LAPLACE TRANSFORMS

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First Edition, May 1993.
(Reprinted 2022)

[1]The authors were supported by Enterprise funding.

Contents

1 Preamble

1.1 About this package

This package is for people who need to solve differential equations using Laplace transforms. It doesn't contain a lot of theory. It isn't really designed for pure mathematicians who require a course discussing the existence, uniqueness and inversion of Laplace transforms. You will find that you need a background knowledge of differentiation and integration in order to get the most out of this package. You need to be familiar with the use of partial fractions and to have some background knowledge about first and second order differential equations. If you are a bit rusty, don't worry - but it would be sensible to do some revision either at the start or as the need arises. Reasonable revision texts are given in the bibliography (Section 10).

If you complete the whole package you should be able to

- understand and reproduce the **definition** of the Laplace transform,
- obtain Laplace transforms of easy functions **directly** from the definition,
- obtain the Laplace transforms of first and second **derivatives**,
- **invert** transforms using tables in conjunction with algebraic manipulations such as **partial fractions** and **completing the square**,
- use Laplace transforms to **solve** first and second order linear differential equations with appropriate initial conditions,
- use Laplace transforms to solve **simultaneous** differential equations,
- deal with discontinuous input functions expressible in terms of the **unit step function** and use the **second shift theorem** to obtain transforms and their inverses,
- deal with problems involving the **unit impulse "function"** $\delta(t)$,
- deal with problems involving **periodic** input functions such as square waves and sawtooths,
- solve **integral** and **integral-differential** equations using Laplace transforms.

Depending on your own programme of study you may not need to cover everything in this package. Your tutor will advise you what, if anything, can be omitted.

1.2 How to use this package

You MUST do examples! Doing lots of examples for yourself is generally the most effective way of learning the contents of this package and covering the objectives listed above. We recommend that you

- first read the theory - make your own notes where appropriate,
- then work through the worked examples - compare your solutions with the ones in the notes,
- finally do similar examples yourself in a workbook.

The original printing of these notes leaves every other page blank. Use the spare space for your own comments, notes and solutions. You will see certain symbols appearing in the right hand margin from time to time:

$\bigcirc$ denotes the end of a worked example,

$\square$ denotes the end of a proof,

V denotes a reference to videos (see below for details),

EX highlights a point in the notes where you should try examples.

By the time you have reached a package like this one you will probably have realised that learning mathematics rarely goes smoothly! When you get stuck, use your accumulated wisdom and cunning to get around the problem. You might try:

- re-reading the theory/worked examples,
- putting it down and coming back to it later,
- reading ahead to see if subsequent material sheds any light,
- talking to a fellow student,
- looking in a textbook (see the bibliography),
- watching the appropriate video (see the video summaries),
- raising the problem at a tutorial.

1.3 Videos, tutorials and self-help

The videos cover the main points in the notes. The areas covered are indicated in the notes, usually at the ends of sections and subsections. To resolve a particular difficulty you may not need to watch a whole video (they are each about 30 minutes long). They are broken up into sections prefaced with titles which can be read on fast scan. In addition, a summary of the videos associated with this package appears as an appendix to these notes.

Your tutor will tell you about the arrangements for viewing the videos. Try the worked examples **before** watching the solution unfold on the screen. Make notes of any points you cannot follow so that you can explain the difficulty in a subsequent tutorial session. If you are viewing a video individually, remember the rewind button! Unlike a lecture you can get instant and 100 percent accurate replay of what was said.

Your tutor will tell you about tutorial arrangements. These may be related to assessment arrangements. If attendance at tutorials is compulsory then make sure you know the details! The tutorials provide you with individual contact with a tutor. Use this time wisely - staff time is the most expensive of all our resources.

You should come to tutorials in a prepared state. This means that you should have read the notes and the worked examples. You should have tried appropriate examples for yourself. If you have had difficulty with a particular section then you should watch the corresponding video. If your tutor finds that you haven't done these things then s/he may refuse to help you. Your tutor will find it easier to assist you if you can make any queries as specific as possible.

Your fellow students are an excellent form of self-help. Discuss problems with one another and compare solutions. Just be careful that

1. any assessed coursework submitted by you is yours alone,
2. you yourself do really understand solutions worked out jointly with colleagues.

Familiarize yourself with the layout and contents of these notes; scan them before reading them more carefully. The contents page will help you find your way about - use it. The bibliography will point you to textbooks covering the same material as these notes.

When you graduate, your future employer will be just as interested in your capacity for learning as in what you already know. If you can learn mathematics from this package and from textbooks then you will not only have learnt a particular mathematical topic. You will also (and more importantly) have learnt **how to learn** mathematics.

2 Introduction

Laplace transforms are used as a method of solution of differential equations. In this section we shall define what we mean by a Laplace transform and look at some easy examples. Basic properties will be examined in section 3. The usefulness of these ideas will become apparent in section 4.

Definition 2.1

Suppose that the function $f(t)$ is defined for $t \geq 0$. The **Laplace transform** of $f(t)$ is defined to be the function $F(s)$ given by

$$F(s) = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

This definition only makes sense when the improper integral on the right-hand side of (1) exists. Therefore, the function $F(s)$ may not be defined for all values of s. In all the practical examples which follow, we shall see that the integral does converge provided that s is sufficiently large; thus $F(s)$ will be defined for all sufficiently large s.

Note that the transform of the **function** $f(t)$ is another **function** $F(s)$. Strictly speaking it does not matter what we call the variables but t and s are the letters used conventionally. Various notations exists for Laplace transforms. The transform of f may be denoted by F, $\bar{f}$ or $L\{f\}$. Sometimes the variable is made explicit as in $F(s)$, $\bar{f}(s)$ or $L\{f\}(s)$.

It may be appropriate to remind you that an improper integral such as that given in (1) is defined by a limiting process, namely

$$\int_0^\infty g(t) dt = \lim_{R \to \infty} \int_0^R g(t) dt \tag{2}$$

The left-hand side of (2) is said to exist or converge when the limit on the right-hand side exists.

Laplace transforms can be determined directly from the definition (1) by interpreting the improper integral as in (2). It is worth noting that for any value of n,

$$x^n e^{-ax} \to 0 \text{ as } x \to \infty$$

provided only that $a > 0$. Bearing this in mind we shall now determine a selection of Laplace transforms.

Example 2.1 If $f(t) = 1$ for all t then we get

$$L\{1\} = \int_0^\infty e^{-st}.1.dt$$

Note that

$$\int_0^R e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^R = -\frac{e^{-Rs}}{s} + \frac{1}{s}$$

but $e^{-Rs} \to 0$ as $R \to \infty$ provided $s > 0$. Hence

$$L\{1\} = \frac{1}{s} \qquad (\text{if } s > 0)$$

Example 2.2

$$L\{t\} = \int_0^\infty e^{-st} t dt$$

Note that integration by parts gives

$$\begin{aligned}
\int_0^R e^{-st} t dt &= \left[\frac{e^{-st} t}{-s}\right]_0^R - \int_0^R \frac{e^{-st}}{-s} dt \\
&= -\frac{Re^{-Rs}}{s} + \left[\frac{e^{-st}}{-s^2}\right]_0^R \\
&= -\frac{Re^{-Rs}}{s} - \frac{e^{-Rs}}{s^2} + \frac{1}{s^2} \\
&\to \frac{1}{s^2} \quad \text{as} \quad R \to \infty \qquad (\text{provided } s > 0)
\end{aligned}$$

Hence

$$L\{t\} = \frac{1}{s^2} \qquad (\text{if } s > 0)$$

Example 2.3

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$$

Note that

$$\begin{aligned}
\int_0^R e^{-(s-a)t} dt &= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^R \\
&= -\frac{e^{-(s-a)R}}{s-a} + \frac{1}{s-a} \\
&\to \frac{1}{s-a} \quad \text{as} \quad R \to \infty \qquad (\text{provided } s > a)
\end{aligned}$$

Hence

$$L\{e^{at}\} = \frac{1}{s-a} \qquad (\text{if } s > a)$$

Example 2.4

$$L\{\sin \omega t\} = \int_0^\infty e^{-st} \sin \omega t dt$$

With $\Im$ denoting the imaginary part, note that

$$\begin{aligned}
\int_0^R e^{-st} \sin \omega t dt &= \Im \int_0^R e^{-st} e^{i\omega t} dt \\
&= \Im \int_0^R e^{-(s-i\omega)t} dt \\
&= \Im \left[\frac{e^{-(s-i\omega)t}}{-(s-i\omega)} \right]_0^R \\
&= \Im \left[-\frac{e^{-(s-i\omega)R}}{s-i\omega} + \frac{1}{s-i\omega} \right]
\end{aligned}$$

Now provided $s > 0$,

$$\begin{aligned}
e^{-(s-i\omega)R} &= e^{-sR} e^{i\omega R} \\
&= e^{-sR}(\cos \omega R + i \sin \omega R) \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

Hence, provided $s > 0$,

$$\begin{aligned}
L\{\sin \omega t\} &= \Im \frac{1}{s-i\omega} \\
&= \Im \frac{s+i\omega}{s^2+\omega^2} \\
&= \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

In a similar way with $\Re$ denoting the real part,

$$L\{\cos \omega t\} = \Re \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} \qquad (\text{provided } s > 0)$$

Not all functions have a Laplace transform. For example, the function e^{t^2} grows so rapidly with t that the improper integral defining the transform does not exist. In fact, if we consider

$$\int_0^R e^{-st}e^{t^2}dt = \int_0^R e^{t(t-s)}dt$$

clearly the integrand satisfies

$$e^{t(t-s)} \geq e^0 = 1 \qquad \text{whenever } t > s.$$

For **any** particular value of s therefore,

$$\int_0^R e^{-st}e^{t^2}dt \to \infty \quad \text{as} \quad R \to \infty$$

None of the functions we examine below will have such a rapid rate of growth as e^{t^2}.

(The video covers the definition of the Laplace transform, notation, some of the examples above and the use of tables. V

You should now try using the definition to determine the Laplace transform of some easy functions. You should also familiarise yourself with the tables (section 12) at the end of this set of notes.) EX

3 Basic properties

For our purposes these fall into two categories:
a) Those which enable us to obtain further Laplace transforms relatively easily.
b) Those which are needed so that we can apply Laplace transforms to solve differential equations.
The former are covered in subsections 3.1 and 3.2, the latter in 3.3. Here and subsequently we shall assume that the rate of growth of f is sufficiently slow to ensure that all the Laplace transforms involved exist for all appropriately large values of s.

3.1 The First Shift Theorem

(This section is not covered in the videos.)

Theorem 3.1.

$$L\{e^{-at}f(t)\} = \bar{f}(s+a)$$

(Here we are using $\bar{f}$ to denote the Laplace transform of $f(t)$.)

Proof

$$L\{e^{-at}f(t)\} = \int_0^\infty e^{-st}e^{-at}f(t)dt = \int_0^\infty e^{-(s+a)t}f(t)dt$$

But this latter integral is just the same as the integral defining the Laplace transform of f **except** that the s in that integral has been replaced here by $(s+a)$. Therefore our integral is simply $\bar{f}(s+a)$. □

Example 3.1 Because

$$L\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$$

it follows that

$$L\{e^{-at}\cos \omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$ ○

3.2 Multiplication by t

(This section is not covered in the videos.)

Theorem 3.2.

$$L\{tf(t)\} = -\frac{d}{ds}\{\bar{f}(s)\}$$

Proof The right hand side is

$$
\begin{aligned}
-\frac{d}{ds}\{\bar{f}(s)\} &= -\frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)dt\right\} \\
&= -\int_0^\infty \frac{d}{ds}\{e^{-st} f(t)\}dt
\end{aligned}
$$

assuming that the order of the two operations:
i) integration with respect to t
ii) differentiation with respect to s
can be interchanged. In fact this is legitimate here provided that f is "reasonable" (such as only having a finite number of discontinuities in any finite interval and growing no faster than e^{at} for some constant a). Carrying out the differentiation **with respect to** s gives

$$
\begin{aligned}
-\frac{d}{ds}\{\bar{f}(s)\} &= -\int_0^\infty -te^{-st} f(t)dt \\
&= \int_0^\infty e^{-st} t f(t)dt \\
&= L\{tf(t)\}
\end{aligned}
$$

as required. □

Example 3.2 Because

$$L\{e^{at}\} = \frac{1}{s-a}$$

it follows that

$$L\{te^{at}\} = -\frac{d}{ds}\left\{\frac{1}{s-a}\right\} = \frac{1}{(s-a)^2}$$ ○

Note that this result can be reapplied, e.g.

$$L\{t^2 f(t)\} = \frac{d^2}{ds^2}\{\bar{f}(s)\}$$

In general

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\}$$

Roughly speaking, multiplying $f(t)$ by t^n corresponds to differentiating $\bar{f}(s)$ n times. The next property shows that (again speaking roughly) the reverse holds, i.e. differentiating $f(t)$ n times corresponds to multiplying $\bar{f}(s)$ by s^n. It is this property which is so useful in solving differential equations.

3.3 Laplace transforms of derivatives

Theorem 3.3.

