# FOURIER SERIES

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# Contents

# 1 Preamble

## 1.1 About this package

This package is for people who need to be able to find the Fourier series of certain types of functions. It doesn't contain a lot of theory. It isn't really designed for pure mathematicians who would require a course discussing the convergence of Fourier series in greater detail.

You will find that you need a background knowledge of differentiation, integration and curve-sketching in order to get the most out of this package. In particular, you need to be able to integrate by simple substitutions and by parts. If you are a bit rusty, don't worry - but it would be sensible to do some revision either at the start or as the need arises. Reasonable revision texts are given in the bibliography (Section 11).

If you complete the whole package you should be able to

- recognize **odd, even and periodic** functions,
- understand what is meant by the **Fourier series** of a function,
- understand what is meant by the **Fourier coefficients** of a function,
- obtain the **Fourier coefficients** of a function,
- obtain the **Fourier series of period** $2\pi$ of a function and obtain the $m^{th}$ **partial sum** of this Fourier series,
- use **the properties of odd or even functions** to reduce the work in finding Fourier coefficients,
- understand **Dirichlet's Theorem** concerning the convergence of Fourier series,
- determine whether a function satisfies the **Dirichlet conditions**,
- apply Dirichlet's Theorem to **sketching the graph of the function to which a Fourier series converges,**
- apply Dirichlet's Theorem to **summation of series**,
- obtain the **Fourier sine or cosine series of period** $2\pi$ of a function,
- understand how to **generalise the previous concepts** to find the **Fourier series, sine or cosine series of period** $2l$ of a function,

Depending on your own programme of study you may not need to cover everything in this package. Your tutor will advise you what, if anything, can be omitted.

## 1.2 How to use this package

You MUST do examples! Doing lots of examples for yourself is generally the most effective way of learning the contents of this package and covering the objectives listed above. We recommend that you

- first read the theory - make your own notes where appropriate,
- then work through the worked examples - compare your solutions with the ones in the notes,
- finally do similar examples yourself in a workbook.

The original printing of these notes leaves every other page blank. Use the spare space for your own comments, notes and solutions. You will see certain symbols appearing in the right hand margin from time to time:

○ denotes the end of a worked example,

□ denotes the end of a proof,

V denotes a reference to videos (see below for details),

EX highlights a point in the notes where you should try examples.

By the time you have reached a package like this one you will probably have realised that learning mathematics rarely goes smoothly! When you get stuck, use your accumulated wisdom and cunning to get around the problem. You might try:

- re-reading the theory/worked examples,
- putting it down and coming back to it later,
- reading ahead to see if subsequent material sheds any light,
- talking to a fellow student,
- looking in a textbook (see the bibliography),
- watching the appropriate video (see the video summaries),
- raising the problem at a tutorial.

## 1.3 Videos, tutorials and self-help

The videos cover the main points in the notes. The areas covered are indicated in the notes, usually at the ends of sections and subsections. To resolve a particular difficulty you may not need to watch a whole video (they are each about 30 - 40 minutes long). They are broken up into sections prefaced with titles which can be read on fast scan. In addition, a summary of the videos associated with this package appears as an appendix to these notes.

Your tutor will tell you about the arrangements for viewing the videos. Try the worked examples **before** watching the solution unfold on the screen. Make notes of any points you cannot follow so that you can explain the difficulty in a subsequent tutorial session. If you are viewing a video individually, remember the rewind button! Unlike a lecture you can get instant and 100 percent accurate replay of what was said.

Your tutor will tell you about tutorial arrangements. These may be related to assessment arrangements. If attendance at tutorials is compulsory then make sure you know the details! The tutorials provide you with individual contact with a tutor. Use this time wisely - staff time is the most expensive of all our resources.

**You should come to tutorials in a prepared state.** This means that you should have read the notes and the worked examples. You should have tried appropriate examples for yourself. If you have had difficulty with a particular section then you should watch the corresponding video. If your tutor finds that you haven't done these things then s/he may refuse to help you. Your tutor will find it easier to assist you if you can make any queries as specific as possible.

Your fellow students are an excellent form of self-help. Discuss problems with one another and compare solutions. Just be careful that

1. any assessed coursework submitted by you is yours alone,
2. you yourself do really understand solutions worked out jointly with colleagues.

Familiarize yourself with the layout and contents of these notes; scan them before reading them more carefully. The contents page will help you find your way about - use it. The bibliography will point you to textbooks covering the same material as these notes.

When you graduate, your future employer will be just as interested in your capacity for learning as in what you already know. If you can learn mathematics from this package and from textbooks then you will not only have learnt a particular mathematical topic, you will also (and more importantly) have learnt **how to learn** mathematics.

# 2 Introduction

Given a suitable function $f$ we can generate the Taylor Series for $f$ about the point $a$ and, providing $x$ is sufficiently close to $a$, we may write

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

By letting $a = 0$ in this expression we obtain the Maclaurin Series of $f$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$

In essence, this amounts to knowing that there are real constants $A_0, A_1, A_2, \ldots$ such that

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots$$

That is, we can write $f(x)$ as a linear combination of the algebraic polynomials

$$\{1, x, x^2, x^3, \ldots\}$$

In fact, finite linear combinations of algebraic polynomials are often used in order to approximate continuous functions. This is partly due to some of the advantageous manipulative properties of algebraic polynomials: they are easily integrated and differentiated, and shifts in co-ordinate system (i.e. changing $x$ to $\alpha x + \beta$) when applied to an algebraic polynomial will still result in an algebraic polynomial. However, most importantly, it is known that any continuous function can be approximated arbitrarily closely on a real interval $[a, b]$ by a linear combination of algebraic polynomials. Thus, algebraic polynomials have much to recommend them as a set of approximating functions.

However, there are occasions when, due to the nature of the function we wish to approximate, approximation by another set of functions would seem more natural. In particular, if the function we wish to approximate exhibits the repetitive wave forms associated with trigonometric functions then it would seem reasonable to suppose that it might be easier to represent such a function as a linear combination of the trigonometric functions

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}.$$

Certain combinations of this type are known as Fourier series.

This package deals with developing the Fourier series of a suitable function $f$. This series has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are all real constants whose definition depends on $f$. Expressing the first term in the series as $\frac{1}{2}a_0$ is a technical device which enables us to provide one formula for all the $a_n, n = 0, 1, 2, \ldots$.

We shall examine when the Fourier series of a suitable function is convergent and see the advantage provided by a Fourier series as a means of approximating suitable discontinuous functions. Our initial discussions will concern functions that are defined for $-\pi \leq x < \pi$. We will then extend our results to deal with functions defined for $0 \leq x \leq \pi$ and eventually to functions defined for $-l \leq x < l$ where $l$ is any positive real number.

Fourier series are of enormous practical help. They are of fundamental importance in constructing solutions of certain types of ordinary and partial differential equations. As an example, a second order inhomogeneous differential equation with constant coefficients of the form

$$Ay'' + By' + Cy = g(x)$$

can be solved easily when $g(x) = a\cos nx + b\sin nx$. However, in other cases, $g(x)$ may still be a function which displays a repetitive wave form but for which this differential equation is difficult to solve. Suppose we could represent $g(x)$ by a Fourier series. Then since the equation is linear the solution could be obtained by summing the solutions of

$$Ay'' + By' + Cy = a_n \cos nx + b_n \sin nx$$

for all values of $n$. In many practical situations the contributions to the solution from small values of $n$ are the most important.

*(The video introduces the concept of approximation of a function by its Fourier series)* V

# 3 Some preliminaries

In developing the definition of the Fourier Series of a function we shall be making use of the properties of odd, even and periodic functions.

## 3.1 Odd and even functions.

**Definition 3.1** An **even** function is a function such that

$$f(x) = f(-x), \quad \text{for all } x \text{ in the domain of } f.$$

Geometrically this means that the graph of $f$ is symmetrical about the $y$ axis.

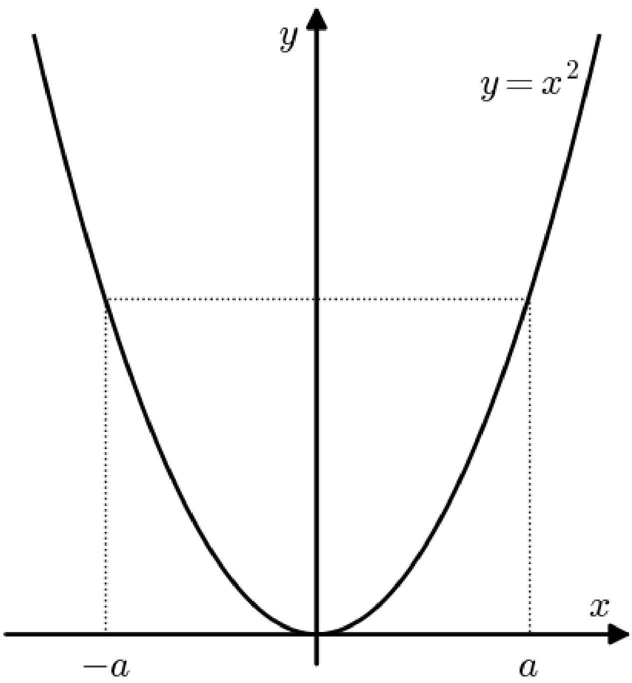



Figure 1: $x^2$ is an even function.

**Definition 3.2** An **odd** function is a function such that

$$f(x) = -f(-x), \quad \text{for all } x \text{ in the domain of } f.$$

Geometrically this means that the graph of $f$ for $x < 0$ may be obtained by rotating the graph of $f$ for $x > 0$ through 180° about the origin.

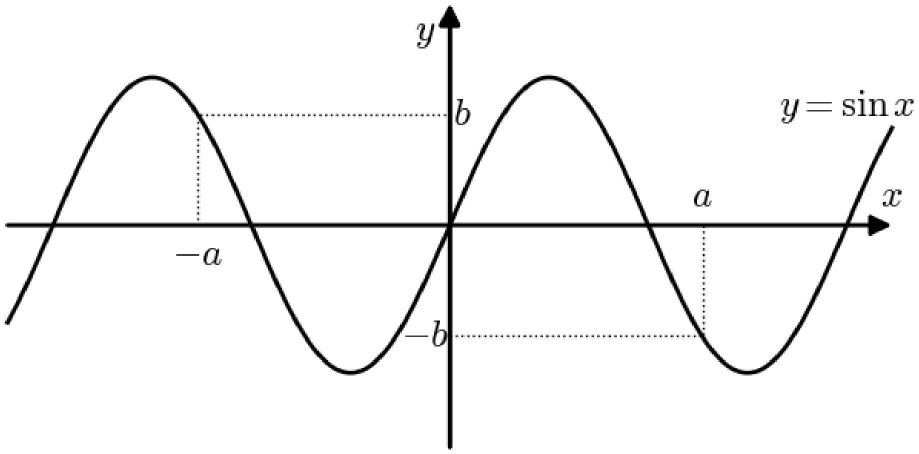



Figure 2: $\sin x$ is an odd function.

