FIRST ORDER DIFFERENTIAL EQUATIONS

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Second Edition, March 1994.
(Reprinted 2022)

[1]The authors were supported by Enterprise funding.

Contents

1 Preamble

1.1 About this package

This package is for people who need to solve first order differential equations. It is an introductory package and not an encyclopaedia. It covers relatively easy types of equations and some basic numerical techniques. It doesn't contain a lot of theory. It isn't really designed for pure mathematicians who require a course discussing existence and uniqueness of solutions.

You will find that you need a background knowledge of differentiation and integration in order to get the most out of this package. In particular, you need to be able to differentiate using the product, quotient and function-of-a-function rules. You also need to be able to carry out integrations by simple substitutions, by parts, and using partial fractions. If you are a bit rusty, don't worry - but it would be sensible to do some revision either at the start or as the need arises. Reasonable revision texts are given in the bibliography (Section 13).

If you complete the whole package you should be able to

- recognise a **first order differential equation,**
- sketch the **isoclines** and **direction field** for a first order differential equation,
- obtain an approximate numerical solution using **Euler's method**,
- recognise **separable** and **linear** equations,
- **solve separable equations** by separating the variables,
- **solve linear equations** by computing an appropriate integrating factor,
- solve some other types of first order equations by reducing them to either separable or linear form,
- understand how first order equations arise from **one-parameter families of curves**,
- explain how first order equations can be used to **model birth/death processes**.

Depending on your own programme of study you may not need to cover everything in this package. Your tutor will advise you what, if anything, can be omitted.

1.2 How to use this package

You **MUST** do examples! Doing lots of examples for yourself is generally the most effective way of learning the contents of this package and covering the objectives listed above. The introductory sections which cover basic terminology, notation and background material may only need to be scanned briefly. (How about investing in a highlighter pen?) But the remaining sections need to be tackled in a different manner:

- read the theory - make your own notes where appropriate,
- work through the worked examples - compare your solutions with the ones in the notes,
- do similar examples yourself in a workbook.

The original printing of these notes leaves every other page blank. Use the spare space for your own comments, notes and solutions. You will see certain symbols appearing in the right hand margin from time to time:

◯ denotes the end of a worked example,

□ denotes the end of a proof,

V denotes a reference to videos (see the next subsection for details),

EX highlights a point in the notes where you should try examples.

By the time you have reached a package like this one you will probably have realised that learning mathematics rarely goes smoothly! When you get stuck, use your accumulated wisdom and cunning to get around the problem. You might try:

- re-reading the theory/worked examples,
- putting it down and coming back to it later,
- reading ahead to see if subsequent material sheds any light,
- talking to a fellow student,
- looking in a textbook (see the bibliography).

These notes are in fact only a part of a larger support provision. The notes are intended to be the primary teaching resource but they are backed-up by

- video lectures and examples,
- tutorials.

If you are fortunate you may find that the notes provide all the assistance you need. However, many people find that watching someone work through the theory and examples is extremely useful. This is where the videos come in. You may also find that some particular point is impeding your progress or causing you undue difficulty. If these notes and the videos do not help then you should be able to get the issue resolved via the tutorial provision.

1.3 How to use the videos

The videos have been designed to cover the main points in the notes. They don't cover everything. The areas covered are indicated in the notes, usually at the ends of sections and subsections. If you can easily follow the notes then there is little point in viewing the videos. The videos do not introduce any new material. If, however, there is some point in the notes where you get stuck, then look at the appropriate point in the video. You may not need to watch a whole video (most of them are about 30 minutes long). The videos are broken up into sections prefaced with titles which can be read on fast scan. In addition, a summary of the videos associated with this package appears as an appendix to these notes.

Your tutor will tell you about the arrangements for viewing the videos. Because of problems with theft of materials and equipment, there are likely to be restrictions on viewing and loan facilities. The videos were designed to be viewed individually or in very small groups (so that **you** can target the parts which **you** need).

Viewing the videos, like reading the notes, is not just a spectator sport. You should have a pen and paper handy and be prepared to use them. Stop the video from time to time. Try the worked examples **before** watching the solutions unfold on the screen. Make notes of any points you cannot follow so that you can explain the difficulty in a subsequent tutorial session. Remember the rewind button! Unlike a lecture you can get instant and 100 percent accurate replay of what was said.

You may find the videos useful for revision purposes towards the end of your course. Again, you are likely to find that scanning the tape for the highlights will prove more effective than dozing off for 30 minutes in front of the television!

1.4 How to use the tutorials

Your tutor will tell you about tutorial arrangements. These may be related to assessment arrangements. If attendance at tutorials is compulsory then make sure you know the details!