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

Proof

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$

Note that integration by parts gives

$$\begin{aligned}\int_0^R e^{-st} f'(t)dt &= \left[e^{-st} f(t)\right]_0^R - \int_0^R (-se^{-st} f(t))dt \\ &= e^{-sR} f(R) - f(0) + s\int_0^R e^{-st} f(t)dt\end{aligned}$$

As R tends to infinity, the last term on the right-hand side approaches $sL\{f(t)\} = s\bar{f}(s)$. Noting our assumption about the rate of growth of f we have $e^{-sR} f(R) \to 0$ as $R \to 0$ for all sufficiently large s. Hence

$$L\{f'(t)\} = -f(0) + s\bar{f}(s)$$

□

The result of the theorem can be reapplied to deal with $L\{f''\}$ as follows

$$\begin{aligned}L\{f''\} &= sL\{f'(t)\} - f'(0) \qquad \text{(first application)} \\ &= s[s\bar{f}(s) - f(0)] - f'(0) \qquad \text{(second application)} \\ &= s^2\bar{f}(s) - sf(0) - f'(0)\end{aligned}$$

In a similar way we can deduce that for any positive integer n,

$$L\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

It is well worth committing these result to memory - they are very important. There is a variety of different notations e.g.

$L\{x'\} = s\bar{x} - x_0$
$L\{x''\} = s^2\bar{x} - sx_0 - x'_0$

Here the function $f(t)$ has been replaced by $x(t)$. Its transform is $\bar{x} = \bar{x}(s)$ and $x_0,\ x'_0$ denote the values of x and of $x' = \dfrac{dx}{dt}$ at $t = 0$.

(The video deals with the transforms of first and second derivatives.) V

3.4 Linearity of Laplace transforms

Theorem 3.4. $L\{af(t)+bg(t)\} = a\bar{f}(s)+b\bar{g}(s)$ *whenever* $a,\ b$ *are constants.*

Proof We have

$$\begin{aligned} L\{af(t)+bg(t)\} &= \int_0^\infty e^{-st}(af(t)+bg(t))dt \\ &= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt \\ &= a\bar{f}(s)+b\bar{g}(s) \end{aligned}$$

as required. □

Of course this result relies on the linearity of the integration process (i.e. $\int(ap(t)+bq(t))dt = a\int p(t)dt + b\int q(t)dt$). Although fairly trivial to establish, it is very useful when dealing with linear differential equations. Note that a similar result does **NOT** apply to products i.e. $L\{f(t)g(t)\} \neq \bar{f}(s)\bar{g}(s)$. (Compare this with $\int p(t)q(t)dt \neq \left(\int p(t)dt\right)\left(\int q(t)dt\right)$.)

(The video deals with the linearity property.) V

3.5 Inverse Laplace transforms

Given a Laplace transform $\bar{f}(s)$, the question arises: "can we find a function $f(t)$ whose transform is $\bar{f}(s)$?" A slightly more awkward question is "could we find two **different** functions $f_1(t),\ f_2(t)$ both of whose transforms are the same $\bar{f}(s)$?"

The first question can be put into mathematical terms as requesting an **inverse transform** for a given $\bar{f}(s)$. The second question askes if the inverse transform (where it exists at all) is unique. We shall determine inverse Laplace transforms by inspection - in all the cases we examine the inverse exists. The uniqueness question is trickier.

Firstly, the Laplace transform of $f(t)$ only depends on the values of $f(t)$ for $t \geq 0$; the inverse transform of $\bar{f}(s)$ can therefore only be defined for $t \geq 0$. Secondly, if we alter the values of $f(t)$ at a number of isolated points we will not affect the value of the integral $\int_0^\infty e^{-st}f(t)dt$; consequently $\bar{f}(s)$ is unaltered. It follows from this second observation that, given $\bar{f}(s),\ f(t)$ is not uniquely determined. We can in fact get around this problem by regarding two functions $f_1(t)$ and $f_2(t)$ (defined for $t \geq 0$) as being equivalent if they only differ in value at isolated points. The functions $f(t)$ which we consider will be continuous except possibly at isolated points. In such cases it can be shown that $\bar{f}(s)$ determines $f(t)$ (for $t \geq 0$) uniquely apart from its values at the discontinuities.

In practice, obtaining inverse transforms is relatively mechanical and you do not need to be overly concerned with questions of existence and uniqueness. There are indeed formulae for inverting the Laplace transform but these involve knowledge of complex variable theory.

The notation employed for the inverse transform is L^{-1}. Thus if

$$L\{f\} = \bar{f}$$

we write

$$f = L^{-1}\{\bar{f}\}$$

Example 3.3 Determine

$$L^{-1}\left\{\frac{3s-2}{s^2+6s+25}\right\}$$

Solution Examples like this rely on the use of tables of transforms. A table is attached as an appendix at the end of these notes. (Section 12.) You will see (to your disappointment) that the ratio $(3s-2)/(s^2+6s+25)$ does not appear in the table. It is necessary to carry out some manipulation:

$$\begin{aligned}\frac{3s-2}{s^2+6s+25} &= \frac{3s-2}{(s+3)^2+16} \quad \text{(completing the square)}\\ &= \frac{3(s+3)-11}{(s+3)^2+4^2} \quad \text{(rewriting top and bottom)}\\ &= 3.\frac{(s+3)}{(s+3)^2+4^2} - \frac{11}{4}.\frac{4}{(s+3)^2+4^2}\end{aligned}$$

The purpose of writing the expression in this form is that both

$$\frac{(s+3)}{(s+3)^2+4^2} \quad \text{and} \quad \frac{4}{(s+3)^2+4^2}$$

do appear in the table of transforms. The former is the transform of $e^{-3t}\cos 4t$, the latter is the transform of $e^{-3t}\sin 4t$. Hence

$$L^{-1}\left\{\frac{3s-2}{s^2+6s+25}\right\} = 3e^{-3t}\cos 4t - \frac{11}{4}e^{-3t}\sin 4t$$

For anyone who is a little puzzled about the algebra involved in the above example we revise below in section 3.6 the technique known as completing the square. The other algebraic manipulations frequently required involve the use of partial fractions. The example below shows how these can be useful. We shall also revise these in section 3.6.

Example 3.4 Determine

$$L^{-1}\left\{\frac{2}{s(s+1)(s^2+1)}\right\}$$

Solution We put the expression in $\{\ \}$ into partial fractions by requiring constants $A,\ B,\ C,\ D$ such that

$$\begin{aligned}\frac{2}{s(s+1)(s^2+1)} &\equiv \frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1}\\ &\equiv \frac{A(s+1)(s^2+1)+Bs(s^2+1)+(Cs+D)s(s+1)}{s(s+1)(s^2+1)}\end{aligned}$$

Hence

$$2 \equiv A(s+1)(s^2+1)+Bs(s^2+1)+(Cs+D)s(s+1) \qquad (3)$$

If this holds for $s=0$ then

$$2 = A$$

Likewise, putting $s=-1$ gives

$$2=-2B$$

and so $B=-1$. Comparing coefficients of s^3 in equation (3) gives

$$0=A+B+C$$

Hence $C=-1$. Finally, comparing coefficients of s^2 in equation (3) gives

$$0=A+C+D$$

and so $D=-1$. It follows that

$$L^{-1}\left\{\frac{2}{s(s+1)(s^2+1)}\right\}=L^{-1}\left\{\frac{2}{s}-\frac{1}{s+1}-\frac{s+1}{s^2+1}\right\}$$

The expression $(s+1)/(s^2+1)$ can be split to form

$$\frac{s}{s^2+1}+\frac{1}{s^2+1}$$

Thus we require

$$L^{-1}\left\{\frac{2}{s}-\frac{1}{s+1}-\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}$$

all the components of which appear in the Laplace transform tables. We obtain the inverse transform as

$$2 - e^{-t} - \cos t - \sin t$$ ○

(The video covers the existence and uniqueness of the inverse Laplace transform, and both the examples in the preceding section. V

You should now try inverting some Laplace transforms. If you have trouble with the partial fractions, completing the square or other algebraic manipulations, then the following section should provide some help.) EX

3.6 Revision of completing the square and partial fractions

(This section is not covered in the videos.)

3.6.1 Completing the square

Given a quadratic expression in s:

$$as^2 + bs + c$$

with $a,\ b,\ c$ as constants and $a \neq 0$ (so that it really is a quadratic) we can re-write it as follows:

$$\begin{aligned} as^2 + bs + c &= a\left(s^2 + \frac{b}{a}s + \frac{c}{a}\right) \\ &= a\left(\left[s + \frac{b}{2a}\right]^2 + \left[\frac{c}{a} - \frac{b^2}{4a^2}\right]\right) \end{aligned}$$

Note that the term $\left[s + \frac{b}{2a}\right]^2$ is the only term containing s and that it is a square. The expression $\frac{b}{2a}$ added to s in this term is precisely half the coefficient of s on the previous line. The other term $\left[\frac{c}{a} - \frac{b^2}{4a^2}\right]$ may be positive or negative. If it is positive we can write

$$as^2 + bs + c = a\left(\left[s + \frac{b}{2a}\right]^2 + \left[\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}\right]^2\right)$$

If it is negative we can write

$$as^2 + bs + c = a\left(\left[s + \frac{b}{2a}\right]^2 - \left[\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\right]^2\right)$$

Thus the original quadratic can be written as

$$as^2 + bs + c = a \times [\text{the sum or difference of two squares, the first of which contains all the } s \text{ terms and the second is constant}]$$

There is no point in remembering the formula, it is easy to deal with particular cases as they arise.

Example 3.5

$$\begin{aligned} s^2 + s + 1 &= \left[s + \frac{1}{2}\right]^2 \pm \text{something} \\ &= s^2 + s + \frac{1}{4} \pm \text{something} \end{aligned}$$

so the $\pm$ something is $+\frac{3}{4}$, i.e.

$$s^2 + s + 1 = \left[s + \frac{1}{2}\right]^2 + \frac{3}{4} = \left[s + \frac{1}{2}\right]^2 + \left[\sqrt{\frac{3}{4}}\right]^2$$

Example 3.6

$$\begin{aligned} s^2 - 4s + 1 &= [s - 2]^2 \pm \text{something} \\ &= s^2 - 4s + 4 \pm \text{something} \end{aligned}$$

so the $\pm$ something is -3, i.e.

$$s^2 - 4s + 1 = [s - 2]^2 - 3 = [s - 2]^2 - \left[\sqrt{3}\right]^2$$

Example 3.7

$$\begin{aligned} 2s^2 + s - 3 &= 2\left(s^2 + \frac{s}{2} - \frac{3}{2}\right) \\ &= 2\left(\left[s + \frac{1}{4}\right]^2 \pm \text{something}\right) \\ &= 2\left(s^2 + \frac{s}{2} + \frac{1}{16} \pm \text{something}\right) \end{aligned}$$

so the $\pm$ something is $-\frac{1}{16} - \frac{3}{2} = -\frac{25}{16}$, i.e.

$$2s^2 + s - 3 = 2\left(\left[s + \frac{1}{4}\right]^2 - \frac{25}{16}\right) = 2\left(\left[s + \frac{1}{4}\right]^2 - \left[\frac{5}{4}\right]^2\right)$$

Looking back to example 3.3 you will see that we had not only to complete the square, we also had to re-write the numerator thus

$$3s - 2 = 3(s + 3) - 11$$

This type of manipulation is commonly required. In general

$$ds + e = d(s + f) + (e - df)$$

for any constants $d,\ e,\ f$. For examples:

$$5s + 11 = 5(s + 2) + 1$$

and

$$5s + 11 = 5(s - 3) + 26$$

3.6.2 Partial fractions

Our aim is to take an expression such as

$$\frac{s + 11}{(s + 3)(s - 1)}$$

and to write it in the form

$$\frac{A}{s + 3} + \frac{B}{s - 1}$$

for suitably chosen constants A and B. In fact we can determine A and B by expressing the latter form over the common denominator $(s+3)(s-1)$ as

$$\frac{A(s - 1) + B(s + 3)}{(s + 3)(s - 1)}$$

For this to equal the original expression we require

$$s + 11 \equiv A(s - 1) + B(s + 3)$$

($\equiv$ means that the two expressions are identical for all values of s).

We can evaluate $A,\ B$ by choosing "convenient" values of s or by comparing coefficients or by a combination of the two methods. Here there are two convenient values of s, namely $s = 1$ which gives

$$1 + 11 = A.0 + B.(1 + 3), \quad \text{and so} \quad B = \frac{12}{4} = 3$$

and $s = -3$ which gives

$$-3 + 11 = A.(-3 - 1) + B.0, \quad \text{and so} \quad A = -\frac{8}{4} = -2$$

If we chose to compare coefficients we would first write

$$s + 11 \equiv A(s - 1) + B(s + 3) \equiv (A + B)s + (-A + 3B)$$

Comparing the coefficients of s on the extreme left-hand side and the extreme right-hand side we see that $1 = A + B$. Similarly, comparing constants we get $11 = -A + 3B$. Solving these two equations for $A,\ B$ gives (again) $A = -2,\ B = 3$.

Whichever method is used for determining $A,\ B$, we obtain

$$\frac{s + 11}{(s + 3)(s - 1)} = \frac{-2}{s + 3} + \frac{3}{s - 1}$$

In general we proceed as follows.
An expression such as

$$r(s) = \frac{p(s)}{q(s)}$$

where $p,\ q$ are polynomials in s is called a **rational function** of s. If the degree of $p(s)$ is **strictly less than** the degree of $q(s)$ then the expression is said to be a proper fraction. Otherwise it is said to be an improper fraction. Easy examples are:

a) $\dfrac{2s + 7}{s(s + 1)} = \dfrac{2s + 7}{s^2 + s}$ (a proper fraction)

b) $\dfrac{2s^3 + 3s + 2}{(s - 1)(s + 2)} = \dfrac{2s^3 + 3s + 2}{s^2 + s - 2}$ (an improper fraction)

c) $\dfrac{s^2}{s^2 + 1}$ (an improper fraction-just!)