The reason for the titles "odd" and "even" is that odd powers of $x$ form odd functions whilst even powers form even functions. Note also that the standard power series of the odd function $\sin x$ contains only odd powers of $x$. Likewise the expansion of the even function $\cos x$ contains only even powers of $x$. This is typical of even and odd functions and their MacLaurin Series. A function is not necessarily even or odd. By considering the graph of $e^x$ we can tell that this function is neither even nor odd.

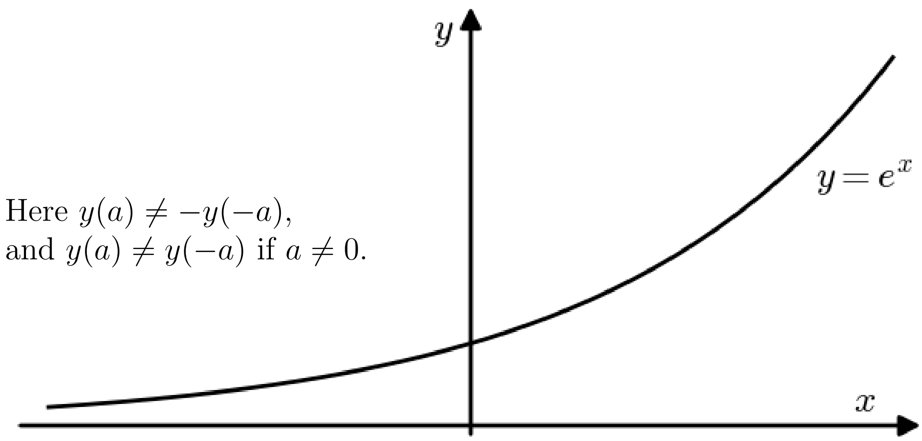



Figure 3: $e^x$ is a function that is neither even nor odd.

## 3.2 Properties of odd and even functions.

**Theorem 3.1.** *The sum, difference, product and quotient of two even functions are even.*

**Proof** Let $f$ and $g$ be two even functions; so, for $x$ in both the domain of $f$ and the domain of $g$, $f(x) = f(-x)$ and $g(x) = g(-x)$. Then,

$$\begin{aligned}(f+g)(x) &= f(x)+g(x) = f(-x)+g(-x) = (f+g)(-x),\\ (f-g)(x) &= f(x)-g(x) = f(-x)-g(-x) = (f-g)(-x),\\ (f.g)(x) &= f(x).g(x) = f(-x).g(-x) = (f.g)(-x),\end{aligned}$$

and, providing $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{f(-x)}{g(-x)} = \left(\frac{f}{g}\right)(-x).$$

□

**Theorem 3.2.** *The sum and difference of two odd functions are odd. The product and quotient of two odd functions are even.*

**Proof** Let $f$ and $g$ be two odd functions; so, for $x$ in both the domain of $f$ and the domain of $g$, $f(x) = -f(-x)$ and $g(x) = -g(-x)$. Then,

$$\begin{aligned}(f+g)(x) &= f(x)+g(x) = -f(-x)+\{-g(-x)\}\\ &= -\{f(-x)+g(-x)\} = -(f+g)(-x),\\ (f-g)(x) &= f(x)-g(x) = -f(-x)-\{-g(-x)\} = -(f-g)(-x),\\ (f.g)(x) &= f(x).g(x) = \{-f(-x)\}.\{-g(-x)\} = (f.g)(-x),\end{aligned}$$

and, providing $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{-f(-x)}{-g(-x)} = \left(\frac{f}{g}\right)(-x).$$

$\square$

**Theorem 3.3.** *The product and quotient of an odd function and an even function are odd.*

**Proof** Let $f$ be an odd function and $g$ be an even function; so, for $x$ in both the domain of $f$ and the domain of $g$, $f(x) = -f(-x)$ and $g(x) = g(-x)$. Then,

$$(f.g)(x) = f(x).g(x) = \{-f(-x)\}.g(-x) = -(f.g)(-x),$$

and, providing $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{-f(-x)}{g(-x)} = -\left(\frac{f}{g}\right)(-x).$$

$\square$

You should note that the sum and difference of an odd function and an even function are not necessarily odd or even.

**Example 3.1** Let functions $f$ and $g$ be defined by

$$f(x) = x^2 + 1, \qquad g(x) = x,$$

for all $x$ in $\Re$, then $f$ is an even function and $g$ is an odd function. However, their sum is neither odd nor even; consider

$$(f+g)(3) = f(3) + g(3) = 13,$$

whereas

$$(f+g)(-3) = f(-3) + g(-3) = 7$$

and

$$(f-g)(3) = f(3) - g(3) = 7,$$

whereas

$$(f-g)(-3) = f(-3) - g(-3) = 13.$$

We shall find the following result particularly useful when dealing with Fourier Series

**Theorem 3.4.** *If $f$ is an odd function,*

$$\int_{-a}^{a} f(x)dx = 0.$$

*If $f$ is an even function,*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

**Proof** : We can assume $a > 0$. Consideration of the graphs of odd and even functions illustrates these results (see figure 4).

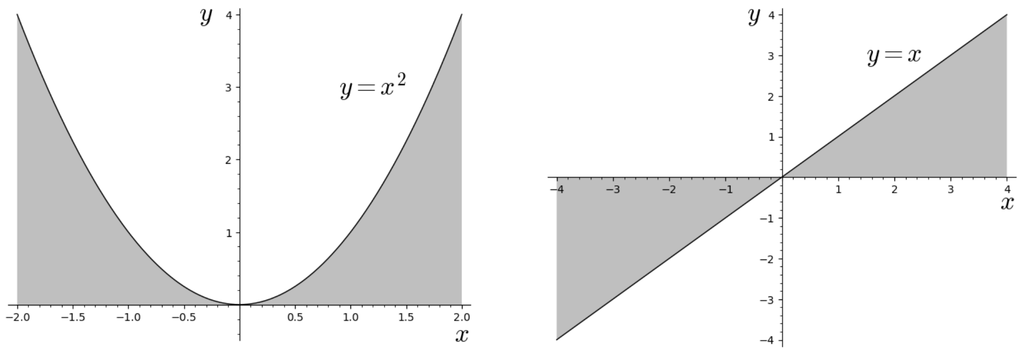



Figure 4: Theorem 3.4 illustrated using the even function $x^2$ and the odd function $x$.

If f is an odd function, then $f(x) = -f(-x)$ and

$$\begin{aligned}
\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\
& \quad \text{(now, employing the substitution } x = -t \\
& \quad \text{in the first integral)} \\
&= -\int_{a}^{0} f(-t)dt + \int_{0}^{a} f(x)dx \\
&= \int_{a}^{0} -f(-t)dt + \int_{0}^{a} f(x)dx \\
&= \int_{a}^{0} f(t)dt + \int_{0}^{a} f(x)dx \qquad \text{(since } -f(-t) = f(t)) \\
&= -\int_{0}^{a} f(t)dt + \int_{0}^{a} f(x)dx = 0.
\end{aligned}$$

A similar technique may be used to prove the second result. □

Notice that by combining the above properties we can yield results concerning integrals of sums, differences, products and quotients of even and odd functions. For instance, referring to Theorems 3.3 and 3.4 we know that if $f$ is an odd function and $g$ is an even function then

$$\int_{-a}^{a} f(x).g(x)dx = 0, \quad \text{for } a > 0.$$

*(The video discusses odd and even functions and examines their properties.)* V

## 3.3 Periodic functions

**Definition 3.3** A function is said to be **periodic** if $f(x+p) = f(x)$, for all $x$, where $p$ is a constant. The constant $p$ is known as a **period** of $f$ and $f$ is sometimes referred to as a $p$**-periodic function**. The smallest positive value of $p$ for which the condition is satisfied is called the **primitive period** of $f$ (or more loosely the **period** of $f$).

The diagram overleaf (figure 5) illustrates a function with primitive period 2. Of course, any integer multiple of 2 is also a period of this function.

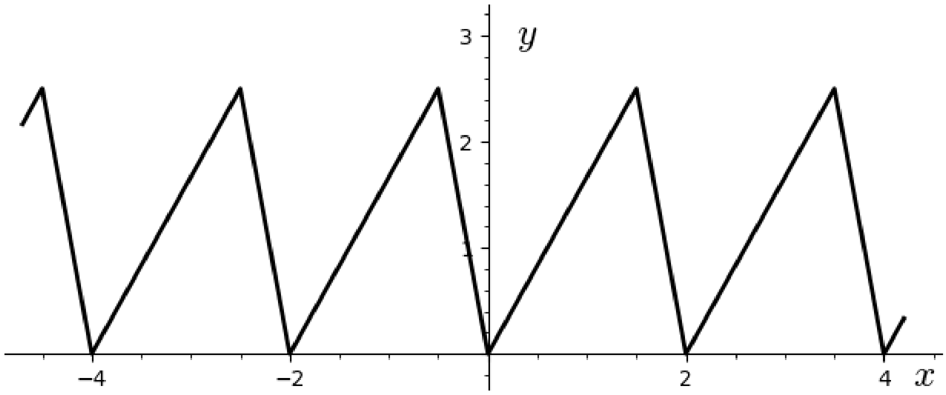
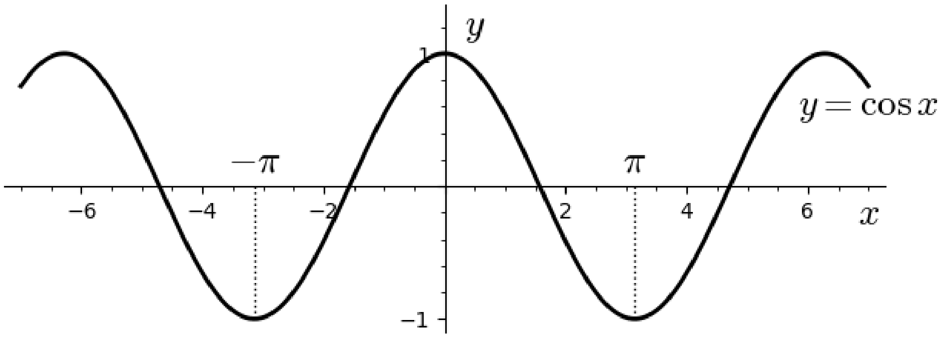


