by $z = x^2 + y^2$ and $z = 8 - (x^2 + y^2)$.

Video title: Multiple Integrals (part 5). (28 minutes)

Summary

1. Definition of **spherical polar coordinates.** Element of volume in spherical polar coordinates.

2. Evaluation of integrals in spherical polar coordinates. Determination of limits.

3. The example

$$\iiint_V \frac{1}{(x^2+y^2+z^2)^{1/2}} dV$$

where V is the region between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ where $a > b > 0$.

4. The example

$$\iiint_V xyz dV$$

where V is the volume in the first octant bounded by the coordinate planes and the sphere $x^2 + y^2 + z^2 = 1$.

Video title: Multiple Integrals (part 6). (19 minutes)

Summary

1. Application of triple integrals to determining the **mass** of a body.

2. Application of triple integrals to determining the **centre of mass and moments of inertia** of a body.

3. The example: Determine the mass, centre of mass and moment of inertia about the axis of symmetry of a cone of height h and base radius a, whose density at a point at distance w from its axis of symmetry is $\rho = \dfrac{\rho_0 w}{a}$.