Example a) is proper because the degree of the numerator is 1 whilst that of the denominator is 2 (and $1 < 2$). Example b) is improper because the numerator has degree 3 and the denominator degree 2 (and $3 \not< 2$). Example c) is also improper because $2 \not< 2$.

We shall only deal with proper fractions.

For such expressions we firstly factorize the denominator. We shall assume that this has been done and has resulted in a denominator containing some or all of the following:

a) unrepeated linear factors $as + b$,

b) repeated linear factors $(cs + d)^r, \quad (r > 1)$,
c) unrepeated quadratic factors $es^2 + fs + g$,
d) repeated quadratic factors $(hs^2 + is + j)^t, \quad (t > 1)$.
For example, the denominator might be

$$(3s + 7)(2s - 3)^3(s^2 + s + 1)(s^2 + 4)^2$$

This contains the unrepeated linear factor $3s + 7$, the (3 times) repeated linear factor $2s - 3$, the unrepeated quadratic factor $s^2 + s + 1$, and the (2 times) repeated quadratic factor $s^2 + 4$.

We then apply the following rules:
a) Each unrepeated linear factor $as + b$ gives rise to a partial fraction of the form

$$\frac{A}{as + b} \qquad \text{(where } A \text{ is a constant).}$$

b) Each repeated linear factor $(cs + d)^r, \quad (r > 1)$ gives rise to r partial fractions of the form

$$\frac{B_1}{cs + d} + \frac{B_2}{(cs + d)^2} + \cdots + \frac{B_r}{(cs + d)^r}$$

(where $B_1, \ B_2, \cdots, \ B_r$ are constants).
c) Each unrepeated quadratic factor $es^2 + fs + g$ gives rise to a partial fraction of the form

$$\frac{Cs + D}{es^2 + fs + g} \qquad \text{(where } C, \ D \text{ are constants).}$$

d) Each repeated quadratic factor $(hs^2 + is + j)^t, \quad (t > 1)$ gives rise to t partial factions of the form

$$\frac{E_1 s + F_1}{hs^2 + is + j} + \frac{E_2 s + F_2}{(hs^2 + is + j)^2} + \cdots + \frac{E_t s + F_t}{(hs^2 + is + j)^t}$$

(where $E_1, \ E_2, \cdots, \ E_t$ and $F_1, \ F_2, \cdots, \ F_t$ are constants).

As an example, let us consider the following expression. (You would not be expected to handle anything as awful as this in practice.)

Example 3.8 The form of the partial fractions for

$$\frac{3s^7 + 1}{(3s + 7)(2s - 3)^3(s^2 + s + 1)(s^2 + 4)^2}$$

is

$$\underbrace{\frac{A}{3s+7}}_{\text{unrepeated linear}} + \underbrace{\frac{B}{2s-3} + \frac{C}{(2s-3)^2} + \frac{D}{(2s-3)^3}}_{\text{3-times repeated linear}}$$
$$+ \underbrace{\frac{Es+F}{s^2+s+1}}_{\text{unrepeated quadratic}} + \underbrace{\frac{Gs+H}{s^2+4} + \frac{Is+J}{(s^2+4)^2}}_{\text{twice repeated quadratic}}$$

The constants $A,\ B, \cdots,\ J$ can be determined by expressing the partial fractions over the common denominator

$$(3s+7)(2s-3)^3(s^2+s+1)(s^2+4)^2$$

and then equating numerators. Note that this denominator is **NOT** simply the product of all the denominators of the partial fractions - this would contain $(2s-3)^6$ and $(s^2+4)^3$. The denominator quoted is the lowest common denominator for the partial fractions; it is the same as the original denominator. ○

Example 3.9 Determine partial fractions for

$$\frac{2s+1}{s^2(s^2+1)}$$

Solution Here s^2 is a repeated linear factor and s^2+1 an unrepeated quadratic. We write

$$\begin{aligned}\frac{2s+1}{s^2(s^2+1)} &\equiv \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}\\ &\equiv \frac{As(s^2+1) + B(s^2+1) + (Cs+D)s^2}{s^2(s^2+1)}\end{aligned}$$

Hence

$$2s+1 \equiv As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

The "convenient" value $s=0$ gives $1=B$.
Equating s terms gives $2=A$.
Equating s^2 terms gives $0=B+D$, so $D=-B=-1$.
Equating s^3 terms gives $0=A+C$, so $C=-A=-2$.
Hence

$$\frac{2s+1}{s^2(s^2+1)} \equiv \frac{2}{s} + \frac{1}{s^2} + \frac{-2s-1}{s^2+1}$$ ○

(If you haven't already done so, now try examples on the inversion of Laplace transforms.) EX

4 Solution of differential equations

The basic method is as follows:
i) take the Laplace transform of each term in the differential equation,
ii) solve the transformed equation (which is an algebraic equation),
iii) determine the inverse transform.

The method can be applied to linear constant-coefficient differential equations. The results of subsections 3.3 and 3.4 are particularly important we recap them here:

$$\begin{aligned} L\{y'\} &= s\bar{y} - y_0 \\ L\{y''\} &= s^2\bar{y} - sy_0 - y_0' \\ L\{af + bg\} &= a\bar{f} + b\bar{g} \qquad \text{(linearity)} \end{aligned}$$

where $y_0 = y(0),\ y_0' = y'(0)$ and $a,\ b$ are constants.
Example 4.1 Solve the differential equation

$$y'' + 3y' + 2y = 6$$

using Laplace transforms, given that $y(0) = 0$ and $y'(0) = 2$.
Solution Taking Laplace transforms and using the linearity property

$$L\{y''\} + 3L\{y'\} + 2L\{y\} = 6L\{1\}$$

Therefore

$$s^2\bar{y} - sy_0 - y_0' + 3\left[s\bar{y} - y_0\right] + 2\bar{y} = \frac{6}{s}$$

(We have obtained $L\{1\} = \dfrac{1}{s}$ from the tables of transforms in the appendix.)
Substituting the values $y_0 = 0$ and $y_0' = 2$ and collecting the $\bar{y}$ terms together gives

$$(s^2 + 3s + 2)\bar{y} - 2 = \frac{6}{s}$$

and so

$$(s^2 + 3s + 2)\bar{y} = 2 + \frac{6}{s} = \frac{2s + 6}{s}$$

It should be noted that the bracketed expression (s^2+3s+2) multiplying $\bar{y}$ bears a strong resemblance to the left hand side of the original differential equation $(y''+3y'+2y)$. This is not coincidental and it is relatively easy to see that this will happen for any linear constant-coefficient differential equation - it is always worth checking at this point for obvious numerical slips. Those of you familiar with the method of complementary functions and particular integrals for solving differential equations will notice that a similar expression arises in the auxiliary equation which here is $m^2 + 3m + 2 = 0$.

Solving the equation for $\bar{y}$ gives

$$\begin{aligned}\bar{y} &= \frac{2s+6}{s(s^2+3s+2)} \\ &= \frac{2s+6}{s(s+1)(s+2)} \\ &= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\end{aligned}$$

in partial fractions where

$$2s + 6 \equiv A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Here $s = 0$ gives $6 = 2A$, $s = -1$ gives $4 = -B$ and $s = -2$ gives $2 = 2C$. Hence $A = 3,\ B = -4,\ C = 1$. Therefore we can write $\bar{y}$ in the form

$$\bar{y} = \frac{3}{s} - \frac{4}{s+1} + \frac{1}{s+2}$$

Taking inverse transforms we obtain

$$y(t) = 3 - 4e^{-t} + e^{-2t}$$

As this is our first example of solving a differential equation using Laplace transforms we shall check that our solution really does satisfy the differential equation and the initial conditions. Firstly, when $t = 0$ our solution takes the value $y(0) = 3 - 4e^0 + e^0 = 0$, as required. Differentiating our formula for $y(t)$ gives

$$y'(t) = 4e^{-t} - 2e^{-2t}$$

and this gives $y'(0) = 4 - 2 = 2$, as required. Differentiating a second time we obtain

$$y''(t) = -4e^{-t} + 4e^{-2t}$$

Combining our expressions for $y,\ y',\ y''$ gives

$$y'' + 3y' + 2y = -4e^{-t} + 4e^{-2t} + 3(4e^{-t} - 2e^{-2t}) + 2(3 - 4e^{-t} + e^{-2t}) = 6$$

as required. Thus our solution really does satisfy the initial conditions and the original differential equation. In future examples we will not bother to verify the solution. ○

Example 4.2 Solve the differential equation

$$y'' + 9y = \cos 3t$$

given that $y(0) = 1$ and $y'(0) = -1$.
Solution Taking Laplace transforms

$$\begin{aligned} s^2\bar{y} - sy_0 - y_0' + 9\bar{y} &= \frac{s}{s^2+9} \\ \text{Therefore} \qquad (s^2+9)\bar{y} &= \frac{s}{s^2+9} + sy_0 + y_0' \\ &= \frac{s}{s^2+9} + s - 1 \\ \text{Hence} \qquad \bar{y} &= \frac{s}{(s^2+9)^2} + \frac{s}{s^2+9} - \frac{1}{s^2+9} \\ &= \frac{1}{6}.\frac{6s}{(s^2+3^2)^2} + \frac{s}{s^2+3^2} - \frac{1}{3}.\frac{3}{s^2+3^2} \end{aligned}$$

Using the transform tables we see that

$$y(t) = \frac{1}{6} t \sin 3t + \cos 3t - \frac{1}{3} \sin 3t$$

The Laplace transform method can be used to reduce **systems** of linear differential equations to systems of ordinary algebraic linear equations.
Example 4.3 Solve the simultaneous differential equations

$$\frac{dx}{dt} = 2x + y, \qquad \frac{dy}{dt} = 3x + 4y$$

given that $x(0) = 1, \;\; y(0) = 0$.
Solution Taking Laplace transforms we obtain

$$s\bar{x} - x_0 = 2\bar{x} + \bar{y}, \qquad s\bar{y} - y_0 = 3\bar{x} + 4\bar{y}$$

Substituting the values $x_0 = 1$ and $y_0 = 0$ and grouping together the $\bar{x}$ and $\bar{y}$ terms in each of these two equations gives

$$(s-2)\bar{x} - \bar{y} = 1, \qquad 3\bar{x} - (s-4)\bar{y} = 0$$

Elimination of, say, $\bar{x}$ between these two equations can be achieved by multiplying the former by 3, the latter by $(s-2)$ and subtracting. This gives

$$\begin{aligned}
[-3+(s-2)(s-4)]\bar{y} &= 3 \\
\text{i.e.} \qquad (s^2-6s+5)\bar{y} &= 3 \\
\text{Therefore} \qquad \bar{y} &= \frac{3}{s^2-6s+5} \\
&= \frac{3}{(s-1)(s-5)} \\
&= \frac{A}{s-1}+\frac{B}{s-5}
\end{aligned}$$

in partial fractions, where

$$3 \equiv A(s-5)+B(s-1)$$

Putting $s=1$ gives $3=-4A$, so $A=-\frac{3}{4}$. Putting $s=5$ gives $3=4B$, so $B=\frac{3}{4}$. Hence

$$\bar{y}=-\frac{3}{4}.\frac{1}{s-1}+\frac{3}{4}.\frac{1}{s-5}$$

Inverting gives

$$y(t)=-\frac{3}{4}e^t+\frac{3}{4}e^{5t}$$

To complete the problem we must also find $x(t)$. There are various ways of doing this. We could return to the simultaneous linear equations for $\bar{x}$ and $\bar{y}$, eliminate $\bar{y}$, solve for $\bar{x}$ and invert to obtain x. In many problems (including this one) an easier alternative is to use our solution for y with one or other of the original differential equations to determine x. In this case the second of the two original differential equations can be re-written as

$$x=\frac{1}{3}\left(\frac{dy}{dt}-4y\right)$$

We know

$$y(t)=-\frac{3}{4}e^t+\frac{3}{4}e^{5t}$$

and so

$$\frac{dy}{dt}=-\frac{3}{4}e^t+\frac{15}{4}e^{5t}$$

Therefore

$$\begin{aligned}
x(t) &= \frac{1}{3}\left(-\frac{3}{4}e^t+\frac{15}{4}e^{5t}+\frac{12}{4}e^t-\frac{12}{4}e^{5t}\right) \\
&= \frac{1}{3}\left(\frac{9}{4}e^t+\frac{3}{4}e^{5t}\right)
\end{aligned}$$

and finally this gives

$$x(t) = \frac{3}{4}e^{t} + \frac{1}{4}e^{5t}$$ ○

(The video covers the method of solution of differential equations using Laplace transforms and two of the above examples. V

You should now try solving some differential equations by this method, including some sets of simultaneous equations.) EX

5 The unit step function

In this section we look at differential equations which involve the unit step function. This function, $u(t)$, is defined by

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

We will not be concerned by the value of $u(t)$ at $t = 0$. The graph of $u(t)$ is shown below (figure 1).

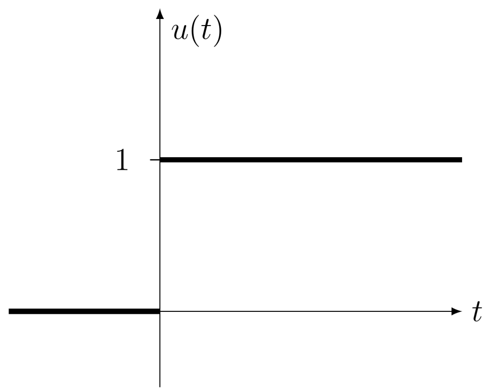


Figure 1: The unit step function $u(t)$.

The function is sometimes called **Heaviside's unit step function** and denoted by $H(t)$.