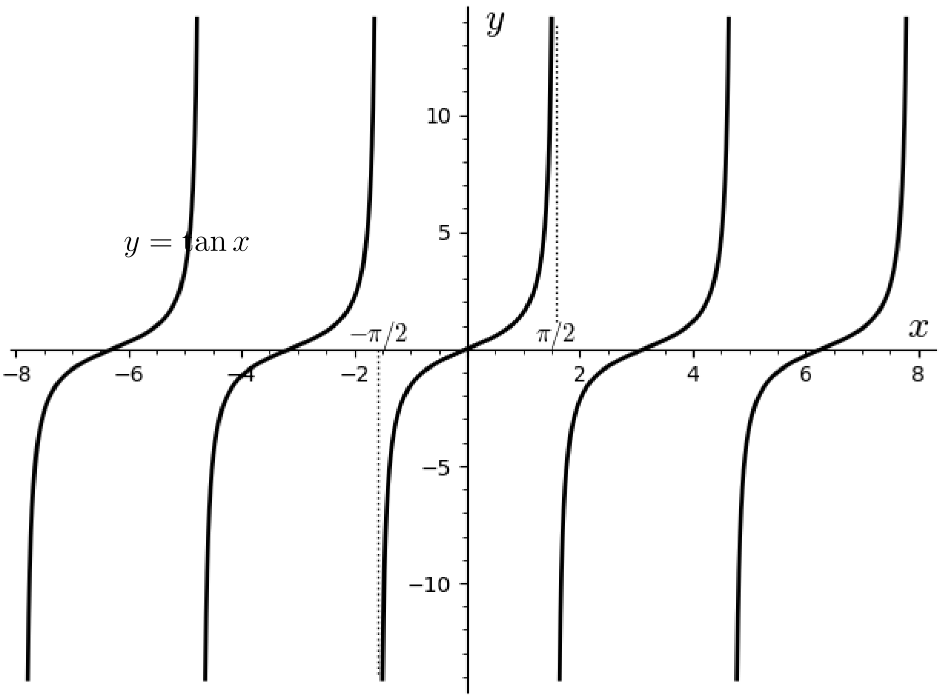

Figure 5: An example of a 2-periodic function.

The functions $\sin x$, $\cos x$ and $\tan x$ are all examples of $2\pi$-periodic functions. The function $\tan x$ is also $\pi$-periodic.




(This function repeats itself after every interval of length $2\pi$.)

Figure 6: The function $\cos x$ is $2\pi$-periodic.




(This function repeats itself after every interval of length $\pi$.)

Figure 7: The function $\tan x$ is $\pi$-periodic.

If $f$ and $g$ are both $p$-periodic then their sum is also $p$-periodic. For,

$$(f+g)(x) = f(x) + g(x) = f(x+p) + g(x+p) = (f+g)(x+p).$$

In a similar fashion we can show that their difference, product and quotient are also $p$-periodic.

*(The video discusses periodic functions and examines their properties.)* V

# 4 Fourier series of period $2\pi$

As a preliminary we shall consider some integrals which we will require in the course of our discussion.

**Theorem 4.1.** *Suppose that $m$ and $n$ are positive integers, then*

1. $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0.$
2. $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$
3. $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$

**Proof**

1. Since $\sin mx$ is an odd function and $\cos nx$ is an even function then referring to Theorems 3.3 and 3.4 yields the result.

2. If $m \neq n$ then using the trig identity
$$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\},$$
gives
$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \sin nx dx &= \frac{1}{2}\int_{-\pi}^{\pi}\{\cos(mx-nx) - \cos(mx+nx)\}dx \\
&= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} \\
&= 0,
\end{aligned}$$
since $\sin k\pi = 0$ for any integer $k$. Note that the denominator $(m-n)$ in the first fraction in square brackets means that we can't use this formula when $m = n$.

   If $m = n$ then
$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \sin nx dx &= \int_{-\pi}^{\pi} \sin^2 mx dx = \frac{1}{2}\int_{-\pi}^{\pi}(1 - \cos 2mx)dx \\
&= \frac{1}{2}\left[x - \frac{\sin 2mx}{2m}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\{(\pi - 0) - (-\pi - 0)\} = \pi.
\end{aligned}$$

3. If $m \neq n$ then using the trig identity

$$\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\},$$

gives

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx \cos nx dx &= \frac{1}{2}\int_{-\pi}^{\pi} \{\cos(mx-nx) + \cos(mx+nx)\}dx \\
&= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0,
\end{aligned}$$

since $\sin k\pi = 0$ for any integer $k$. As before, the denominator $(m-n)$ in the first fraction in square brackets means that we can't use this formula when $m = n$.

If $m = n$ then

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx \cos nx dx &= \int_{-\pi}^{\pi} \cos^2 mx dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2mx)dx \\
&= \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\{(\pi + 0) - (-\pi + 0)\} = \pi.
\end{aligned}$$

□

As was discussed in section 2, we would like to be able to express a function $f$ defined for $-\pi \leq x < \pi$, as a trigonometric series of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\} \tag{1}$$

where $a_0, a_1, \ldots, b_1, b_2, \ldots$ are real constants. These constants are known as the **Fourier coefficients of f** and the series in (1) is called the **Fourier series of f.**

Since each of the functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ is $2\pi$-periodic then, if the Fourier series of $f$ converges, its sum will be $2\pi$-periodic.

You may wonder why we don't try to expand $f$ just using sine functions as in

$$\sum_{n=1}^{\infty} c_n \sin nx \tag{2}$$

The answer is that the collection $\{\sin x, \sin 2x, \sin 3x, \ldots\}$ contains only odd functions and so anything of the form (2) will be an odd function. Since most functions aren't odd, most functions can't be expanded just using sine functions. For similar reasons we can't just use cosine functions.

You may also wonder why the series starts with the $n = 1$ term and not $n = 0$. Well, the cosine term when $n = 0$ contains $\cos 0x = \cos 0 = 1$ and this is the reason for the constant term $\frac{1}{2}a_0$ which sits in the front of the summation sign. This doesn't explain the $\frac{1}{2}$ but you'll see the reason for that shortly. What about the $n = 0$ sine term? This would contain $\sin 0x = \sin 0 = 0$ so there isn't a lot of point in recording a term which is guaranteed to be zero.

Of course none of this <u>proves</u> that you can expand $f$ in the form given by (1). Is the collection $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$ good enough? (The technical word is "complete"). In fact it is for all "reasonable" $f$ but the proof is well beyond the scope of these notes.

*(The video covers the proof of Theorem 4.1.)* V

## 4.1 Finding the Fourier coefficients

For the time being we shall suppose that $f$ is a suitable function of period $2\pi$ and that it has a representation as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\} \qquad (3)$$

Now, assuming we can integrate the series on the right hand side term-by-term, we can integrate (3) between $-\pi$ and $\pi$ to obtain

$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} 1dx + \sum_{n=1}^{\infty}\left\{a_n \int_{-\pi}^{\pi} \cos nxdx + b_n \int_{-\pi}^{\pi} \sin nxdx\right\}.$$

Since

$$\int_{-\pi}^{\pi} \cos nxdx = 0 \text{ and } \int_{-\pi}^{\pi} \sin nxdx = 0, \qquad n = 1, 2, \ldots$$

we have

$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} 1dx = a_0.\pi$$

So

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Thus we have determined the value of $a_0$. To find the values of the remaining $a_n, n = 1, 2, \ldots$ we multiply both sides of (3) by $\cos mx$, where $m$ is a fixed positive integer, so

$$f(x)\cos mx = \frac{a_0}{2}\cos mx + \sum_{n=1}^{\infty}\{a_n \cos mx \cos nx + b_n \cos mx \sin nx\}$$

and again integrate term-by-term, giving

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos mxdx &= \frac{a_0}{2}\int_{-\pi}^{\pi} \cos mxdx \\ &+ \sum_{n=1}^{\infty}\left\{a_n \int_{-\pi}^{\pi} \cos mx \cos nxdx + b_n \int_{-\pi}^{\pi} \cos mx \sin nxdx\right\}\end{aligned}$$

We saw earlier that

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0, \qquad n = m \text{ and } n \neq m$$

Also, since $\int_{-\pi}^{\pi} \cos mx dx = 0$ for $m = 1, 2, \ldots$, this means that the only non-zero term on the right-hand side of the equation occurs when $n = m$ in the summation. The equation reduces to

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = a_m \int_{-\pi}^{\pi} \cos mx \cos mx dx = a_m \pi.$$

Hence,

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \qquad m = 1, 2, \ldots$$

Finally, to yield the values of $b_n, n = 1, 2, \ldots$ we apply a similar technique by multiplying (3) by $\sin mx$ and integrating term-by-term. This gives

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin mx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \sin mx dx \\ &\quad + \sum_{n=1}^{\infty} \left\{ a_n \int_{-\pi}^{\pi} \sin mx \cos nx dx + b_n \int_{-\pi}^{\pi} \sin mx \sin nx dx \right\} \end{aligned}$$

Since

$$\int_{-\pi}^{\pi} \sin mx dx = 0 \quad \text{and} \int_{-\pi}^{\pi} \sin mx \cos nx dx = 0 \qquad n, m = 1, 2, \ldots$$

and

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

then the only non-zero term in the right hand side of the equation occurs when $n = m$ in the summation. The equation reduces to

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = b_m \int_{-\pi}^{\pi} \sin mx \sin mx dx = b_m \pi.$$

So,

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx \qquad m = 1, 2, \ldots$$

In the original expression for the series (3) the constants had the form $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ whereas we have derived formulae for $a_m$ and $b_m$. Of course, this is simply because $m$ is a dummy variable; we could just as well write the above expressions replacing all occurences of $m$ by $n$. So, to summarise, the Fourier coefficients of $f$ are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad n = 1, 2, \ldots$$

Notice that it has been possible to incorporate the definition for $a_0$ into the definition for the other $a_n$ since, when $n = 0$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 0 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx = a_0$$

This was the reason for the $\frac{1}{2}$ in front of the $a_0$ term in (1). If $a_0$ was not scaled by $\frac{1}{2}$ in (1) then our previous work would have derived the constant term in the series as $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx$ which would not have fitted into the general formula we have derived for the other $\{a_n\}_{n=1}^{\infty}$.