Leaving aside the assessment component which may vary from course to course and from year to year, the tutorials provide you with individual contact with a tutor. Use this time wisely - staff time is the most expensive of all our resources.

You should come to tutorials in a prepared state. This means that you should have read (re-read many times if necessary) the notes and the worked examples. You should have tried appropriate examples for yourself. If you have had difficulty with a particular section then you should watch the corresponding video.

It will help your tutor enormously if you can make any queries as specific as possible:

> "I don't understand example 7 in subsection 4.3"

is infinitely preferable to

> "I had some trouble with section 4."

Your tutor may, quite properly, refuse to help you if you haven't tried to help yourself. If he/she finds that you haven't read the notes, tried examples or looked at the videos then you may be told to do these things before any individual help is offered. It seems to be the nature of things that the people who are most conscientious about self-help are the ones most embarassed about asking tutors questions. You can demonstrate that you have tried self-help by referring to specific points.

1.5 Other forms of self-help

Your fellow students are an excellent form of self-help. Discuss problems with one another and compare solutions. Just be careful that

1. any assesed coursework submitted by you is yours alone,
2. you yourself do really understand solutions worked out jointly with colleagues.

Familiarize yourself with the layout and contents of these notes; scan them before reading them more carefully. The contents page will help you find your way about - use it.

These notes also contain a bibliography. This references books which cover specific points and more general textbooks which cover whole sections of material, sometimes in greater depth and giving a slightly different approach. Don't be afraid of textbooks; bear in mind that they can't usually be read like novels. Go straight to the section of immediate interest and work outwards. In addition to the texts mentioned in the bibliography, the University library contains dozens of texts which cover the material in this package. You need never be short of an alternative approach or more questions to try!

1.6 The hidden agenda!

If you can learn mathematics from this package and from textbooks then you will not only have learnt a particular mathematical topic. You will also (and more importantly) have learnt **how to learn** mathematics.

2 Introduction

A differential equation is an equation connecting one or more derivatives of an unknown function $y = y(x)$. The equation may involve explicit references to y and x themselves but must include at least one of the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$. The following are examples of differential equations:

1. $\dfrac{dy}{dx} = x^3$

2. $x^2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = \sin x$

3. $\sin(\dfrac{d^2y}{dx^2}) + \cos(yx^2) = \dfrac{1}{1+y^2}$

The principal objective will be to solve differential equations. This means that we would like to obtain y in terms of x. In many cases it is downright impossible to obtain a nice tidy formula for y. Sometimes, even though such a formula may not exist, we can obtain an implicit formula. [By an implicit formula we mean one involving x and y which cannot be manipulated to a form where y is the subject of the formula - an example is $y + x = \sin(xy)$.] For most equations which arise in practice it is possible to obtain numerical solutions, i.e. for each given x we can obtain a numerical approximation to the corresponding value of y.

Our three examples of differential equations given above all contain just two variables x and y. Sometimes x will be referred to as the independent variable and y as the dependent variable. You can tell which is which by examining the derivatives:

$$\frac{dy}{dx} : \begin{cases} y & \text{(on the top) is the dependent variable} \\ x & \text{(on the bottom) is the independent variable} \end{cases}$$

Differential equations like these are sometimes called **ordinary** differential equations to distinguish them from **partial** differential equations which have two or more independent variables as in:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y^2$$

You will be relieved to know that we won't be considering partial differential equations here. In fact, even the ordinary differential equations we shall look at will be fairly simple - they will all be **first-order** equations.

(The video gives examples of differential equations.) V

3 The order of a differential equation

A differential equation such as

$$(\frac{d^2y}{dx^2})^3 = \sin((\frac{dy}{dx})^7 + y^5)$$

is said to be of **second-order** since the highest order derivative which appears is the second derivative $\frac{d^2y}{dx^2}$

In general, an equation is described as nth order if the highest order derivative which appears is the nth derivative.

Let us consider a very simple first order equation:

$$\frac{dy}{dx} = x^3$$

Integrating with respect to x we obtain

$$y = \int x^3 dx = \frac{x^4}{4} + c$$

where c is a constant. Thus the original equation has solution

$$y = \frac{x^4}{4} + c$$

In rough and ready terms, the appearance of the constant c in the solution is tied up with the order of the equation:

$$\text{first order} \Rightarrow \text{one integration} \Rightarrow \text{one constant.}$$

For a second order equation we might reasonably expect to do two integrations (to eliminate $\frac{d^2y}{dx^2}$) and consequently obtain a solution with two independent arbitrary constants.