This type of function occurs very commonly in the practical world - it corresponds to a switch being turned on. Since the switch may be turned on at times other than $t = 0$ it is worth looking at $u(t - a)$, where a is a constant. From the definition of $u(t)$,

$$u(t-a) = \begin{cases} 0 & \text{if } t - a < 0 \quad (\text{ i.e. } t < a) \\ 1 & \text{if } t - a > 0 \quad (\text{ i.e. } t > a) \end{cases}$$

The graph of $u(t - a)$ is shown below (figure 2).

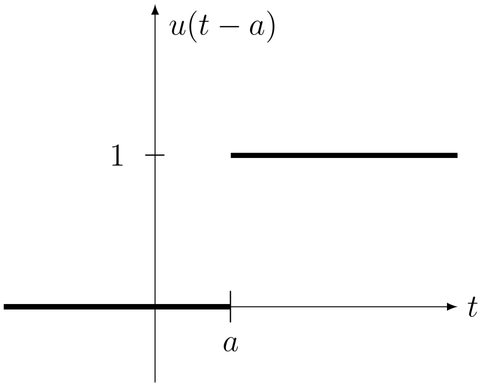


Figure 2: The unit step function $u(t-a)$.

This corresponds to a switch being turned on at time $t = a$.

The input after the switch is turned on may not be a constant of size 1 and it is worth looking at the product of $u(t-a)$ with an arbitrary function $f(t)$. Since $u(t-a)$ is zero for $t < a$, $f(t)u(t-a)$ is also zero for $t < a$. For $t > a$, $u(t-a)$ has the value 1 and so $f(t)u(t-a)$ has the value $f(t)$. Therefore

$$f(t)u(t-a) = \begin{cases} 0 & \text{if } \ t < a \\ f(t) & \text{if } \ t > a \end{cases}$$

The graphs of a "typical" function $f(t)$ and the resulting product $f(t)u(t-a)$ are shown below (figure 3).

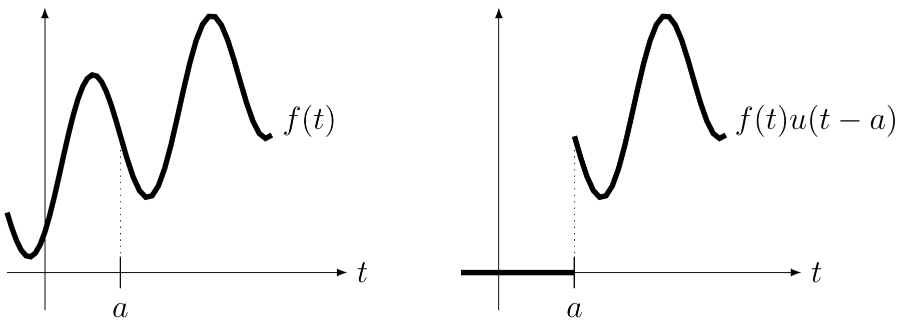


Figure 3: A "typical" $f(t)$ and the product $f(t)u(t-a)$.

By combining products of the above type we can generally express functions defined in a piecewise manner. Take, for example,

$$g(t) = \begin{cases} 0 & \text{if } \ t < 1 \\ 4 & \text{if } \ 1 < t < 2 \\ 0 & \text{if } \ t > 2 \end{cases}$$

The graph of $g(t)$ is shown below (figure 4).

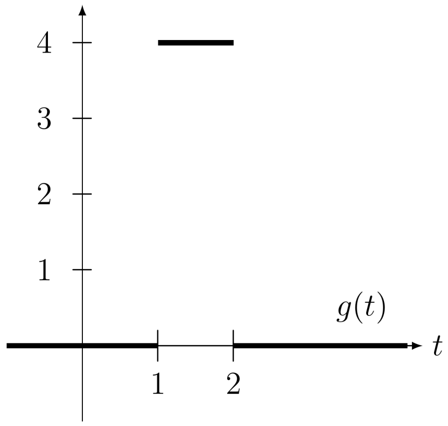


Figure 4: The function $g(t)$.

We can build up $g(t)$ by working from left to right across the graph. Initially (for $t < 1$) $g(t) = 0$. If we wrote $g(t) = 0$ (for all t) we would be correct for $t < 1$ but incorrect for $t > 1$. At $t = 1$ we want to add in (switch on) a step of height 4. We can do this by adding $4u(t-1)$ so that our formula now reads $g(t) = 0 + 4u(t-1)$. This formula is still correct for $t < 1$ because here $u(t-1) = 0$ and it is also correct for $1 < t < 2$ because here $u(t-1) = 1$. The formula is incorrect for $t > 2$ because it provides no step down at $t = 2$. We can allow for this by subtracting $4u(t-2)$ (switching off a step of height 4 at $t = 2$). The formula now reads

$$\begin{aligned} g(t) &= 0 + 4u(t-1) - 4u(t-2) \\ &= 4u(t-1) - 4u(t-2) \end{aligned}$$

Example 5.1 With $h(t)$ defined by

$$h(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 < t < 1 \\ 1 & \text{if } t > 1 \end{cases}$$

find an expression for $h(t)$ involving the unit step function.
Solution The graph of $h(t)$ is shown below (figure 5). Working from left to right across the graph we have $h(t)$ initially at 0. At $t = 0$ we switch on t using $tu(t)$. At $t = 1$ we switch off t using $-tu(t-1)$ and switch on 1 using $u(t-1)$. The resulting expression for $h(t)$ is

$$h(t) = 0 + tu(t) - tu(t-1) + u(t-1) = tu(t) - (t-1)u(t-1)$$

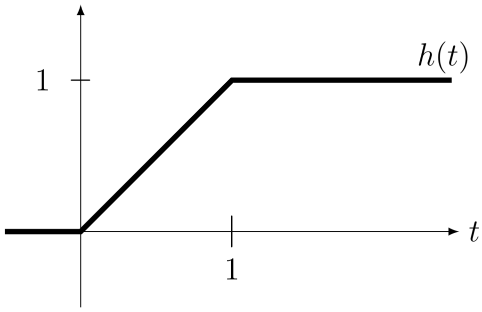


Figure 5: The function $h(t)$.

There is no real necessity to sketch the graph.

Example 5.2 If

$$g(t) = \begin{cases} 3 & \text{if } t < 2 \\ -3 & \text{if } 2 < t < 4 \\ e^{-t} & \text{if } t > 4 \end{cases}$$

then we can represent $g(t)$ as follows. Initially $g(t)$ is 3. At $t = 2$ this is switched off using $-3u(t-2)$ and -3 is switched on using $(-3)u(t-2)$. At $t = 4$ this is switched off using $-(-3)u(t-4)$ and e^{-t} is switched on using $e^{-t}u(t-4)$. Therefore

$$g(t) = 3 - 3u(t-2) + (-3)u(t-2) - (-3)u(t-4) + e^{-t}u(t-4)$$

which reduces to $g(t) = 3 - 6u(t-2) + (3 + e^{-t})u(t-4)$.

In order to solve differential equations involving the unit step function we shall look at the transform of $u(t-a)$ and at the transform of $f(t)u(t-a)$ (both for $a > 0$).

For $a > 0$ we have

$$\begin{aligned} L\{u(t-a)\} &= \int_0^\infty e^{-st}u(t-a)dt \qquad \text{(definition)} \\ &= \int_0^a e^{-st}u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt \end{aligned}$$

We can split any integral at any point in the range of integration - here we choose to make the split at a. We do this because if $0 < t < a$ then $u(t-a) = 0$ and so the first integral is zero. If $t > a$ then $u(t-a) = 1$ and so the second integral is just

$$\int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-sa}}{s} \qquad (s > 0)$$

(The upper limit is to be interpreted as $\lim_{R\to\infty} \frac{e^{-sR}}{-s} = 0$ for $s > 0$.) Hence

$$\boxed{L\{u(t-a)\} = \frac{e^{-sa}}{s}}$$

You will see that this appears in the table of Laplace transforms in the appendix. A slightly more complicated argument allows us to deal with $L\{g(t)u(t-a)\}$. The result is known as the second shift theorem.

Theorem 5.1. *(The Second Shift Theorem) For $a > 0$*

$$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$$

Proof For $a > 0$ we have

$$\begin{aligned} L\{g(t)u(t-a)\} &= \int_0^\infty e^{-st}g(t)u(t-a)dt \qquad \text{(definition)} \\ &= \int_0^a e^{-st}g(t)u(t-a)dt + \int_a^\infty e^{-st}g(t)u(t-a)dt \end{aligned}$$

Here, as before, we have split the range of integration into two parts. In the former part the integrand is zero (because $u(t-a) = 0$ if $t < a$) whilst in the latter part the integrand is $e^{-st}g(t)$ (because $u(t-a) = 1$ if $t > a$). Hence

$$L\{g(t)u(t-a)\} = \int_a^\infty e^{-st}g(t)dt$$

We now change the integration variable from t to $w = t - a$. Note that when $t = a$ (the lower integration limit) $w = 0$, whilst as t tends to infinity (the upper limit) we also have w tending to infinity. If $w = t - a$ then $t = w + a$ and $dt = dw$. Therefore

$$\begin{aligned} L\{g(t)u(t-a)\} &= \int_0^\infty e^{-s(w+a)}g(w+a)dw \\ &= \int_0^\infty e^{-sw}e^{-sa}g(w+a)dw \\ &= e^{-sa}\int_0^\infty e^{-sw}g(w+a)dw \end{aligned}$$

In the latter integral, the variable w is just a "dummy" variable - we can replace it by any other symbol we choose - for example by t. All we are saying here is that

$$\int_0^\infty e^{-sw}g(w+a)dw = \int_0^\infty e^{-st}g(t+a)dt$$

in the same way that we might say

$$\int_0^1 w^3 dw = \int_0^1 t^3 dt$$

Hence

$$L\{g(t)u(t-a)\} = e^{-sa} \int_0^\infty e^{-st} g(t+a) dt$$

We can now identify the latter integral as a Laplace transform, namely that of $g(t+a)$. Therefore

$$L\{g(t)u(t-a)\} = e^{-as} L\{g(t+a)\}$$

□

If we replace $g(t)$ in the above result by $f(t-a)$, so that $g(t+a) = f(t+a-a) = f(t)$ we can write the result as

$$L\{f(t-a)u(t-a)\} = e^{-as} L\{f(t)\} = e^{-as} \bar{f}(s)$$

Both forms are useful; the original version is generally of greatest use in determining the Laplace transform of a combination whilst the latter is generally of greatest use in inverting a given transform. We summarize them here for future reference.

Second shift theorem

$L\{g(t)u(t-a)\} = e^{-as} L\{g(t+a)\}$
$L\{f(t-a)u(t-a)\} = e^{-as} \bar{f}(s)$

We use the result to obtain Laplace transforms as shown in the following examples.

Example 5.3

$$\begin{aligned} L\{t^2 u(t-3)\} &= e^{-3s} L\{(t+3)^2\} \\ &= e^{-3s} L\{t^2 + 6t + 9\} \\ &= e^{-3s} \left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right] \end{aligned}$$

(the terms in the [ ] brackets being obtained from tables.) ○

Example 5.4

$$\begin{aligned} L\{e^{3t}u(t-2)\} &= e^{-2s}L\{e^{3(t+2)}\} \\ &= e^{-2s}L\{e^{3t+6}\} \\ &= e^{-2s}e^{6}L\{e^{3t}\} \\ &= e^{-2s}.e^{6}.\frac{1}{s-3} \qquad \text{(from tables)} \\ &= \frac{e^{6-2s}}{s-3} \end{aligned}$$

We use the result to obtain inverse transforms as shown in the following examples.

Example 5.5 Obtain $L^{-1}\left\{\dfrac{e^{-2s}}{s^2+1}\right\}$

Solution We note that the factor e^{-2s} in the transform corresponds to a factor $u(t-2)$ in the original function of t. For the other component $L^{-1}\{\dfrac{1}{s^2+1}\} = \sin t$. Using the second version of the shift theorem we have

$$L\{\sin(t-2)u(t-2)\} = e^{-2s}L\{\sin t\}$$

Hence

$$L^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\} = \sin(t-2)u(t-2)$$

In effect what we have done is to replace e^{-2s} by $u(t-2)$, and to replace $\frac{1}{s^2+1}$ by its inverse $\sin t$ but with t "shifted" to $t-2$.

Example 5.6 Obtain $L^{-1}\left\{\dfrac{e^{-3s}}{(s+2)^2}\right\}$

Solution e^{-3s} corresponds to $u(t-3)$.

For the other component $L^{-1}\{\dfrac{1}{(s+2)^2}\} = te^{-2t}$. Hence

$$L^{-1}\left\{\frac{e^{-3s}}{(s+2)^2}\right\} = (t-3)e^{-2(t-3)}u(t-3)$$

We now try solving some differential equations involving the unit step function.