In fact, for any suitable function $f$ we define the **Fourier series of f** to be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad n = 1, 2, \ldots$$

Notice that only information concerning $f$ in $[-\pi, \pi]$ is used to derive $\{a_n\}_0^{\infty}$ and $\{b_n\}_1^{\infty}$. Hence, if $f$ is not $2\pi$-periodic then we would not expect the series to be a good candidate to represent $f$ outside $[-\pi, \pi]$.

If we truncate the summation at $n = m$, giving

$$\frac{1}{2}a_0 + \sum_{n=1}^{m}\{a_n \cos nx + b_n \sin nx\},$$

we obtain the $m^{th}$ **partial sum of the Fourier series of f.** For instance, the 3rd partial sum of the Fourier series of $f$ is

$$\begin{aligned} & \frac{1}{2}a_0 + \sum_{n=1}^{3}\{a_n \cos nx + b_n \sin nx\} \\ = \; & a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x. \end{aligned}$$

The term $a_n \cos nx + b_n \sin nx$ is called the $n^{th}$ **harmonic.**

Remember that in developing the Fourier series for $f$ our approach was to assume that $f$ could be expressed as such a series and to use this assumption in order to determine the Fourier coefficients of $f$. As yet we have not considered the questions of convergence of the series and whether or when it might be reasonable to regard the Fourier series of a function as being equivalent to that function. In fact, there is a wide class of $2\pi$-periodic functions for which the Fourier series of a function in the class will converge, in some sense, to the function itself. We will return to a discussion of convergence later on. First of all we will determine the Fourier series of various functions.

*(The video covers the development of the formulae for the Fourier coefficients.)* V

# 5 Finding the Fourier series of a function

In this section we will find the Fourier series of some particular functions. Some, but not all, of the functions will be $2\pi$-periodic. Of course, as mentioned previously, in all cases the resulting Fourier series will be $2\pi$-periodic. We will look at graphs of the functions and some of their Fourier series partial sums in order to obtain some feel for the nature of the approximation being provided by the partial sums.

**Example 5.1** Let $f(x)$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Determine the Fourier series of $f(x)$.

**Solution**

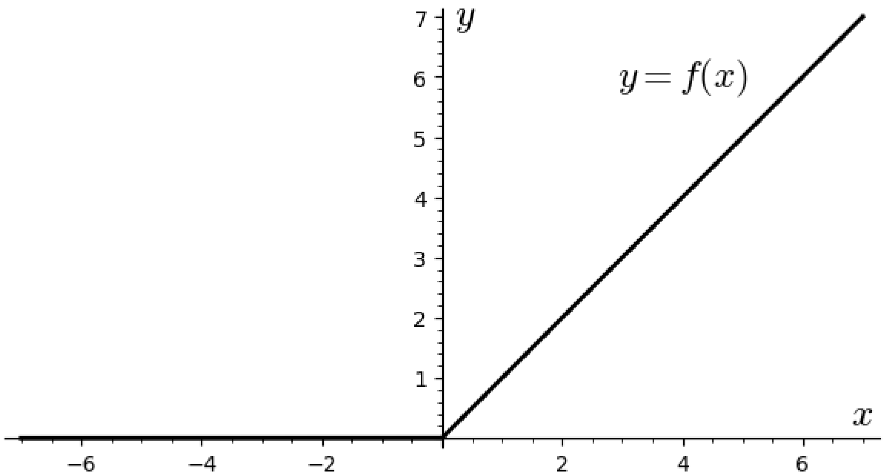



Figure 8: The graph of $f(x)$.

The Fourier series of $f(x)$ is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$, where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} xdx = \frac{\pi}{2}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx \\
&= \frac{1}{\pi}\left\{\left[\frac{x}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx dx\right\} \\
&= \frac{1}{\pi}.\frac{1}{n^2}\left[\cos nx\right]_0^{\pi} = \frac{1}{\pi n^2}\{\cos n\pi - 1\}
\end{aligned}$$

Using the fact that $\cos n\pi = (-1)^n$, this gives

$$a_n = \frac{1}{\pi n^2}\{(-1)^n - 1\}$$

We also have

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx \\
&= \frac{1}{\pi}\left\{\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos nx dx\right\}
\end{aligned}$$

Now, there is no need to "do" the integral $\displaystyle\int_0^{\pi}\frac{1}{n}\cos nx dx$ since it will clearly contain $\sin nx$ which is zero for $x = 0$ and $x = \pi$. (You should be on the lookout to save work in such ways.) Noting this, we have here

$$b_n = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\right]_0^{\pi} = \frac{1}{\pi}\left(-\frac{\pi}{n}\cos n\pi\right) = \frac{(-1)^{n+1}}{n}.$$

Hence the Fourier series of $f(x)$ is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{\pi n^2}\{(-1)^n - 1\}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right\}$$

A point worth noting is that we calculated $a_0$ separately from $a_1, a_2, \ldots;$ this is generally necessary because the integral does not really contain a trigonometric function (on account of $\cos 0x = 1$). The diagram overleaf (figure 9) illustrates the previous example. It shows the graphs of $f(x)$ and the $4^{th}$ and $12^{th}$ partial sums of the Fourier series of $f(x)$.

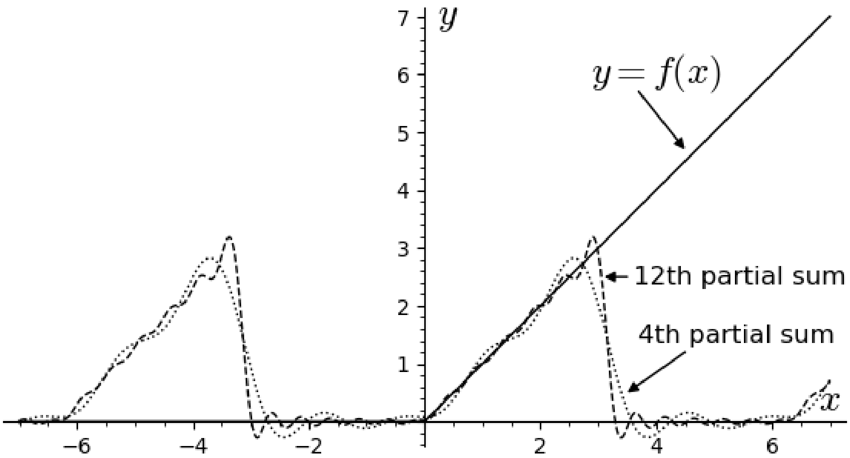



Figure 9: $f(x)$ and the fourth and twelfth partial sums of its Fourier series.

Remember that we find the $m^{th}$ partial sum of the Fourier series of $f(x)$ by truncating the summation at the $n = m$ term. For instance, the $4^{th}$ partial sum of this Fourier series is

$$\frac{\pi}{4} + \sum_{n=1}^{4} \left\{ \frac{1}{\pi n^2}\{(-1)^n - 1\} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right\}$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left(-2\cos x - \frac{2}{9}\cos 3x\right) + \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x$$

The Fourier series of $f(x)$, and all its partial sums, are $2\pi$-periodic. Thus we couldn't expect our partial sums to provide good approximations to $f(x)$ outside $(-\pi, \pi)$. The $12^{th}$ partial sum approximates $f(x)$ better than the $4^{th}$ partial sum between $-\pi$ and $\pi$. As long as the function we are approximating is reasonably well-behaved (we'll explain what we mean by this later in the notes) then we would expect to obtain better approximations to our functions as we increase the number of terms in the partial sums.

We've mentioned that the partial sums are $2\pi$-periodic. So we can only hope to provide a good approximation to a function along the whole $x$-axis if the function is itself $2\pi$-periodic. Let's consider such a function. Let $g(x)$ be the $2\pi$-periodic function defined for $-\pi \leq x < \pi$ by

$$g(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ x & \text{if } 0 \leq x < \pi \end{cases}$$

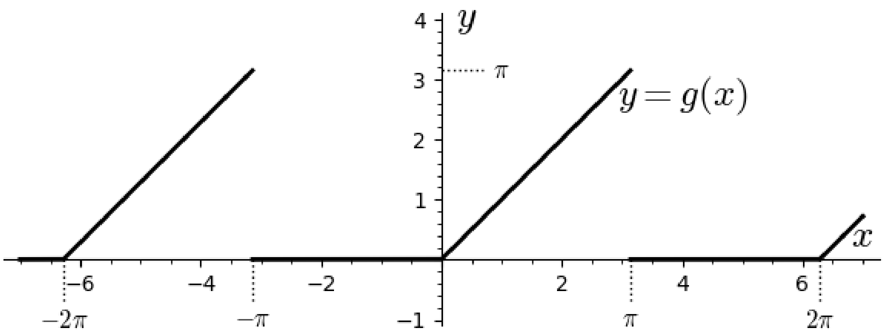



Figure 10: The graph of $g(x)$.

Notice that for $-\pi \leq x < \pi$ the function $g(x)$ and the previous function $f(x)$ are identical. Since the definition of the Fourier coefficients only uses the definition of the function in $(-\pi, \pi)$ the Fourier coefficients, and hence the Fourier series, of $f(x)$ and $g(x)$ are the same. For, the Fourier coefficients of $g(x)$ are:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \end{aligned}$$

The diagram overleaf (figure 11) shows the graph of $g(x)$ and its $4^{th}$ and$12^{th}$ partial sums.

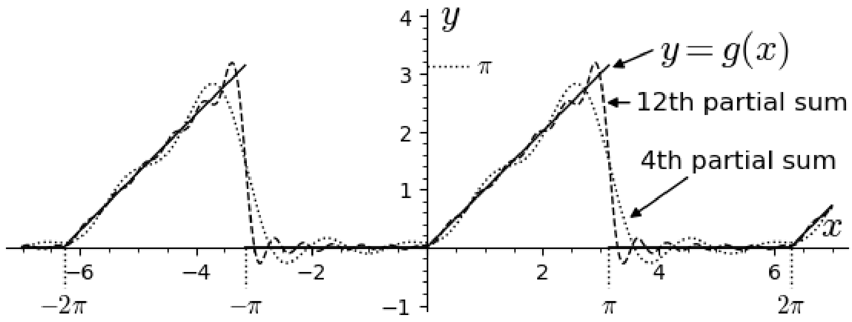



Figure 11: $g(x)$ and the fourth and twelfth partial sums of its Fourier series.