For equations which arise in practice it is normally the case that an nth order equation will give rise to a solution containing n independent arbitrary constants. Additional information (such as the values of y and $\frac{dy}{dx}$ when $x = 0$) may enable us to evaluate these constants.

The equations that we shall consider in this module will be first order equations and all of them (except the one just below!) will give rise to solutions containing a single arbitrary constant.

To conclude this section we give an example of a badly-behaved differential equation which is first order but whose solution contains no arbitrary constants:

$$(\frac{dy}{dx})^2 + y^2 = 0$$

The only solution of this (for real values of y) is $y = 0$. None of the equations we consider below will behave like this.

(The video discusses the order of a differential equation and the number of constants in a solution.) V

4 Solutions

The most general form of a first order differential equation is

$$G(x, y, y') = 0 \qquad \text{(where } y' \text{ denotes } \tfrac{dy}{dx}\text{)}$$

Here G is some function of three variables. We will only consider cases where y' can be expressed in terms of x and y. Consequently we take our general form as

$$\boxed{y' = F(x, y)}$$

where F is some function of two variables. An example is

$$\frac{dy}{dx} = \sin(x + y)$$

There are results which guarantee the existence and the uniqueness of solutions to the general form given above. These results prove that under certain conditions on $F(x, y)$, the equation has a solution y and that this solution is unique (up to an arbitrary constant). An example of such a result is Picard's theorem (see the bibliography for references). In the cases which we consider there will be few (if any) problems with existence and uniqueness. However, just to show that it is easy to write down equations with no solutions consider:

$$(\frac{dy}{dx})^2 + 1 = 0$$

This has no solution (for real values of y).

A further complication is the range of validity of a solution. An equation such as:

$$\frac{dy}{dx} = \sqrt{x}$$

is hardly likely to have a solution valid for $x < 0$ (because negative numbers do not have real square roots). The solution

$$y = \frac{2}{3}x^{\frac{3}{2}} + c$$

obtained by integration is only valid for $x > 0$. When we speak of a solution we mean a function that satisfies the differential equation for values of x lying in some interval, not necessarily for all x.

Returning to an earlier example:

$$\frac{dy}{dx} = x^3$$

has (by integration) the solution

$$y = \frac{x^4}{4} + c$$

If we have some extra piece of information, then we may be able to evaluate c. For example, if we were told that $y(0) = 1$ (by which we mean that y has the value 1 when x has the value 0) then we can find c. To do this we simply substitute $x = 0,\ y = 1$ into the solution. This gives

$$1 = \frac{0^4}{4} + c$$

ie

$$1 = 0 + c$$

so

$$c = 1$$

Thus the only solution of this differential equation which satisfies $y(0) = 1$ is

$$y = \frac{x^4}{4} + 1$$

In order to clarify things we will, in future, refer to a solution which contains an arbitrary constant as a **general** solution. Solutions obtained by giving the constant a particular value (such as 1 in the case above) will be called **particular** solutions (or particular integrals). In the example above, the general solution is:

$$y = \frac{x^4}{4} + c$$

and a particular solution is

$$y = \frac{x^4}{4} + 1$$

It is perhaps worth remarking that two apparently different general solutions can really represent the same solution. For example

$$y = \frac{x^4}{4} + 1 + d$$

would be a general solution to our example. We would regard this as the **same** general solution as that given earlier; we can translate from one to the other by equating c and $(1 + d)$:

$$c = 1 + d$$

All we are really doing here is changing the **name** of the constant from c to $1 + d$.

(The video discusses explicit, implicit and numerical solutions.) V

5 How they arise

Differential equations arise in modelling a wide variety of phenomena. Situations in which rates of change are related to the value of a variable will give rise to such equations. As an example, consider bacteria growing in a culture medium. If p denotes the population of bacteria at time t, then in the early phase of growth the rate of population increase will be proportional to the size of the population i.e.

$$\frac{dp}{dt} \text{ is proportional to } p$$

This gives a differential equation of the form

$$\frac{dp}{dt} = kp$$

where k is a positive constant.

There are other, more complicated, birth/death processes which can be modelled in this way. Further examples include chemical reactions, solubility, radioactive decay, vibrations, heat transfer, fluid flow and economics. Some of these are described in section 11 of these notes.

First order differential equations can be interpreted geometrically as representing families of curves. We shall see this in the next section.

(The video discusses briefly the modelling of bacterial growth.) V

6 One-parameter families of curves

6.1 The one-parameter family $y = ax^2$

The equations

$$y = x^2, y = 2x^2, y = -x^2, y = -2x^2$$

all represent parabolas with vertices at the origin. They are sketched below in figure 1.