Example 5.7 Solve the differential equation

$$y'' + y = f(t)$$

subject to $y(0) = 1, \quad y'(0) = 0$ where

$$f(t) = \begin{cases} 0 & \text{if } \ t < 1 \\ 4 & \text{if } \ 1 < t < 2 \\ 0 & \text{if } \ t > 2 \end{cases}$$

Solution $f(t)$ can be expressed as

$$\begin{aligned} f(t) &= 0 + 4u(t-1) - 4u(t-2) \\ &= 4u(t-1) - 4u(t-2) \end{aligned}$$

Hence

$$\bar{f}(s) = \frac{4e^{-s}}{s} - \frac{4e^{-2s}}{s}$$

Taking the transform of the differential equation gives

$$s^2\bar{y} - sy_0 - y_0' + \bar{y} = \bar{f}(s)$$

Substituting for $\bar{f}(s)$ and putting $y_0 = 1, \quad y_0' = 0$ this gives

$$(s^2+1)\bar{y} = s + \frac{4e^{-s}}{s} - \frac{4e^{-2s}}{s}$$

Hence

$$\bar{y} = \frac{s}{s^2+1} + \frac{4}{s(s^2+1)}e^{-s} - \frac{4}{s(s^2+1)}e^{-2s}$$

Before proceeding further we resolve $\dfrac{1}{s(s^2+1)}$ into partial fractions. Put

$$\frac{1}{s(s^2+1)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+1} \equiv \frac{A(s^2+1) + (Bs+C)s}{s(s^2+1)}$$

Hence

$$1 \equiv A(s^2+1) + (Bs+C)s$$

Then $s = 0$ gives $1 = A$. Equating s^2 terms gives $0 = A + B$, so $B = -A = -1$. Equating s terms gives $0 = C$. Therefore

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

It follows that $\bar{y}$ can be expressed as

$$\bar{y} = \frac{s}{s^2+1} + 4\left[\left(\frac{1}{s} - \frac{s}{(s^2+1)}\right)e^{-s} - \left(\frac{1}{s} - \frac{s}{(s^2+1)}\right)e^{-2s}\right]$$

We have

$$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t \qquad \text{and} \qquad L^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t$$

therefore

$$L^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s}\right\} = u(t-1)\left[1 - \cos(t-1)\right]$$

and

$$L^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-2s}\right\} = u(t-2)\left[1 - \cos(t-2)\right]$$

Hence

$$y(t) = \cos t + 4u(t-1)\left[1 - \cos(t-1)\right] - 4u(t-2)\left[1 - \cos(t-2)\right]$$

Example 5.8 The function $g(t)$ is given by

$$g(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 < t < 1 \\ 0 & \text{if } t > 1 \end{cases}$$

Solve the differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = g(t)$$

given that $x(0) = x'(0) = 0$.
Solution The function $g(t)$ can be expressed by

$$g(t) = 0 + tu(t) - tu(t-1)$$

Its Laplace transform is therefore

$$\begin{aligned} \bar{g}(s) &= \frac{1}{s^2} - e^{-s}L\{(t+1)\} \\ &= \frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right] \end{aligned}$$

Taking the transform of the equation we obtain

$$s^2\bar{x} - sx_0 - x_0' + 3(s\bar{x} - x_0) + 2\bar{x} = \bar{g}(s)$$

Substituting for $\bar{g}(s)$ and the values $x_0 = x_0' = 0$ gives

$$(s^2 + 3s + 2)\bar{x} = \frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

Hence, noting $(s^2 + 3s + 2) = (s+1)(s+2)$,

$$\bar{x} = \frac{1}{s^2(s+1)(s+2)} - e^{-s}\left[\frac{1}{s^2(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}\right]$$

The form of the partial fractions for $\dfrac{1}{s^2(s+1)(s+2)}$ is

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

and that for $\dfrac{1}{s(s+1)(s+2)}$ is

$$\frac{E}{s} + \frac{F}{s+1} + \frac{G}{s+2}$$

We will **not** evaluate $A,\ B,\ C,\ D,\ E,\ F,\ G$ here [**you** can do it if you like!]. Whatever these values turn out to be, we have

$$\begin{aligned} x(t) &= A + Bt + Ce^{-t} + De^{-2t} \\ &\quad - u(t-1)\left[A + B(t-1) + Ce^{-(t-1)} + De^{-2(t-1)}\right. \\ &\qquad\qquad \left. + E + Fe^{-(t-1)} + Ge^{-2(t-1)}\right] \end{aligned}$$

(The video covers most of the material in this section. V

You should do some examples to ensure that you understand how to represent piecewise continuous functions using the unit step function and how to use the second shift theorem to obtain their Laplace transforms. You should then try solving differential equations which involve the unit step function and the use of the second shift theorem to invert the resulting transforms.) EX

6 The unit impulse "function"

In the physical world an object can be set in motion by applying a modest force over a period of time, or by applying a sharp blow at an instant of time. The **unit impulse "function"** is a mathematical model of the sharp blow applied at an instant of time. You will note we have described it as a "function" rather than as a function. This is because it is not really a function although it has many characteristics in common with functions. Sometimes it is called the **Dirac delta function**; generally it is denoted by $\delta(t)$. To give a proper rigorous account of this object is well beyond the scope of these notes. In broad terms $\delta(t)$ is defined by its effect on other functions and we give below a very informal treatment. Fortunately it has a Laplace transform which is a perfectly respectable function.

Let ϵ be a small positive quantity. We define the function $\delta_\epsilon(t)$ by

$$\delta_\epsilon(t) = \begin{cases} \dfrac{1}{\epsilon} & \text{if } \ 0 < t < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

The graph of $\delta_\epsilon(t)$ is sketched below (figure 6).

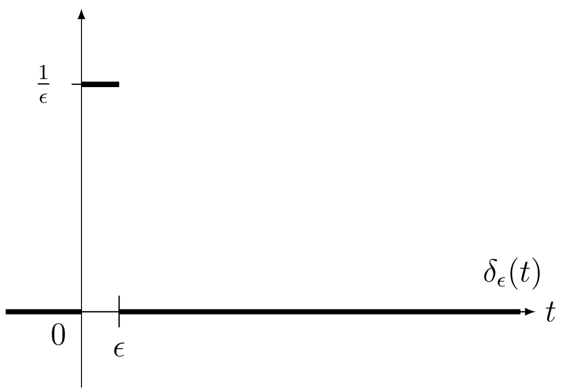


Figure 6: The function $\delta_\epsilon(t)$.

The important features to note are that $\delta_\epsilon(t)$ is zero for most values of t but that near to $t = 0$ the value is so large ($\frac{1}{\epsilon}$) that the area under the graph is precisely 1 unit.

Informally you can think of the unit impulse "function" as being the result of letting ϵ tend to zero in $\delta_\epsilon(t)$. You will obtain a "function" which is zero everywhere except at one point ($t = 0$) and at that point it is so large that the area under the graph is one unit. [Of course if you really do let ϵ tend to zero then $\delta_\epsilon(t) \to 0$ if $t \neq 0$, and $\delta_\epsilon(0) \to +\infty$]

The "unit impulse" referred to in the title is the unit area under the graph. In $\delta_\epsilon(t)$ this is located near to $t=0$. It is useful to be able to locate the unit impulse at other times (i.e. at other values of t). For this reason we consider the translate $\delta_\epsilon(t-a)$.

$$\delta_\epsilon(t-a) = \begin{cases} \dfrac{1}{\epsilon} & \text{if } \ 0 < t-a < \epsilon \ \text{ i.e. } a < t < a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

The graph of $\delta_\epsilon(t-a)$ is sketched below (figure 7).

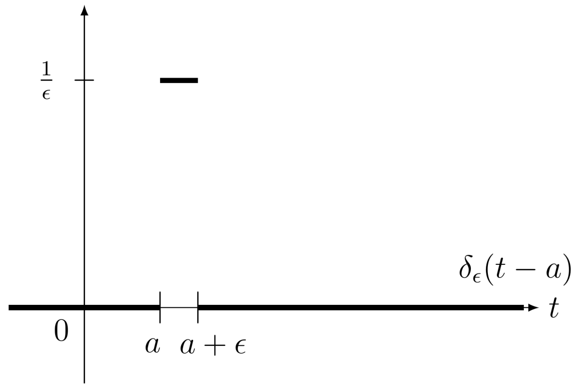


Figure 7: The function $\delta_\epsilon(t-a)$.

In the same way that $\delta(t)$ is regarded as a limiting case of $\delta_\epsilon(t)$, we can think of the limiting case of $\delta_\epsilon(t-a)$ (as ϵ tends to zero) as defining $\delta(t-a)$.

We said that the unit impulse "function" is really defined by its effect on other functions. To see what we mean by this, take any continuous function $f(t)$ defined for $t \geq 0$ and consider (for $a > 0$) the integral

$$\int_0^\infty \delta_\epsilon(t-a)f(t)dt$$

We can split the integral into three parts, namely

$$\int_0^a \quad + \quad \int_a^{a+\epsilon} \quad + \quad \int_{a+\epsilon}^\infty$$

In the first and third of these integrals $\delta_\epsilon(t-a)$ has the value zero. In the second it has the value $\frac{1}{\epsilon}$. Consequently

$$\int_0^\infty \delta_\epsilon(t-a)f(t)dt = 0 + \frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)dt + 0 = \frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)dt$$

We examine this latter integral, bearing in mind that we are interested in small values of ϵ. Consider the diagram below (figure 8).

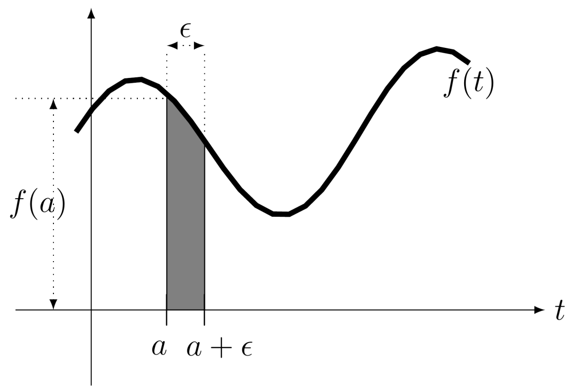


Figure 8: The integral $\int_a^{a+\epsilon} f(t)dt$.

The shaded area represents $\int_a^{a+\epsilon} f(t)dt$. If ϵ is small this will be approximately equal to $f(a) \times \epsilon$ (and the approximation will improve as ϵ tends to zero). Hence

$$\int_0^\infty \delta_\epsilon(t-a)f(t)dt \approx \frac{1}{\epsilon}.f(a).\epsilon = f(a)$$

i.e.

$$\int_0^\infty \delta_\epsilon(t-a)f(t)dt \to f(a) \quad \text{as} \quad \epsilon \to 0$$

We therefore feel justified in writing

$$\int_0^\infty \delta(t-a)f(t)dt = f(a) \qquad \text{(provided } a > 0\text{)}$$

In effect, $\delta(t-a)$ is so large at $t = a$ that it picks out the value of f at a from the range of integration.
If we apply the result to $f(t) = 1$ (for all t), we get (for $a > 0$)

$$\int_0^\infty \delta(t-a)dt = 1$$

(i.e. the delta function has unit area under its graph).
If $f(t)$ is zero at $t = a$ then irrespective of its value elsewhere

$$\int_0^\infty \delta(t-a)f(t)dt = 0$$

(i.e. the delta function $\delta(t-a)$ is zero except at $t=a$).
Applying the result with $f(t)=e^{-st}$ gives the Laplace transform:

$$L\{\delta(t-a)\}=\int_0^\infty \delta(t-a)e^{-st}dt=e^{-sa}$$

Bearing this in mind we can now solve differential equations involving the unit impulse "function".
Example 6.1 Solve the differential equation

$$y''+2y'+y=\delta(t-1)$$

given that $y(0)=0,\ y'(0)=1$.
Solution Before obtaining the solution it is probably worth considering what to expect. The system represented by the left-hand side of the equation is in some state initially (i.e. from $t=0$ to $t=1$) determined by the conditions $y(0)=0$ and $y'(0)=1$. At time $t=1$, a unit impulse is delivered. This will cause an abrupt change in the system. Thus the solution for y is likely to have a step at time $t=1$ (i.e. it will probably involve $u(t-1)$).

Taking Laplace transforms we obtain

$$s^2\bar{y}-sy_0-y_0'+2(s\bar{y}-y_0)+\bar{y}=e^{-s}$$

Putting $y_0=0$ and $y_0'=1$ gives

$$(s^2+2s+1)\bar{y}=1+e^{-s}$$

Noting $s^2+2s+1=(s+1)^2$ we obtain

$$\bar{y}=\frac{1}{(s+1)^2}+\frac{e^{-s}}{(s+1)^2}$$

Now $L^{-1}\left\{\dfrac{1}{(s+1)^2}\right\}=te^{-t}$. Hence, from the second shift theorem, $L^{-1}\left\{\dfrac{e^{-s}}{(s+1)^2}\right\}=u(t-1)(t-1)e^{-(t-1)}$. Therefore

$$y(t)=te^{-t}+u(t-1)(t-1)e^{-(t-1)}$$

The first term (te^{-t}) represents the solution to

$$y''+2y'+y=0, \qquad \text{given}\ \ y(0)=0,\ \ y'(0)=1.$$

The second term ($u(t-1)(t-1)e^{-(t-1)}$) is the response of the system to the unit impulse at $t=1$. Note that this second term is, as expected, zero for $t<1$. ○

Example 6.2 Solve the differential equation

$$y'' + 3y' + 3y = \delta(t-1)t^2$$

given that $y(0) = y'(0) = 0$.
Solution The term t^2 multiplying $\delta(t-1)$ is a bit of a fraud. Since $\delta(t-1)$ is zero everywhere except at $t = 1$, it is only the value of t^2 at $t = 1$ which matters. We might as well replace $\delta(t-1).t^2$ by $\delta(t-1).1^2$. Thus the transform of the right-hand side of the equation is simply $L\{\delta(t-1)\} = e^{-s}$. If you find this distasteful, note in general

$$L\{\delta(t-a)f(t)\} = \int_0^\infty \delta(t-a)f(t)e^{-st}dt = f(a)e^{-sa} = f(a)L\{\delta(t-a)\}$$

Taking the transform of the equation gives

$$s^2\bar{y} - sy_0 - y_0' + 3(s\bar{y} - y_0) + 3\bar{y} = e^{-s}$$

Putting $y_0 = y_0' = 0$ then gives

$$(s^2 + 3s + 3)\bar{y} = e^{-s}$$

and so

$$\bar{y} = \frac{1}{s^2 + 3s + 3}.e^{-s}$$

To invert this, note that completing the square gives

$$s^2 + 3s + 3 = \left(s + \frac{3}{2}\right)^2 + \frac{3}{4} = \left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence

$$\frac{1}{s^2 + 3s + 3} = \frac{\frac{\sqrt{3}}{2}}{(s + \frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2}.\frac{2}{\sqrt{3}}$$

Here we have introduced a factor $\frac{\sqrt{3}}{2}$ on the top because the table of transforms contains the entry

$$L\{e^{-\alpha t}\sin \omega t\} = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

We have of course immediately corrected for this additional factor $\frac{\sqrt{3}}{2}$ by multiplying the whole expression by $\frac{2}{\sqrt{3}}$. It follows that

$$L^{-1}\left\{\frac{1}{s^2 + 3s + 3}\right\} = \frac{2}{\sqrt{3}}e^{-\frac{3}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Hence by the second shift theorem

$$L^{-1}\left\{\frac{e^{-s}}{s^2+3s+3}\right\} = \frac{2}{\sqrt{3}}u(t-1)e^{-\frac{3}{2}(t-1)}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)$$

i.e.

$$y(t) = \frac{2}{\sqrt{3}}u(t-1)e^{-\frac{3}{2}(t-1)}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)$$

In this case the value of y was zero until the impulse was received at time $t = 1$.