The diagram illustrates the $2\pi$-periodic nature of the partial sums. Of course, our function $g$ is discontinuous at $x = n\pi$, for odd values of $n$. Since the partial sums are finite linear combinations of continuous functions then each partial sum is continuous. So we can hardly expect any of the partial sums to mimic the behaviour of $g$ at its points of discontinuity.

However, for a reasonably well-behaved function it is possible to predict the behaviour of its Fourier series at a discontinuity. Before we discuss this further, we will look at some more examples of finding Fourier series. Our next example involves an odd function. We'll see how the work involved in finding its Fourier series is substantially reduced by using the properties of even and odd functions.

**Example 5.2** Let $f(x)$ be defined by

$$f(x) = x \qquad \text{for all real } x$$

Determine the Fourier series of $f(x)$.

**Solution** The Fourier series of $f(x)$ is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$, where for $n = 0, 1, 2, \ldots,$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = 0$$

because $x \cos nx$, the product of the odd function $x$ and the even function

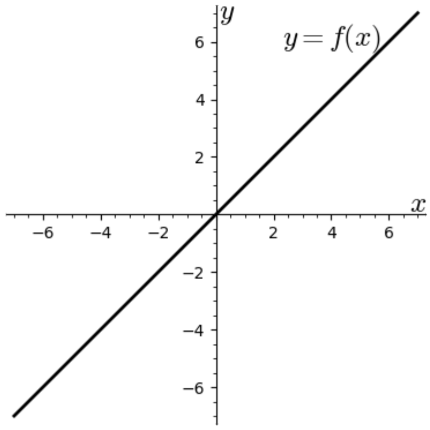



Figure 12: The graph of $f(x)$.

$\cos nx$, is an odd function. The other constants are given by

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx dx \qquad \text{(since } x\sin nx \text{ is an even function)} \\ &= \frac{2}{\pi}\left\{\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos nx dx\right\} \\ &= \frac{2}{\pi}\left\{-\frac{\pi}{n}(-1)^n + 0\right\} = \frac{2}{n}(-1)^{n+1} \end{aligned}$$

Thus the Fourier series of $f(x)$ is

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

It should come as no surprise that the series contains only sine terms. This is because $x$ is an odd function and its series will not contain even functions. The diagram overleaf (figure 13) shows the graph of $f(x)$ and its $4^{th}$ and $12^{th}$ partial sums.

Again, due to the $2\pi$-periodic nature of the partial sums, we don't expect them to approximate $f(x)$ well outside $(-\pi, \pi)$. Increasing the number of terms in the partial sum appears to provide a better approximation to $f(x)$ in $(-\pi, \pi)$.

We'll now find the Fourier series of an even function. Again, the use of the properties of odd and even functions will enable us to reduce the work involved in finding the Fourier coefficients.

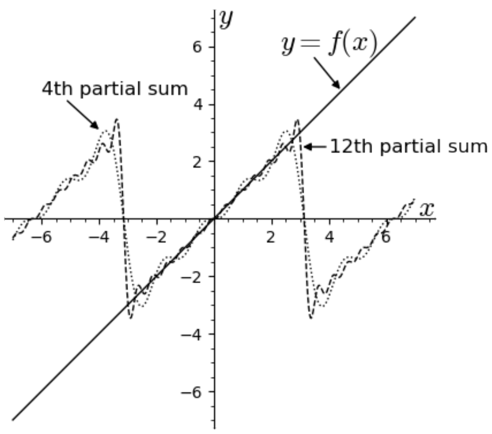



Figure 13: $f(x)$ and the fourth and twelfth partial sums of its Fourier series.

**Example 5.3** Let $f(x)$ be defined by

$$f(x) = x^2 \qquad \text{for all real } x$$

Determine the Fourier series of $f(x)$.

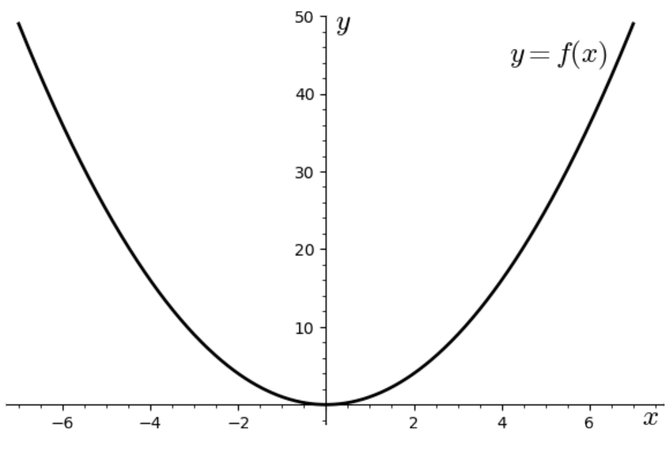



Figure 14: The graph of $f(x)$.

**Solution** The Fourier series of $f(x)$ is $\dfrac{1}{2}a_0 + \displaystyle\sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$, where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx \qquad (\text{since } x^2\cos nx \text{ is an even function}) \\
&= \frac{2}{\pi}\left\{\left[\frac{x^2}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi}\frac{2x}{n}\sin nx dx\right\} \\
&= \frac{2}{\pi}\left\{0 - \frac{2}{n}\int_0^{\pi} x\sin nx dx\right\} \\
&= -\frac{4}{\pi n}\left\{\left[-\frac{x}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos nx}{n} dx\right\} \\
&= -\frac{4}{\pi n}\left\{-\frac{\pi}{n}(-1)^n + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\} = \frac{4}{n^2}(-1)^n
\end{aligned}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx dx = 0$$

since $x^2 \sin nx$ is an odd function. Thus the Fourier series of $f(x)$ is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Note that the series contains only even terms because $x^2$ is an even function. The diagram below (figure 15) shows the graphs of $f$ and its $4^{th}$ partial sum.

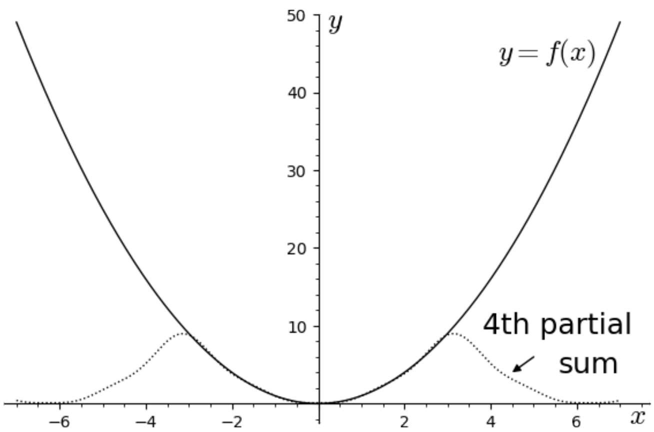



Figure 15: $f(x)$ and the fourth partial sum of its Fourier series.

In the earlier examples above we touched on issues concerning the convergence of Fourier series. In the following section we state conditions under which convergence is assured and deal with the behaviour of a Fourier series at certain types of discontinuity. We will not present any proofs.

*(The video covers all the above examples of finding the Fourier series of a function.* V
*At this point you should try several examples of finding the Fourier series of functions).* EX

# 6 Convergence of Fourier series

## 6.1 Boundedness

Before presenting the main result we require the following definition.

**Definition 6.1** A function $f$ is **bounded** if there is some positive number $M$ such that $|f(x)| \leq M$ for all $x$ in the domain of $f$.

This means that $f$ is bounded if we can find a positive value $M$ such that, if we draw lines parallel to the $x$-axis at height $M$ above the axis and at distance $M$ below the axis, then the graph of $f(x)$ lies entirely between the two lines.

**Example 6.1** The functions $\sin x$ and $\cos x$ are both examples of bounded functions:

$$|\sin x|,\ |\cos x| \leq 1 \text{ for all } x.$$

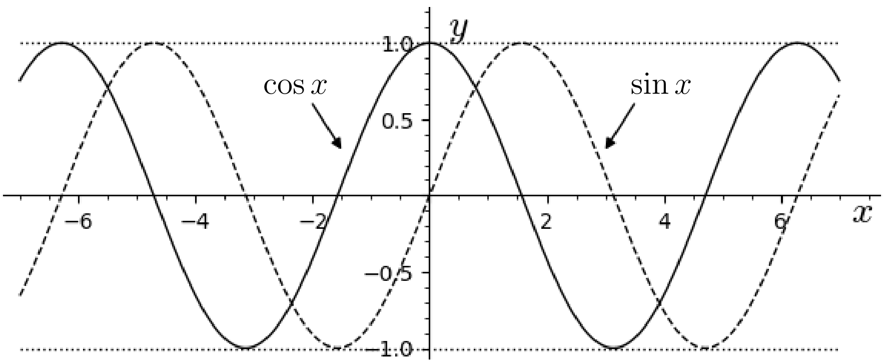



Figure 16: $\sin x$ and $\cos x$ are bounded.

**Example 6.2** The function $f(x) = x$ is not a bounded function. For any positive number $M$ it is always the case that $f(M+1) > M$.

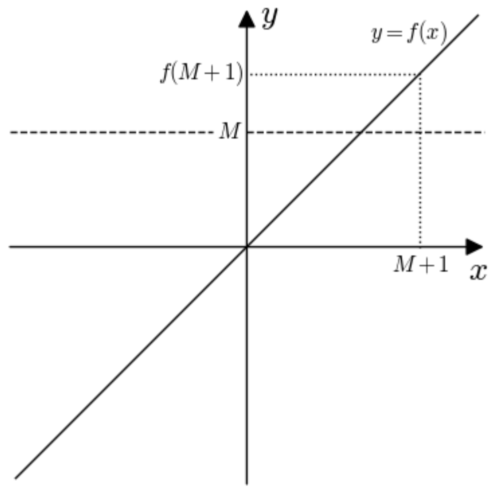



Figure 17: The function $x$ is unbounded.

**Example 6.3** The $2\pi$-periodic function, defined for $-\pi \leq x < \pi$ by $f(x) = x$, is a bounded function. The magnitude of this function is no greater than $\pi$ for $-\pi \leq x < \pi$ and, since the function is $2\pi$-periodic, it follows that $|f(x)| \leq \pi$ for every value of $x$.