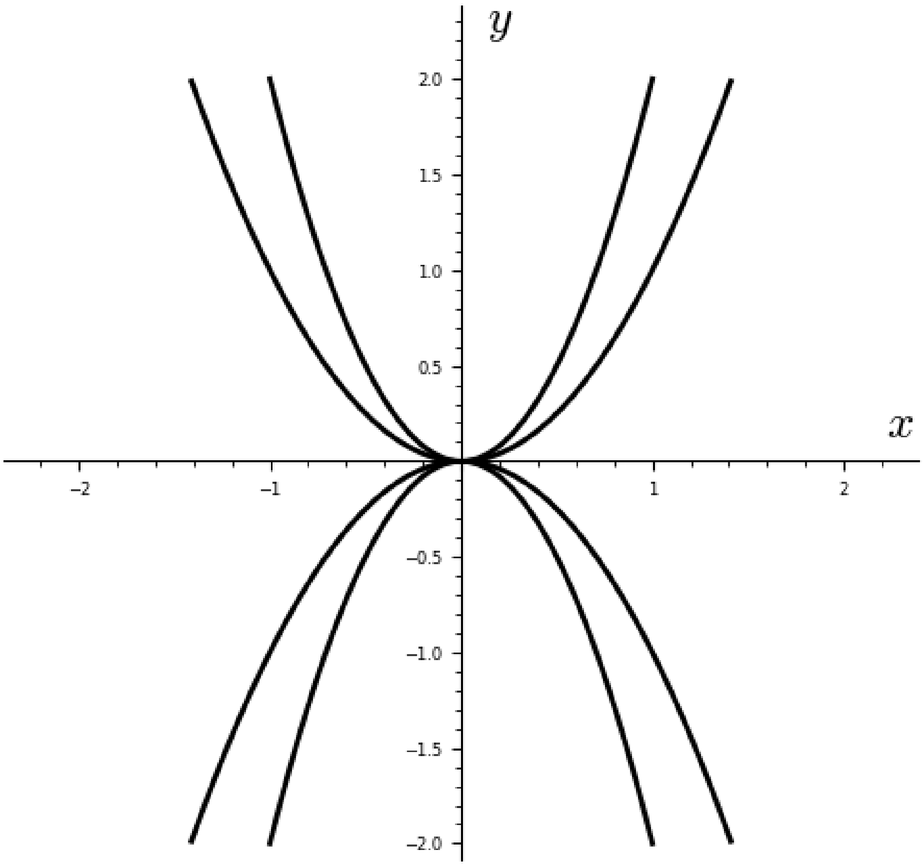

Figure 1: A one-parameter family of parabolas.

More generally, for **each**[1] value of the number a, the equation $y = ax^2$ represents a parabola. We say that $y = ax^2$ gives a one-parameter family of curves; the number a is the parameter.

[1]Well, $a = 0$ is a bit peculiar!

Such a family gives rise to a differential equation. This is obtained by eliminating the parameter between the original equation and the equation obtained from it by differentiating with respect to x. In this case

$$y = ax^2$$

gives

$$\frac{dy}{dx} = 2ax$$

and dividing these two equations to eliminate a gives

$$\frac{dy}{dx} = \frac{2y}{x}$$

This differential equation is in fact equivalent to the original $y = ax^2$. Indeed, we can obtain both the graph (showing the parabolas) and the original $y = ax^2$ equation from the differential equation. The next subsection shows how to get the graph, and the section on separable equations (section 8) shows how to solve the differential equation and obtain $y = ax^2$.

(The video covers the example given above.) V

6.2 Direction Fields

Given a first order differential equation of the form

$$\frac{dy}{dx} = F(x, y)$$

we can draw a diagram to represent the equation. Imagine that you had solved this equation and that you were asked for the **gradient** of your solution at the point (x_0, y_0). Since $F(x, y)$ purports to be the gradient (i.e. $\frac{dy}{dx}$) of the solution, all you would need to do would be to insert the values x_0 and y_0 into $F(x, y)$. Thus $F(x_0, y_0)$ gives the gradient of the solution at the point (x_0, y_0). We can represent this fact on a diagram by drawing a small line segment centred on (x_0, y_0) with gradient $F(x_0, y_0)$. (Figure 2.)