We conclude this section by mentioning the concept of **Transfer Functions**. Consider the differential equation

$$y'' + ay' + by = f(t) \qquad \text{given } y(0) = y'(0) = 0$$

where $a,\ b$ are constants and $f(t)$ is a function of t. Taking Laplace transforms gives

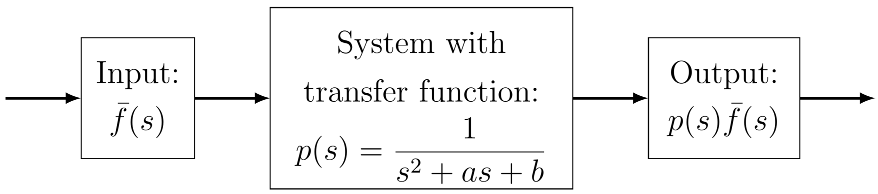
$$s^2\bar{y} + as\bar{y} + b\bar{y} = \bar{f}(s)$$

and so

$$\begin{aligned} \bar{y} &= \frac{1}{s^2+as+b}\bar{f}(s) \\ &= p(s)\bar{f}(s), \quad \text{say.} \end{aligned}$$

The function $p(s) = \dfrac{1}{s^2+as+b}$ is called the **Transfer Function** of the system described by the differential equation. If we consider the case when $f(t) = \delta(t)$ then $\bar{f}(s) = 1$ and so $\bar{y} = p(s)$. Thus the transfer function of the system is just the Laplace transform of the response y of the system to the unit impulse at $t = 0$. (This response is called the **impulse response**).

Schematically we can represent the physical system by means of the diagram below.



(The video covers most of the material in this section and it includes an on-location session in the student union. V

You should now try some examples of differential equations which involve the unit impulse function.) EX

7 Periodic functions

(This section is not covered in the videos.)

A function f is said to have period p if $f(t+p) = f(t)$ for all values of t. Geometrically this means that the graph of f repeats in blocks of length p. Many functions which arise in practice are periodic - the leading examples are probably the sine and cosine functions - but digital equipment often generates other periodic functions such as sawtooths and square wares. Calculation of the Laplace transforms of such functions is simplified by the following result.

Theorem 7.1. *If f is periodic with period p then*

$$\bar{f}(s) = \frac{1}{1-e^{-sp}} \int_0^p f(t)e^{-st}dt$$

Proof From the definition of the Laplace transform

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^p f(t)e^{-st}dt + \int_p^\infty f(t)e^{-st}dt \end{aligned}$$

In the latter integral we make the substitution $u = t - p$ (so that $t = u + p$ and $dt = dp$). Note that the limit $t = p$ corresponds to $u = 0$ and that $u \to \infty$ as $t \to \infty$. Hence

$$\bar{f}(s) = \int_0^p f(t)e^{-st}dt + \int_0^\infty f(u+p)e^{-s(u+p)}du$$

But $f(u+p) = f(u)$ because f is periodic with period p, and $e^{-s(u+p)} = e^{-su}e^{-sp}$. Hence

$$\bar{f}(s) = \int_0^p f(t)e^{-st}dt + e^{-sp}\int_0^\infty f(u)e^{-su}du$$

The latter integral is just the same as

$$\int_0^\infty f(t)e^{-st}dt$$

i.e. it is simply $\bar{f}(s)$. Hence we have

$$\bar{f}(s) = \int_0^p f(t)e^{-st}dt + e^{-sp}\bar{f}(s)$$

Therefore

$$\bar{f}(s) - e^{-sp}\bar{f}(s) = \int_0^p f(t)e^{-st}dt$$

i.e.

$$(1 - e^{-sp})\bar{f}(s) = \int_0^p f(t)e^{-st}dt$$

Hence

$$\bar{f}(s) = \frac{1}{1 - e^{-sp}} \int_0^p f(t)e^{-st}dt \qquad \square$$

We now use the result to produce Laplace transforms for some simple periodic functions.

Example 7.1 Determine the Laplace transform of the square wave of period $2a$ given by

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < a \\ 0 & \text{if } a \le t < 2a \end{cases}$$

Solution The graph of $f(t)$ has the following form (figure 9).

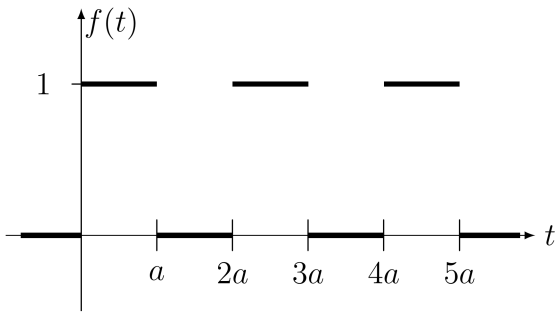


Figure 9: The squarewave $f(t)$.

Using the result of theorem 7.1 with $p = 2a$ we get

$$\bar{f}(s) = \frac{1}{1 - e^{-2as}} \int_0^{2a} f(t)e^{-st}dt$$

We can split the integral into

$$\int_0^a \quad + \quad \int_a^{2a}$$

In the former integral $f(t) = 1$, whilst in the latter integral $f(t) = 0$ (and so the integral is zero). Therefore in this case

$$\begin{aligned}
\bar{f}(s) &= \frac{1}{1-e^{-2as}} \int_0^a e^{-st} dt \\
&= \frac{1}{1-e^{-2as}} \left[\frac{e^{-st}}{-s} \right]_0^a \\
&= \frac{1}{1-e^{-2as}} \left[\frac{1-e^{-as}}{s} \right] \\
&= \frac{1}{s} . \frac{1-e^{-as}}{1-e^{-2as}}
\end{aligned}$$

In fact this can be simplified because $1 - e^{-2as} = (1 - e^{-as})(1 + e^{-as})$ (difference of two squares formula). Hence

$$\bar{f}(s) = \frac{1}{s} . \frac{1-e^{-as}}{(1-e^{-as})(1+e^{-as})} = \frac{1}{s} . \frac{1}{1+e^{-as}}$$

Example 7.2 Determine the Laplace transform of the sawtooth wave of period a given by

$$f(t) = \frac{t}{a} \quad \text{if} \quad 0 \leq t < a$$

Solution The graph of $f(t)$ has the following form (figure 10).

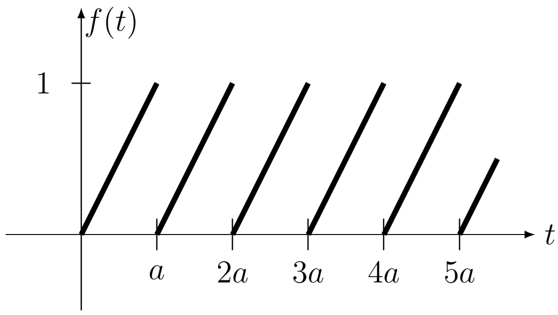


Figure 10: The sawtooth $f(t)$.

Applying theorem 7.1 with $p = a$ gives

$$\begin{aligned}
\bar{f}(s) &= \frac{1}{1-e^{-as}} \int_0^a f(t) e^{-st} dt \\
&= \frac{1}{1-e^{-as}} \int_0^a \frac{t}{a} e^{-st} dt \\
&= \frac{1}{a} . \frac{1}{1-e^{-as}} \int_0^a t e^{-st} dt
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
\int_0^a te^{-st}dt &= \left[t\left(\frac{e^{-st}}{-s}\right) - \int\left(\frac{e^{-st}}{-s}\right)dt\right]_0^a \\
&= -\frac{ae^{-as}}{s} - \left[\frac{e^{-st}}{s^2}\right]_0^a \\
&= -\frac{ae^{-as}}{s} + \frac{1}{s^2} - \frac{e^{-as}}{s^2} \\
&= \frac{1}{s^2}(1-e^{-as}) - \frac{a}{s}e^{-as}
\end{aligned}$$

Hence

$$\bar{f}(s) = \frac{1}{a}.\frac{1}{1-e^{-as}}\left[\frac{1}{s^2}(1-e^{-as}) - \frac{a}{s}e^{-as}\right] = \frac{1}{as^2} - \frac{e^{-as}}{s(1-e^{-as})}$$

Example 7.3 Determine the Laplace transform of the full-wave rectification of the sine wave $\sin(\omega t)$, i.e. $f(t) = |\sin(\omega t)|$ (period $\frac{\pi}{\omega}$).
Solution The graph of $f(t)$ has the following form (figure 11).

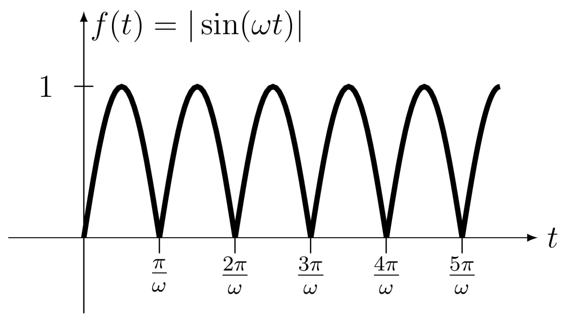


Figure 11: The rectified sine wave $|\sin(\omega t)|$.

Applying theorem 7.1 with $p = \frac{\pi}{\omega}$ gives

$$\begin{aligned}
\bar{f}(s) &= \frac{1}{1-e^{-\pi s/\omega}}\int_0^{\pi/\omega} f(t)e^{-st}dt \\
&= \frac{1}{1-e^{-\pi s/\omega}}\int_0^{\pi/\omega} \sin(\omega t)e^{-st}dt
\end{aligned}$$

because $f(t) = \sin(\omega t)$ for $0 \leq t \leq \frac{\pi}{\omega}$. To evaluate the integral substitute $u = \omega t$ so that the integral becomes

$$I = \frac{1}{\omega}\int_0^{\pi} \sin u e^{-su/\omega}du$$

Replacing $\sin u$ by $\Im(e^{iu})$ (where $\Im$ denotes the imaginary part) gives

$$\begin{aligned} I &= \frac{1}{\omega}.\Im\left(\int_0^{\pi} e^{(i-s/\omega)u}du\right) \\ &= \frac{1}{\omega}.\Im\left(\left[\frac{e^{(i-s/\omega)u}}{(i-s/\omega)}\right]_0^{\pi}\right) \\ &= \frac{1}{\omega}.\Im\left(\frac{1-e^{(i\pi-\pi s/\omega)}}{s/\omega - i}\right) \end{aligned}$$

noting $e^{i\pi} = -1$ we obtain

$$\begin{aligned} I &= \frac{1}{\omega}.\Im\left(\frac{1+e^{-\pi s/\omega}}{s/\omega - i}\right) \\ &= \frac{1}{\omega}.\Im\left(\frac{(1+e^{-\pi s/\omega})(s/\omega + i)}{s^2/\omega^2+1}\right) \\ &= \frac{1}{\omega}.\frac{(1+e^{-\pi s/\omega})}{(s^2/\omega^2+1)} \\ &= \frac{\omega}{s^2+\omega^2}.(1+e^{-\pi s/\omega}) \end{aligned}$$

Hence

$$\bar{f}(s) = \frac{\omega}{s^2+\omega^2}.\frac{1+e^{-\pi s/\omega}}{1-e^{-\pi s/\omega}}$$

○

Working with periodic functions often involves inverting expressions of the forms

$$\bar{f}(s).\frac{1}{1+e^{-as}} \qquad \text{and} \qquad \bar{f}(s).\frac{1}{1-e^{-as}}$$

These (and similar) forms can be inverted by using the expansions

$$\begin{aligned} \frac{1}{1+X} &= 1 - X + X^2 - X^3 + \cdots \qquad (\text{for } |X| < 1) \\ \frac{1}{1-X} &= 1 + X + X^2 + X^3 + \cdots \qquad (\text{for } |X| < 1) \end{aligned}$$

and then applying the second shift theorem.