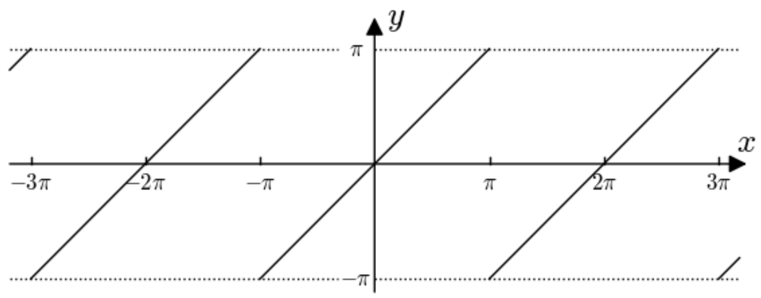



Figure 18: The $2\pi$-periodic function defined as $x$ for $-\pi \leq x < \pi$ is bounded.

## 6.2 Dirichlet's Theorem

We can now provide conditions for convergence of a Fourier series. Notice that although so far we have dealt only with $2\pi$-periodic functions, the theorem allows for the possibility of determining the Fourier series of a periodic function of period other than $2\pi$. We shall return to this matter later in the notes.

**Theorem 6.1. (Dirichlet's Theorem)**

*Let $f$ be a suitable function whose Fourier series has been obtained. If $f$ is a bounded, periodic function which in any one period has at most a finite number of points of discontinuity and a finite number of local maxima and minima, then the Fourier series of $f$ converges to $f(x)$ at all points $x$ at which $f$ is continuous, and converges to the average of the right and left-hand limits of $f(x)$ at each point $x$ at which $f$ is discontinuous.*

The conditions on $f$ specified in the theorem are known as "Dirichlet conditions"; the boundedness of $f$ severely restricts the types of discontinuity which are possible, so much so that such a function will always have left and right-hand limits at each point. The proof of this fact and of the theorem itself is well beyond the scope of these notes. However, in the light of the result we can re-examine the functions whose Fourier series we found previously and draw some definite conclusions concerning convergence.

First of all we will consider the $2\pi$-periodic function defined by

$$f(x) = x^2, \qquad -\pi \leq x < \pi$$

This function is bounded, $2\pi$-periodic, has no discontinuities and has one local maximum and one local minimum in $[-\pi, \pi)$. Thus, according to Dirichlet's Theorem, the Fourier series of $f$ converges to $f(x)$, for all $x$.

We'll see how Dirichlet's Theorem applies to a function with discontinuities by considering the $2\pi$-periodic function defined by

$$g(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ x & \text{if } 0 \leq x < \pi \end{cases}$$

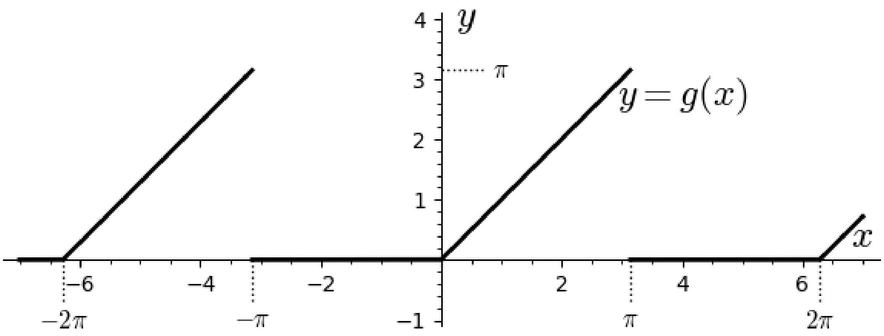



Figure 19: The graph of $g(x)$.

This function is bounded, $2\pi$-periodic, has no local maxima or local minima and has discontinuities at $x = (2n+1)\pi$, for each integer $n$. According to Dirichlet's Theorem, the Fourier series of $g$ converges to

$$\begin{array}{ll} g(x) & \text{for} \quad x \neq (2n+1)\pi \\ \dfrac{\pi}{2} & \text{for} \quad x = (2n+1)\pi \end{array}$$

for each integer $n$. Note the distinction between $g(x)$ and its Fourier series at the points of discontinuity. The Fourier series of $g$ has the graph shown below (figure 20).

If we look at a picture of $g$ and some of its partial sums we can see how the convergence of the Fourier series of $g$ at the discontinuities starts to manifest itself (figure 21).

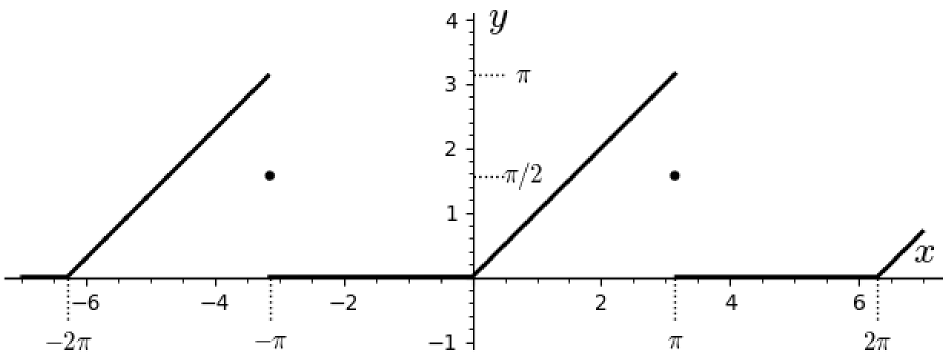



Figure 20: The graph of the Fourier series of $g(x)$.

It is beyond the scope of this discussion to provide the background to the derivation of the Dirichlet conditions or the proof of the theorem. However,

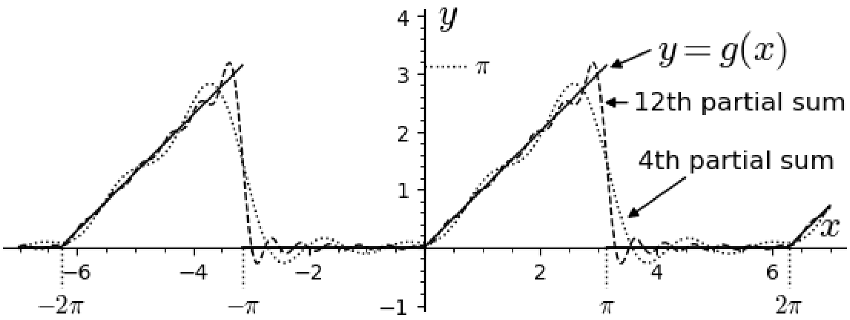



Figure 21: $g(x)$ and the fourth and twelfth partial sums of its Fourier series.

it is important to realise that you should check that the conditions are satisfied by a function before you assume convergence of its Fourier series. For instance, an example does exist of a $2\pi$-periodic function, which does not satisfy all the Dirichlet conditions, whose Fourier series diverges at a point.

Note that if $f$ satisfies the Dirichlet conditions then, if we alter its value at a single point, the modified function will continue to satisfy the Dirichlet conditions and the Fourier series is unaltered. For this reason it is irrelevant whether we require a $2\pi$-periodic function to be defined for $-\pi \leq x < \pi$, or for $-\pi < x < \pi$, or for $-\pi \leq x \leq \pi$, etc.

*(The video covers the definition of bounded functions, Dirichlet's Theorem and its application.* V
*At this point you should try examples concerning bounded functions and the convergence of the Fourier series of a function).* EX

## 6.3 Summation of series using Fourier series

Certain arithmetic series may be summed by substituting an appropriate value of $x$ into a convergent Fourier series. For instance, we showed that the Fourier series of the $2\pi$-periodic function defined by

$$g(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ x & \text{if } 0 \leq x < \pi \end{cases}$$

is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\pi n^2}\{(-1)^n - 1\} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right\}$$

When $x = \pi$, $g$ is discontinuous and its Fourier series converges to $\dfrac{\pi}{2}$. Thus

$$\begin{aligned} \frac{\pi}{2} &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\pi n^2}\{(-1)^n - 1\} \cos n\pi + \frac{(-1)^{n+1}}{n} \sin n\pi \right\} \\ &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\pi n^2}\{(-1)^n - 1\}(-1)^n \right\} \\ &= \frac{\pi}{4} + \frac{2}{\pi} \left\{ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right\} \end{aligned}$$

From which we can conclude that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

*(The video covers the above example of summation of a series.* V
*At this point you should try examples concerning summation of series).* EX

# 7 Fourier series for even and odd functions

Suppose $f$ is an even function. Then $b_n$, the coefficient of $\sin nx$ in the Fourier series of $f(x)$, is

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0, \qquad n = 1, 2, 3, \ldots,$$

since the integrand is an odd function. Thus the Fourier series of an even function $f(x)$ has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

since $f(x)\cos nx$ is an even function.

The $2\pi$-periodic function, defined for $-\pi \leq x < \pi$ by $f(x) = x^2$ illustrates this situation. Its Fourier series, which we derived in section 5, has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

and for $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx = \frac{4(-1)^n}{n^2}$$

Similarly, if $f$ is an odd function then $a_n$, the coefficient of $\cos nx$ in the Fourier series of $f$, is

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0, \qquad n = 0, 1, 2, \ldots,$$

since the integrand is an odd function. Thus the Fourier series of an odd function $f$ has the form

$$\sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

since $f(x)\sin nx$ is an even function. So considerable work can be saved in evaluating Fourier series if you notice that $f$ is either even or odd before calculating coefficients.

*(The video covers Fourier Series of odd and even functions.* V
*At this point you should try examples of finding the Fourier series of odd and even functions).* EX

# 8 Fourier sine and cosine series

Suppose that $f$ is a suitable function which is defined in $[0, \pi]$ and that we wish to find a trigonometric series expansion for $f$. By extending the definition of $f$, so that it is defined over $[-\pi, \pi]$, it is possible to generate the Fourier series for $f$. Clearly, how we choose to define $f$ over $[-\pi, \pi]$ will affect the form of the series generated from $f$. In particular, extending $f$ to be an even function in $[-\pi, \pi]$ will mean that $f$ can be expanded in a cosine series; and extending $f$ as an odd function in $[-\pi, \pi]$ will result in a sine series expansion for $f$.