Example 7.4 Determine

$$L^{-1}\left\{\frac{1}{s-\omega}.\frac{e^{-as}}{1+e^{-as}}\right\}$$

Solution Note firstly $L^{-1}\{\frac{1}{s-\omega}\} = e^{\omega t}$. Expanding

$$\frac{1}{1+e^{-as}} = 1 - e^{-as} + e^{-2as} - e^{-3as} + \cdots$$

Hence, multiplying by e^{-as},

$$\frac{e^{-as}}{1+e^{-as}} = e^{-as} - e^{-2as} + e^{-3as} - e^{-4as} + \cdots$$

From the second shift theorem

$$\begin{aligned}
L^{-1}\left\{\frac{e^{-as}}{s-\omega}\right\} &= u(t-a)e^{\omega(t-a)} \\
L^{-1}\left\{\frac{e^{-2as}}{s-\omega}\right\} &= u(t-2a)e^{\omega(t-2a)} \\
L^{-1}\left\{\frac{e^{-3as}}{s-\omega}\right\} &= u(t-3a)e^{\omega(t-3a)} \\
&\text{etc.}
\end{aligned}$$

Hence

$$\begin{aligned}
L^{-1}\left\{\frac{1}{s-\omega}.\frac{e^{-as}}{1+e^{-as}}\right\} &= L^{-1}\left\{\frac{e^{-as}}{s-\omega} - \frac{e^{-2as}}{s-\omega} + \frac{e^{-3as}}{s-\omega} - \frac{e^{-4as}}{s-\omega} + \cdots\right\} \\
&= u(t-a)e^{\omega(t-a)} - u(t-2a)e^{\omega(t-2a)} \\
&\quad + u(t-3a)e^{\omega(t-3a)} - u(t-4a)e^{\omega(t-4a)} + \cdots \\
&= \sum_{r=1}^{\infty} u(t-ra)e^{\omega(t-ra)}(-1)^{r-1}
\end{aligned}$$

Note that although the final answer appears to be an infinite series, for any particular value of t only a finite number of the $u(t-ra)$ terms are non-zero. [In fact $u(t-ra) = 0$ for all values of $r > \frac{t}{a}$.] Thus the series is only a finite series for any particular value of t. ○

We now look at some examples of differential equations involving the types of periodic functions considered above

Example 7.5 Solve the differential equation

$$\frac{dy}{dt} + y = f(t)$$

given that $y(0) = 0$, where $f(t)$ is the square wave of period 2 given by

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 2 \end{cases}$$

Solution This function was discussed in an earlier example (Example 7.1) where its Laplace transform was found to be

$$\bar{f}(s) = \frac{1}{s}.\frac{1}{1+e^{-s}}$$

Taking the Laplace transform of the differential equation gives

$$s\bar{y} - y_0 + \bar{y} = \frac{1}{s}.\frac{1}{1+e^{-s}}$$

But $y_0 = 0$ so we obtain

$$(s+1)\bar{y} = \frac{1}{s}.\frac{1}{1+e^{-s}}$$

Hence

$$\bar{y} = \frac{1}{s(s+1)}.\frac{1}{1+e^{-s}}$$

Partial fractions give

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Hence

$$\bar{y} = \frac{1}{s}.\frac{1}{1+e^{-s}} - \frac{1}{s+1}.\frac{1}{1+e^{-s}}$$

Inverting the first term on the right-hand side is easy: it gives the original function $f(t)$. To deal with the second term we note

$$L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

and

$$\frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots$$

Hence

$$\begin{aligned} L^{-1}\left\{\frac{1}{s+1}.\frac{1}{1+e^{-s}}\right\} &= L^{-1}\left\{\frac{1}{s+1} - \frac{e^{-s}}{s+1} + \frac{e^{-2s}}{s+1} - \frac{e^{-3s}}{s+1} + \cdots\right\} \\ &= e^{-t} - u(t-1)e^{-(t-1)} + u(t-2)e^{-(t-2)} \\ &\quad - u(t-3)e^{-(t-3)} + \cdots \end{aligned}$$

using the second shift theorem. Finally we obtain

$$y(t) = f(t) - e^{-t} + u(t-1)e^{-(t-1)} - u(t-2)e^{-(t-2)} + u(t-3)e^{-(t-3)} + \cdots$$

Although the sum on the right-hand side appears to be an infinite series, for any particular value of t all but a finite number of the terms are zero. ○

Example 7.6 Solve the differential equation

$$\frac{d^2y}{dt^2} + 4y = |\sin t|$$

given that $y(0) = y'(0) = 0$.

Solution The function $|\sin t|$ is the full-wave rectification of the sine wave $\sin t$. Again from an earlier example (Example 7.3) the transform of this is given by

$$\frac{1}{s^2+1}.\frac{1+e^{-\pi s}}{1-e^{-\pi s}}$$

Taking the transform of the differential equation we obtain

$$s^2\bar{y} - sy_0 - y_0' + 4\bar{y} = \frac{1}{s^2+1}.\frac{1+e^{-\pi s}}{1-e^{-\pi s}}$$

Substituting $y_0 = y_0' = 0$ this reduces to

$$(s^2+4)\bar{y} = \frac{1}{s^2+1}.\frac{1+e^{-\pi s}}{1-e^{-\pi s}}$$

and so

$$\bar{y} = \frac{1}{(s^2+4)(s^2+1)}.\frac{1+e^{-\pi s}}{1-e^{-\pi s}}$$

Using partial fractions

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

Hence

$$\bar{y} = \frac{1}{3}.\frac{1}{s^2+1}.\frac{1+e^{-\pi s}}{1-e^{-\pi s}} - \frac{1}{3}.\frac{1}{s^2+4}.\frac{1+e^{-\pi s}}{1-e^{-\pi s}}$$

The first term on the right-hand side is just one third of the transform of $|\sin t|$ and so is easily inverted. To deal with the second, note that

$$L^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$$

Also

$$\frac{1}{1-e^{-\pi s}} = 1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + \cdots$$

so

$$\begin{aligned}
\frac{1+e^{-\pi s}}{1-e^{-\pi s}} &= (1+e^{-\pi s})(1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + \cdots) \\
&= 1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + \cdots \\
&\quad\; + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + \cdots \\
&= 1 + 2e^{-\pi s} + 2e^{-2\pi s} + 2e^{-3\pi s} + \cdots
\end{aligned}$$

Hence

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{s^2+4}.\frac{1+e^{-\pi s}}{1-e^{-\pi s}}\right\} &= L^{-1}\left\{\frac{1}{s^2+4}+\frac{2}{s^2+4}e^{-\pi s}+\frac{2}{s^2+4}e^{-2\pi s}+\cdots\right\} \\
&= \frac{1}{2}\sin 2t + u(t-\pi)\sin 2(t-\pi) \\
&\quad + u(t-2\pi)\sin 2(t-2\pi)+\cdots
\end{aligned}
$$

Because of the periodicity of $\sin 2t$, we have that $\sin 2(t-r\pi) = \sin 2t$ for $r = 1,\ 2,\ 3, \ldots$. We can therefore write the previous expression as

$$
\begin{aligned}
&\frac{1}{2}\sin 2t + u(t-\pi)\sin 2t + u(t-2\pi)\sin 2t + \cdots \\
&\qquad = \sin 2t[\frac{1}{2} + u(t-\pi) + u(t-2\pi) + \cdots]
\end{aligned}
$$

Finally, we obtain

$$
y(t) = \frac{1}{3}|\sin t| - \frac{1}{3}\sin 2t[\frac{1}{2} + u(t-\pi) + u(t-2\pi) + \cdots]
$$

Again here the apparently infinite sum on the right-hand side is in fact finite for any particular value of t. ○

(You should now try obtaining the Laplace transforms of some easy periodic functions using the theorem at the start of this section. You should then try solving some differential equations which involve such periodic functions.)

EX

8 Transforms and integrals

(This section is not covered in the videos.)

8.1 Convolution

The convolution of two functions $f(t),\ g(t)$ (defined for $t \geq 0$) is the function of t

$$\int_0^t f(u)g(t-u)du$$

(defined for $t \geq 0$).
This function is denoted by $f \star g$ so that

$$(f \star g)(t) = \int_0^t f(u)g(t-u)du$$

It is easy to show that the convolution operator $\star$ has a number of features reminiscent of multiplication:

$$\begin{aligned} f \star g &= g \star f \\ f \star (g_1 + g_2) &= f \star g_1 + f \star g_2 \\ (f \star g) \star h &= f \star (g \star h) \end{aligned}$$

Example 8.1 If $f(t) = t,\ g(t) = e^t$, determine $(f \star g)(t)$.
Solution

$$\begin{aligned} (f \star g)(t) &= \int_0^t ue^{t-u}du \\ &= e^t \int_0^t ue^{-u}du \\ &= e^t \left[-ue^{-u} + \int e^{-u}du \right]_0^t \\ &= e^t \left[-te^{-t} - e^{-t} + 1 \right] \\ &= -t - 1 + e^t \end{aligned}$$

Hence $(f \star g)(t) = e^t - t - 1$ ○

It turns out that the Laplace transform of $(f \star g)(s)$ is just the product $\bar{f}(s)\bar{g}(s)$. The proof requires a knowledge of double integrals. You should omit the proof if you are unfamiliar with this topic.

Theorem 8.1. *(The convolution theorem)*

$$\overline{(f \star g)}(s) = \bar{f}(s)\bar{g}(s)$$

Proof

$$\begin{aligned}
\overline{(f \star g)}(s) &= \int_0^\infty \left(\int_0^t f(u)g(t-u)du \right) e^{-st} dt \\
&= \int_0^\infty \int_0^t e^{-st} f(u)g(t-u)dudt
\end{aligned}$$

This integral is a double integral over the region shown below (figure 12).

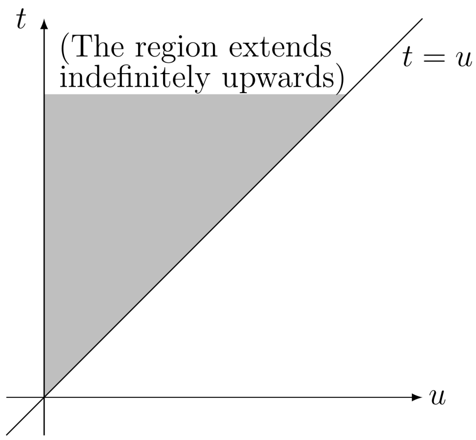


Figure 12: The region of integration.

Changing the order of integration gives

$$\begin{aligned}
\overline{(f \star g)}(s) &= \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u)dtdu \\
&= \int_0^\infty \left(f(u)e^{-su} \int_u^\infty e^{-s(t-u)} g(t-u)dt \right) du \\
&= \int_0^\infty \left(f(u)e^{-su} \int_0^\infty e^{-sv} g(v)dv \right) du
\end{aligned}$$

where there has been a change of variable to $v = t - u$ in the inner integral. But

$$\int_0^\infty e^{-sv} g(v)dv = \bar{g}(s)$$

Hence

$$\begin{aligned}\overline{(f \star g)}(s) &= \int_0^\infty f(u)e^{-su}\bar{g}(s)du \\ &= \bar{g}(s)\int_0^\infty f(u)e^{-su}du \\ &= \bar{g}(s)\bar{f}(s)\end{aligned}$$

and this completes the proof. □

Whether or not you have the knowledge of multiple integrals to follow the proof, the result of the theorem should make it clear why $\star$ is likely to have properties similar to multiplication.

The convolution theorem can be useful in finding inverse transforms.

Example 8.2 Use the convolution theorem to find the inverse transform of $1/[s^2(s-1)]$.

Solution Firstly $1/s^2$ is the inverse transform of t. Secondly $1/(s-1)$ is the inverse transform of e^t. Hence $1/[s^2(s-1)]$ is the inverse transform of the convolution of t and e^t. From the previous example this function is $e^t - t - 1$, i.e.

$$L^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = e^t - t - 1$$ ○

8.2 The transform of an integral

Theorem 8.2.

$$L\left\{\int_0^t f(u)du\right\} = \frac{\bar{f}(s)}{s}$$

Proof Apply the convolution theorem with $g(t) \equiv 1$. This gives

$$L\{f \star g\}(s) = \bar{f}(s)\bar{g}(s)$$

i.e. $L\left\{\int_0^t f(u)du\right\} = \bar{f}(s).\frac{1}{s}$. □

Of course the result can be reapplied to give transforms of iterated integrals. For example

$$L\left\{\int_0^t \left(\int_0^u f(v)dv\right)du\right\} = \frac{\bar{f}(s)}{s^2}$$

Example 8.3 Given that the transform of t is $1/s^2$, the result gives

$$L\left\{\int_0^t udu\right\} = \frac{1}{s^3}$$

i.e. $L\left\{\frac{t^2}{2}\right\} = \frac{1}{s^3}$. ○

8.3 Solution of integral and integral-differential equations

A differential equation is an equation relating derivatives of an unknown function y. An integral equation is an equation relating integrals of an unknown function y. More complicated equations may involve both derivatives and integrals; these are called integral-differential equations.

Just as it is possible to solve certain differential equations by use of Laplace transforms, so it may be possible to solve integral and integral-differential equations using this technique. The two theorems above (theorem 8.1 (the convolution theorem) and theorem 8.2) are generally useful.