## 8.1 Fourier cosine series

Given a function $f$ defined in $[0, \pi]$ we can extend $f$ to become an even function, $g$ say, by letting

$$g(x) = \begin{cases} f(x) & \text{for } 0 \le x \le \pi \\ f(-x) & \text{for } -\pi \le x < 0 \end{cases}$$

Since $g$ is an even function its Fourier series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

The coefficients are given by

$$a_n = \frac{2}{\pi} \int_0^{\pi} g(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

for $n = 0, 1, 2, \ldots$. Since $f(x) = g(x)$ for $0 \le x \le \pi$ then $f$ has the **Fourier cosine series** on the interval $[0, \pi]$:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \qquad n = 0, 1, 2, \ldots$$

Providing $g$ satisfies the Dirichlet conditions then this cosine series will converge as specified by Dirichlet's theorem.

**Example 8.1** Let $f$ be defined for $0 \le x \le \pi$ by $f(x) = x + 2$. Find its Fourier cosine series on the interval $[0, \pi]$

**Solution** To find the Fourier cosine series of $f$ we extend $f$ to an even function. The previous discussion gives the Fourier cosine series of $f$ as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{4}$$

where

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}(x+2)dx \\ &= \frac{2}{\pi}\left[\frac{x^2}{2} + 2x\right]_0^{\pi} = \frac{2}{\pi}\left\{\frac{\pi^2}{2} + 2\pi\right\} = \pi + 4 \end{aligned}$$

and, for $n = 1, 2, \ldots$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi}(x+2)\cos nx dx \\ &= \frac{2}{\pi}\left\{\left[\frac{(x+2)}{n}\sin nx\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx dx\right\} \\ &= \frac{2}{\pi n^2}\left[\cos nx\right]_0^{\pi} = \frac{2}{\pi n^2}\{(-1)^n - 1\} = \left\{\begin{array}{ll} 0 & \text{for even } n \\ -\frac{4}{\pi n^2} & \text{for odd } n \end{array}\right. \end{aligned}$$

We can include this information in formula (4). Notice that every odd integer $n$ can be expressed as $2m+1$, where $m = \frac{1}{2}(n-1)$. If we replace every odd value of $n$ in (4) by $2m+1$ and sum over $m$ from 0 to $\infty$, instead of $n$ from 1 to $\infty$, then all the terms which corresponded to odd values of $n$ in the original formula are retained but those which corresponded to the even values of $n$ disappear. For the odd values of $n$ we have

$$a_n = -\frac{4}{\pi n^2} = -\frac{4}{\pi(2m+1)^2}.$$

This means we can express the Fourier series of $f$ as

$$\frac{\pi}{2} + 2 - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}$$

The Fourier cosine series of $f$ is

$$\frac{\pi}{2} + 2 - \frac{4}{\pi}\left\{\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right\}$$

Below (figure 22) is a graph of the function to which this series converges. Notice that this graph coincides with that of $y = x + 2$ on the interval $[0, \pi]$ but not elsewhere.

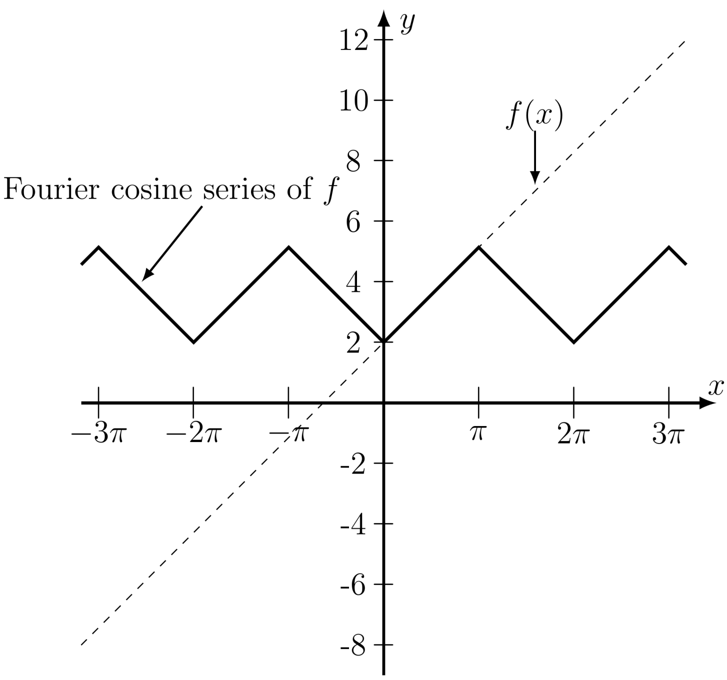



Figure 22: The function $f(x) = x + 2$ and its cosine series on $[0, \pi]$.

## 8.2 Fourier sine series

Given a function $f$ defined for $[0, \pi]$ we can extend $f$ to become an odd function, $g$ say, by letting

$$g(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq \pi \\ -f(-x) & \text{for } -\pi \leq x < 0 \end{cases}$$

Since $g$ is an odd function its Fourier series has the form

$$\sum_{n=1}^{\infty} b_n \sin nx.$$

The coefficients are given by

$$b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin nx dx = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx$$

for $n = 1, 2, \ldots$. Since $f(x) = g(x)$ for $x$ satisfying $0 \le x \le \pi$ then $f$ has the **Fourier sine series** on the interval $[0, \pi]$:

$$\sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx, \qquad n = 1, 2, \ldots$$

Providing $g$ satisfies the Dirichlet conditions then this sine series will converge as specified by Dirichlet's theorem.

For comparison purposes we will find the Fourier sine series of the function whose Fourier cosine series was determined in the previous example.

**Example 8.2** Let $f$ be defined for $0 \le x \le \pi$ by $f(x) = x + 2$. Find its Fourier sine series on the interval $[0, \pi]$

**Solution** To find the Fourier sine series of $f$ we extend $f$ to an odd function. The previous discussion gives the Fourier sine series of $f$ as

$$\sum_{n=1}^{\infty} b_n \sin nx$$

where, for $n = 1, 2, \ldots$,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx = \frac{2}{\pi}\int_0^\pi (x+2)\sin nx dx \\ &= \frac{2}{\pi}\left\{\left[-\frac{(x+2)}{n}\cos nx\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos nx dx\right\} \\ &= -\frac{2}{n\pi}\left\{(\pi+2)(-1)^n - 2\right\} + 0 \\ &= \frac{2}{n\pi}\left\{2 - (\pi+2)(-1)^n\right\} = \left\{ \begin{array}{ll} -\frac{2}{n} & \text{for even } n \\ \frac{2(4+\pi)}{n\pi} & \text{for odd } n \end{array} \right. \end{aligned}$$

The Fourier sine series of $f$ is

$$
\begin{aligned}
&= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{2-(\pi+2)(-1)^n\}\sin nx \\
&= \frac{2(4+\pi)}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right\} \\
&\qquad - 2\left\{\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \dots\right\}
\end{aligned}
$$

Below (figure 23) is a graph of the function to which this series converges. This graph coincides with that of $y = x+2$ on the interval $(0,\pi)$ but not elsewhere. Note the behaviour of the series at $x = \pm\pi, \pm 2\pi, \dots$, where it converges to the value zero.

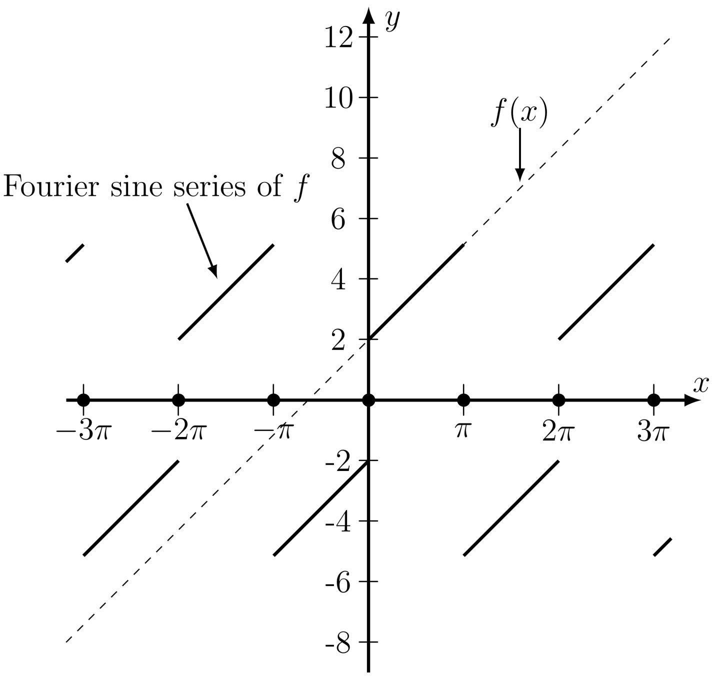



Figure 23: The function $f(x) = x+2$ and its sine series on $[0,\pi]$.

*(The video discusses Fourier cosine and sine series and covers the above examples.* V
*At this point you should try examples of finding the Fourier cosine and sine series of various functions.)* EX

# 9 Fourier series of period $2l$

*(This section is not covered on the videos.)*

In this section we extend the ideas discussed so far to apply to finding Fourier series which are $2l$-periodic, $l > 0$. This would mean that any suitable periodic function which satisfies the Dirichlet conditions could be represented by its Fourier series of period $2l$ (with the usual rules governing convergence at its discontinuities).

## 9.1 Fourier series for functions defined over $-l \leq x < l$

Suppose $f$ is a suitable function defined over $-l \leq x < l$ and that we wish to find the Fourier series of $f$. In order to do so we use a transformation which enables us to treat $f$ in terms of a function $g$ defined over $[-\pi, \pi)$, and hence apply the previous theory. We let

$$t = \frac{\pi . x}{l},$$

then, when $-l \leq x < l$ we have $-\pi \leq t < \pi$, and

$$f(x) = f\left(\frac{l.t}{\pi}\right) = g(t), \qquad \text{say.}$$

Now $g$ is defined in $-\pi \leq t < \pi$ and hence has Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nt + b_n \sin nt\}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt dt \qquad n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin nt dt \qquad n = 1, 2, \ldots$$

Substituting back for $t$ as $\dfrac{\pi . x}{l}$, and noting that $\dfrac{dt}{dx} = \dfrac{\pi}{l}$, then this means that the Fourier series of $f$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\},$$

where

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right).\frac{\pi}{l}dx \\
&= \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \qquad n = 0, 1, 2, \ldots
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right).\frac{\pi}{l}dx \\
&= \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad n = 1, 2, \ldots
\end{aligned}$$

**Example 9.1** Determine the Fourier series of period 4 for the 4-periodic function $f$ defined for $-2 \leq x < 2$ by

$$f(x) = \begin{cases} 0 & \text{if } -2 \leq x < 0 \\ 1 & \text{if } 0 \leq x < 2 \end{cases}$$

**Solution** The required Fourier series has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\},$$

where

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{0}^{2} 1dx = 1.$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{0}^{2} 1.\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}.\frac{2}{n\pi}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 = 0.
\end{aligned}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{0}^{2} 1.\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}.\left(\frac{-2}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 = \frac{-1}{n\pi}\{\cos n\pi - \cos 0\} \\
&= \frac{1}{n\pi}\{1 - (-1)^n\}.
\end{aligned}$$

Thus the Fourier series of $f$ of period 4 is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\{1 - (-1)^n\} \sin\left(\frac{n\pi x}{2}\right). \tag{5}$$

Notice that $\quad \frac{1}{n\pi}\{1 - (-1)^n\} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$

We can include this information in formula (5). Notice that every odd integer $n$ can be expressed as $2m+1$, where $m = \frac{1}{2}(n-1)$. If we replace every odd value of $n$ in (5) by $2m+1$ and sum over $m$ from 0 to $\infty$, instead of $n$ from 1 to $\infty$, then all the terms which corresponded to odd values of $n$ in the original formula are retained but those which corresponded to the even values of $n$ disappear. For the odd values of $n$ we have $b_n = \dfrac{2}{n\pi} = \dfrac{2}{(2m+1)\pi}$. This means we can express the Fourier series of $f$ as

$$\frac{1}{2} + \sum_{m=0}^{\infty} \frac{2}{(2m+1)\pi} \sin\left(\frac{(2m+1)\pi x}{2}\right).$$

Notice that $f$ satisfies the Dirichlet conditions since $f$ is bounded, 4-periodic, has a finite number of discontinuities and no local maxima or minima in $[-2, 2]$. The discontinuities of $f$ occur at $x = 2n$, for any integer $n$. For all such points the average of the right and left hand limits of $f(x)$ is $\dfrac{1}{2}$. Applying Dirichlet's Theorem we know that the Fourier series of $f$ converges to

$$\begin{array}{ll} f(x) & \text{for} \quad x \neq 2n \\ \frac{1}{2} & \text{for} \quad x = 2n \end{array}$$

for each integer $n$. Note the distinction between $f(x)$ and its Fourier series at the points of discontinuity. The Fourier series of $f$ has the graph shown below (figure 24).

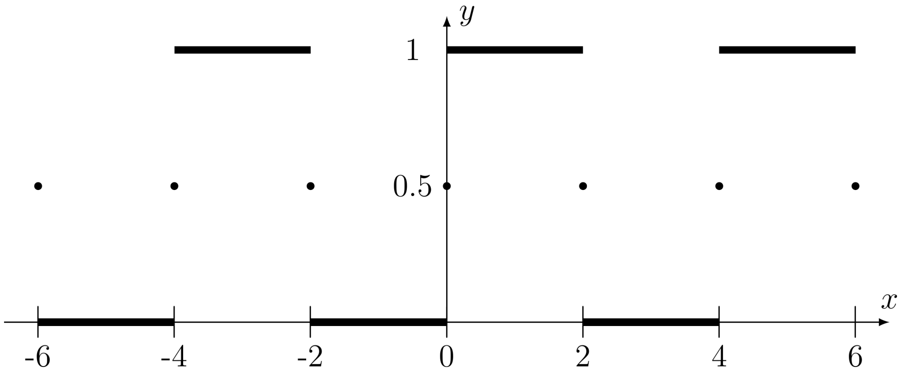



Figure 24: The Fourier series of $f(x)$.

*(At this point you should try an example of finding the Fourier series of a $2l$-periodic function).* EX

## 9.2 Fourier sine and cosine series for functions defined over $0 \leq x \leq l$

For a suitable function defined in $0 \leq x \leq l$ we may use the same transformation, $t = \dfrac{\pi.x}{l}$, that was applied to functions defined in $-l \leq x < l$. Then when $0 \leq x \leq l$ we have $0 \leq t \leq \pi$, and

$$f(x) = f\left(\frac{l.t}{\pi}\right) = g(t), \qquad \text{say.}$$

Since $g$ is defined over $0 \leq t < \pi$ then we may use the techniques of section 8 to find both its Fourier sine series and its Fourier cosine series.

The Fourier cosine series of $g$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(t)\cos nt dt \qquad n = 0, 1, 2, \ldots$$

which means that the Fourier cosine series of $f$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right),$$

where

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right).\frac{\pi}{l}dx \\ &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \qquad n = 0, 1, 2, \ldots \end{aligned}$$

The Fourier sine series of $g$ is

$$\sum_{n=1}^{\infty} b_n \sin nt$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(t)\sin ntdt \qquad n = 1, 2, \ldots$$

which means that the Fourier cosine series of $f$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right).\frac{\pi}{l}dx \\ &= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad n = 1, 2, \ldots \end{aligned}$$

*(At this point you should try examples of finding Fourier cosine and sine series of a $2l$-periodic function).* EX

# 10 Summary

When you have completed this package you should be able to do the things listed below.

1. recognize **odd, even and periodic** functions,
2. understand what is meant by the **Fourier series** of a function,
3. understand what is meant by the **Fourier coefficients** of a function,
4. obtain the **Fourier coefficients** of a function,
5. obtain the **Fourier series of period** $2\pi$ of a function and obtain the $m^{th}$ **partial sum** of this Fourier series,
6. use **the properties of odd or even functions** to reduce the work in finding Fourier coefficients,
7. understand **Dirichlet's Theorem** concerning the convergence of Fourier series,
8. determine whether a function satisfies the **Dirichlet conditions**,
9. apply Dirichlet's Theorem to **sketching the graph of the function to which a Fourier series converges,**
10. apply Dirichlet's Theorem to **summation of series**,
11. obtain the **Fourier sine or cosine series of period** $2\pi$ of a function,
12. understand how to **generalise the previous concepts** to find the **Fourier series, sine or cosine series of period** $2l$ of a function,

# 11 Bibliography

For textbooks that cover the basic prerequisites for this package (differentiation and integration of functions of one variable) see, for example,


**Anton, H.** Calculus with Analytic Geometry (fourth edition), John Wiley, 1992.

**Larson, R. E., Hostetler, R. P., Edwards, B. H.** Calculus (fourth edition), D. C. Heath and Co., 1990.

**Stroud, K. A.** Engineering Mathematics (third edition), Macmillan, 1992.


The following books provide a treatment of the material on Fourier series covered in the videos and notes:


**Gowar, N. W.** Fourier Series, Mathematics for Science and Technology.

**Kreyszig, E.** Advanced Engineering Mathematics (third edition), Wiley International.

**Sanchez, D. A., Allen, R. C., Kyner, W. T.** Differential Equations (second edition), Addison Wesley.

# 12 Appendix - Video Summaries

There are three videos associated with the topic Fourier Series. The presenters are Julie Halton and Mike Grannell from the Department of Mathematics and Statistics at the University of Central Lancashire. We recommend that you read the preamble to these notes which makes some suggestions about how you should approach viewing the videos.

**Video title: Fourier Series (part 1).** (33 minutes)

**Summary**

1. Review of the idea of approximation of a function.
2. Introduction to the idea of approximation at $x$ of a suitable function $f$ by a series of trigonometric functions of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\},$$

   known as the **Fourier series** of the function.

3. **Odd and even functions:** definition and examples of odd and even functions. Properties of odd and even functions.
4. **Periodic functions:** definition and examples of periodic functions. Properties of periodic functions.
5. Some useful preliminaries: the results
   - $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0.$
   - $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$
   - $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$, for integers $m$ and $n$.
6. Finding the **Fourier series** and the **Fourier coefficients** of a suitable function $f$: derivation of the Fourier series at $x$ of a suitable function $f$ as

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad n = 1, 2, \ldots$$

7. Definition of the $m^{th}$ **partial sum of the Fourier series** of $f$.

**Video title: Fourier Series (part 2).** (33 minutes)

**Summary**

1. Recap of the definition of the Fourier series and the Fourier coefficients of a suitable function $f$.

2. Derivation of the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

   Derivation of the $4^{th}$ partial sum of the Fourier series of $f$.

   Comparison of the graphs of $f$ and the graphs of the $4^{th}$ and $12^{th}$ partial sums of the Fourier series of $f$.

   Derivation of the Fourier series of the $2\pi$-periodic function $g$ defined for $-\pi \leq x < \pi$ by

$$g(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ x & \text{if } 0 \leq x < \pi \end{cases}$$

   Relationship between the Fourier coefficients of $g$ and those of the function $f$.

   Comparison of the graphs of $g$ and the graphs of the $4^{th}$ and $12^{th}$ partial sums of the Fourier series of $g$.

3. Derivation of the Fourier series of the function

$$f(x) = x \qquad \text{for all real } x$$

   The consequences of $f$ being an odd function on the form of the Fourier series of $f$.

   Comparison of the graphs of $f$ and the graphs of the $4^{th}$ and $12^{th}$ partial sums of the Fourier series of $f$.

4. Derivation of the Fourier series of the function

$$f(x) = x^2 \qquad \text{for all real } x$$

   The consequences of $f$ being an even function on the form of the Fourier series of $f$.

   Comparison of the graphs of $f$ and the graph of the $4^{th}$ partial sum of the Fourier series of $f$.

**Video title: Fourier Series (part 3).** (33 minutes)

**Summary**

1. **Bounded functions:** definition and examples of bounded functions.
2. **Dirichlet's Theorem** concerning the convergence of Fourier series: statement of the theorem and examples of its application in sketching the Fourier series of a function.
3. **Summation of series** using Dirichlet's Theorem.
4. Recap of the properties of Fourier series of odd or even functions.
5. Definition of the **Fourier sine series and the Fourier cosine series of a suitable function $f$ defined for $0 \leq x \leq \pi$.**

   Derivation of the Fourier sine series and the Fourier cosine series of the function $f$ defined for $0 \leq x \leq \pi$ by

$$f(x) = x + 2.$$

   Comparison of the graph of $f$ and the graphs of its Fourier sine series and its Fourier cosine series.