Example 8.4 Solve the integral equation

$$y(t) = e^{2t} + \int_0^t e^{t-u} y(u) du$$

Solution We recognise the integral as the convolution of e^t and $y(t)$; its transform is therefore $\bar{y}(s)/(s-1)$. Taking the transform of the equation we obtain

$$\bar{y}(s) = \frac{1}{s-2} + \frac{\bar{y}(s)}{s-1}$$

Grouping the $\bar{y}$ terms this gives

$$\bar{y}(s)\left(1 - \frac{1}{s-1}\right) = \frac{1}{s-2}$$

so

$$\bar{y}(s)\left(\frac{s-2}{s-1}\right) = \frac{1}{s-2}$$

Hence

$$\bar{y}(s) = \frac{s-1}{(s-2)^2} = \frac{s-2}{(s-2)^2} + \frac{1}{(s-2)^2} = \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

Therefore, inversion gives

$$y(t) = e^{2t} + te^{2t} = (t+1)e^{2t}$$

○

Example 8.5 Solve the integral-differential equation

$$\frac{dy}{dt} = 1 + \cos t - \int_0^t y(u) du$$

given that $y(0) = 0$.

Solution Taking Laplace transforms we have

$$s\bar{y} - y_0 = \frac{1}{s} + \frac{s}{s^2+1} - \frac{1}{s}\bar{y}$$

Putting $y_0 = 0$, multiplying by s, and grouping the $\bar{y}$ terms gives

$$(s^2+1)\bar{y} = 1 + \frac{s^2}{s^2+1}$$

therefore

$$\bar{y} = \frac{1}{s^2+1} + \frac{s^2}{(s^2+1)^2}$$

To make use of the transform tables we re-write s^2 as $(s^2-1)+1$ in the numerator of the second term on the right-hand side. This gives

$$\bar{y} = \frac{1}{s^2+1} + \frac{s^2-1}{(s^2+1)^2} + \frac{1}{(s^2+1)^2}$$

Hence

$$y(t) = \sin t + t\cos t + \frac{1}{2}(\sin t - t\cos t)$$

and this reduces to

$$y(t) = \frac{3}{2}\sin t + \frac{t}{2}\cos t$$

(You should now try some examples involving the convolution theorem and transforms of integrals. You should then try examples of an integral and an integral- differential equation.) EX

9 Summary

When you have completed this package, you should be able to do the things listed below:

1. understand and reproduce the **definition** of the Laplace transform,
2. obtain Laplace transforms of easy functions **directly** from the definition,
3. obtain the Laplace transforms of first and second **derivatives**,
4. **invert** transforms using tables in conjunction with algebraic manipulations such as **partial fractions** and **completing the square**,
5. use Laplace transforms to **solve** first and second order linear differential equations with appropriate initial conditions,
6. use Laplace transforms to solve **simultaneous** differential equations,
7. deal with discontinuous input functions expressible in terms of the **unit step function** and use the **second shift theorem** to obtain transforms and their inverses,
8. deal with problems involving the **unit impulse "function"** $\delta(t)$,
9. deal with problems involving **periodic** input functions such as square waves and sawtooths,
10. solve **integral** and **integral-differential** equations using Laplace transforms.

10 Bibliography

For textbooks covering the basic prerequisites for this package (differentiation, integration and partial fractions) see, for example, one of the following (although there are dozens of other suitable textbooks many of which are in the University library).

Stroud, K. A. Engineering Mathematics (third edition), Macmillan, 1992.

Jeffrey, A. Mathematics for Engineers and Scientists, Van Nostrand, 1989.

Thomas, G. B. and Finney, R. L. Calculus and Analytic Geometry, Addison-Wesley, 1988.

Larson, R. E., Hostetler, R. P. and Edwards, B. H. Calculus, D. C. Heath and Company, 1990.

Gilbert, J. Guide to Mathematical Methods, Macmillan, 1991.

Anton, H. Calculus with Analytic Geometry, Wiley, 1992.

There are many textbooks which cover Laplace transforms. Of the books listed above that by Jeffrey gives a very elementary treatment. There is a second volume by Stroud which gives a reasonable treatment:

Stroud, K. A. Further Engineering Mathematics (second edition), Macmillan, 1992.

The books listed below are devoted exclusively to the subject of differential equations and they give more detail.

Sanchez, D. A., Allen, R. C. and Kyner, W. T. Differential Equations (second edition), Addison-Wesley, 1988.

Zill, D. G. A first course in Differential Equations with Applications (fourth edition), Prindle, Weber, Schmidt - Kent Publishing Company, 1989.

Boyce, W. E. and DiPrima, R. C. Elementary Differential Equations and Boundary Value Problems (fifth edition), Wiley, 1992.

Of the three books listed, you would probably find the volume by Boyce et al. to be too advanced for general use. All three volumes cover a much wider range of topics than this package - for example first and second order ordinary differential equations, partial differential equations and Fourier series. In addition to these three books there are very many other textbooks covering Laplace transforms and, again, many of these are in the University library. Your tutor should be able to advise you which textbooks are suitable for your own needs, but you need never be short of an alternative approach or more questions to try!

11 Appendix - Video Summaries

There are four videos associated with the topic of Laplace transforms. The presenter is Mike Grannell from the Department of Mathematics and Statistics at the University of Central Lancashire. We recommend that you read the preamble to these notes which makes some suggestions about how you should approach viewing the videos.

Video title: Laplace transforms (part 1). (24 minutes)

Prerequisite: you will need to be familiar with basic integration and differentiation.

Summary

1. **Definition and notation**

   The integral definition, the range of values of the parameter s, improper integrals, various notations including $L\{y\}$, $\bar{y}$, and Y for the transform of $y(t)$. The limit $\lim_{x\to\infty} x^n e^{-ax} = 0$ for $a > 0$.

2. **Examples**

$$
\begin{aligned}
&1. \quad L\{1\} = \frac{1}{s} \qquad (s > 0) \\
&2. \quad L\{t\} = \frac{1}{s^2} \qquad (s > 0) \\
&3. \quad L\{e^{at}\} = \frac{1}{s-a} \qquad (s > a)
\end{aligned}
$$

3. **Use of tables**

4. **Fundamental properties**

   (a) **Linearity**

$$L\{af + bg\} = aL\{f\} + bL\{g\}$$

   where $a,\ b$ are constants and $f,\ g$ are functions of t.

   (b) **Transforms of derivatives**

$$
\begin{aligned}
L\{f'\} &= s\bar{f} - f(0) \\
L\{f''\} &= s^2\bar{f} - sf(0) - f'(0)
\end{aligned}
$$

Video title: Laplace transforms (part 2). (33 minutes)

Prerequisite: you will need to be familiar with basic integration, differentiation and partial fractions, and to have covered the material in the first video.

Summary

1. **Inverse transforms**

   Existence and uniqueness

   Use of tables

   Examples

   1. $L^{-1}\left\{\dfrac{3s-2}{s^2+6s+25}\right\}$ (completing the square)
   2. $L^{-1}\left\{\dfrac{2}{s(s+1)(s^2+1)}\right\}$ (partial fractions)

2. **Solution of differential equations**

   Description of the method.

   Example $y'' + 9y = \cos 3t$ given that $y(0) = 1$ and $y'(0) = -1$.

   Solution of systems of differential equations -the example

   $$\frac{dx}{dt} = 2x + y, \quad \frac{dy}{dt} = 3x + 4y$$

   given that $x(0) = 1$ and $y(0) = 0$.

Video title: Laplace transforms (part 3). (29 minutes)

Prerequisite: you will need to be familiar with basic integration, differentiation and partial fractions, and to have covered the material in the first two videos.

Summary

1. **The unit step function**

   Definition of the Heaviside unit step function $u(t)$ and its translate $u(t-a)$.

2. **Piecewise continuous functions**

   The graph of $f(t)u(t-a)$

   Example Expressing the function

$$g(t) = \begin{cases} 3 & \text{if } t < 2 \\ -3 & \text{if } 2 < t < 4 \\ e^{-t} & \text{if } t > 4 \end{cases}$$

   in terms of the unit step function.

3. **The Laplace transform of** $u(t-a)$ $(a \geq 0)$

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

4. **The second shift theorem**

   $L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$ and the alternative version
   $L\{f(t-a)u(t-a)\} = e^{-as}L\{f(t)\}$

   Examples

   1. $L\{t^2u(t-3)\}$
   2. $L\{e^{3t}u(t-2)\}$
   3. $L^{-1}\left\{\dfrac{e^{-2s}}{s^2+1}\right\}$
   4. $L^{-1}\left\{\dfrac{e^{-3s}}{(s+2)^2}\right\}$

Video title: Laplace transforms (part 4). (28 minutes)

Prerequisite: you will need to be familiar with basic integration, differentiation and partial fractions, and to have covered the material in the first three videos.

Summary

1. **Solution of differential equations involving the unit step function**

   The example $y'' + y = f(t)$ given that $y(0) = 1,\ y'(0)$, where

   $$f(t) = \begin{cases} 0 & \text{if } \ t < 1 \\ 4 & \text{if } \ 1 < t < 2 \\ 0 & \text{if } \ t > 2 \end{cases}$$

2. **The unit impulse"function"**

   On location in the student union: the unit impulse "function" (Dirac delta "function") and its physical interpretation.

3. **Mathematical treatment**

   $$\delta_\epsilon(t) = \begin{cases} \dfrac{1}{\epsilon} & \text{if } \ 0 < t < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

   $$\delta_\epsilon(t-a) = \begin{cases} \dfrac{1}{\epsilon} & \text{if } \ a < t < a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

   Graphs of these functions, areas under the graphs and behaviour as ϵ tends to 0.

4. **Integrals and the unit impulse "function"**

   If f is continuous and $a > 0$

   $$\int_0^\infty \delta(t-a) f(t) dt = f(a)$$

   Simple consequences including

   $$\int_0^\infty \delta(t-a) dt = 1$$

   $$L\{\delta(t-a)\} = e^{-as}$$

5. **Solution of differential equations involving the unit impulse "function"**

The example $y'' + 2y' + y = \delta(t - a)$ given that $y(0) = 0,\ y'(0) = 1.$

12 Appendix - Table of Laplace transforms

If f is a function of t then its Laplace transform, $L\{f(t)\}$ is the function of s given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

Function $f(t)$	**Laplace transform** $\bar{f}(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
$\cosh \alpha t$	$\frac{s}{s^2-\alpha^2}$
$t^n e^{-\alpha t}$	$\frac{n!}{(s+\alpha)^{n+1}}$
$e^{-\alpha t} \sin \omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t} \cos \omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
Continued on next page	

Function $f(t)$	**Laplace transform** $\bar{f}(s)$
$e^{-\alpha t}\sinh\omega t$	$\dfrac{\omega}{(s+\alpha)^2-\omega^2}$
$e^{-\alpha t}\cosh\omega t$	$\dfrac{s+\alpha}{(s+\alpha)^2-\omega^2}$
$t\sin\omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
$t\cos\omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$	$\dfrac{1}{(s^2+\omega^2)^2}$
$te^{-\alpha t}\sin\omega t$	$\dfrac{2\omega(s+\alpha)}{((s+\alpha)^2+\omega^2)^2}$
$te^{-\alpha t}\cos\omega t$	$\dfrac{(s+\alpha)^2-\omega^2}{((s+\alpha)^2+\omega^2)^2}$
Unit step function, step at 0: $u(t)=\begin{cases}1 & \text{if } t>0\\ 0 & \text{if } t<0\end{cases}$	$\dfrac{1}{s}$
Unit step function, step at a, $(a>0)$: $u(t-a)=\begin{cases}1 & \text{if } t>a\\ 0 & \text{if } t<a\end{cases}$	$\dfrac{e^{-as}}{s}$
Unit impulse or delta function, impulse at 0: $\delta(t)$	1
Unit impulse or delta function, impulse at a, $(a>0)$: $\delta(t-a)$	e^{-as}
Second shift theorem: a) $u(t-a)g(t)$ b) $u(t-a)f(t-a)$	a) $e^{-as}L\{g(t+a)\}$ b) $e^{-as}L\{f(t)\}$
Continued on next page	

Function $f(t)$	**Laplace transform** $\bar{f}(s)$
Periodic functions: if f has period p (i.e. $f(t+p)=f(t)$ for all t) then ...	$\bar{f}(s) = \dfrac{1}{1-e^{-sp}} \displaystyle\int_0^p e^{-st} f(t)dt$
Full-wave rectification of the sine wave $\sin \omega t$, i.e. $f(t) = \vert \sin \omega t \vert$, (period $= \pi/\omega$):	$\left(\dfrac{\omega}{s^2+\omega^2}\right)\left(\dfrac{1+e^{-\pi s/\omega}}{1-e^{-\pi s/\omega}}\right)$
Sawtooth wave: $f(t) = t/a$ if $0 < t < a$, period $= a$.	$\dfrac{1}{as^2} - \dfrac{e^{-as}}{s(1-e^{-as})}$
Squarewave (type 1), period $2a$: $f(t) = \begin{cases} 1 & \text{if } 0 \leq t < a \\ 0 & \text{if } a \leq t < 2a \end{cases}$	$\left(\dfrac{1}{s}\right)\left(\dfrac{1}{1+e^{-as}}\right)$
Squarewave (type 2), period $2a$: $f(t) = \begin{cases} 1 & \text{if } 0 \leq t < a \\ -1 & \text{if } a \leq t < 2a \end{cases}$	$\left(\dfrac{1}{s}\right)\left(\dfrac{1-e^{-as}}{1+e^{-as}}\right)$

Note also the formulae for the Laplace transforms of first and second derivatives:

$$\begin{aligned} L\{\frac{dy}{dt}\} &= s\bar{y} - y_0 \\ L\{\frac{d^2y}{dt^2}\} &= s^2\bar{y} - sy_0 - y_0' \end{aligned}$$

where $\bar{y}$ denotes the transform of the function y, $y_0 = y(0)$ and $y_0' = y'(0)$.