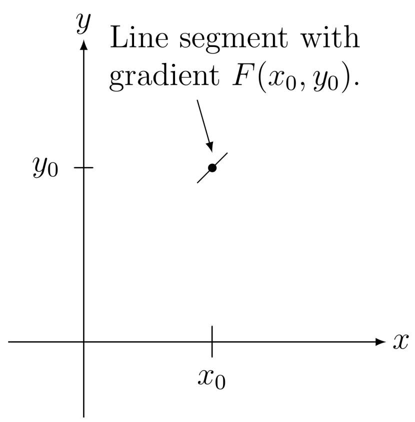

Figure 2: The gradient of the solution at (x_0, y_0) is $F(x_0, y_0)$.

If we carry out this procedure at every point on the diagram then what results is called the **direction field** of the differential equation. Naturally we cannot really draw a line segment at **every** point (the diagram would get rather cluttered) so in practice we take a representative sample. We then obtain a diagram covered in short line segments. We can see this happening if we return to our example

$$\frac{dy}{dx} = \frac{2y}{x}$$

Using this example consider, for instance, the point (1,1). Here the value of $2y/x$ is 2. Therefore, at the point (1,1) we draw a small line segment with gradient 2. (Figure 3.)

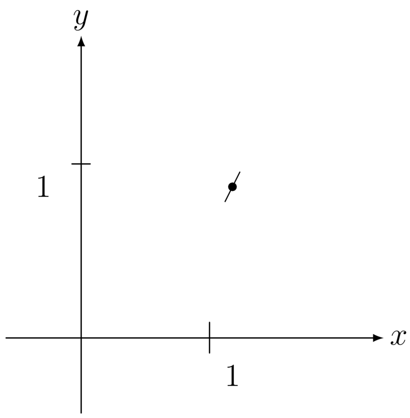

Figure 3: A line segment with gradient 2 at the point (1.1).

Repeating this at a representative sample of points yields the following diagram. (Figure 4.)

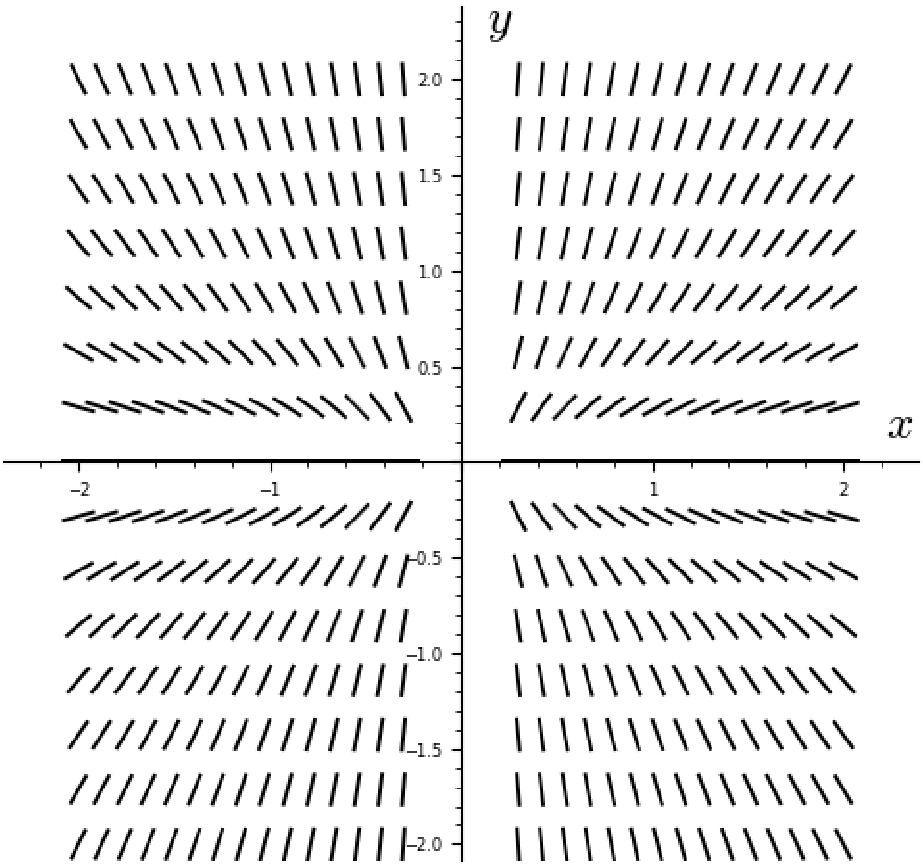

Figure 4: The direction field for $\frac{dy}{dx} = 2y/x$.

If you look carefully at this latest diagram and join up neighbouring line segments in the "obvious" way then the original parabolas described in the previous subsection will be seen to re-emerge. The diagram below illustrates this process. (Figure 5.)

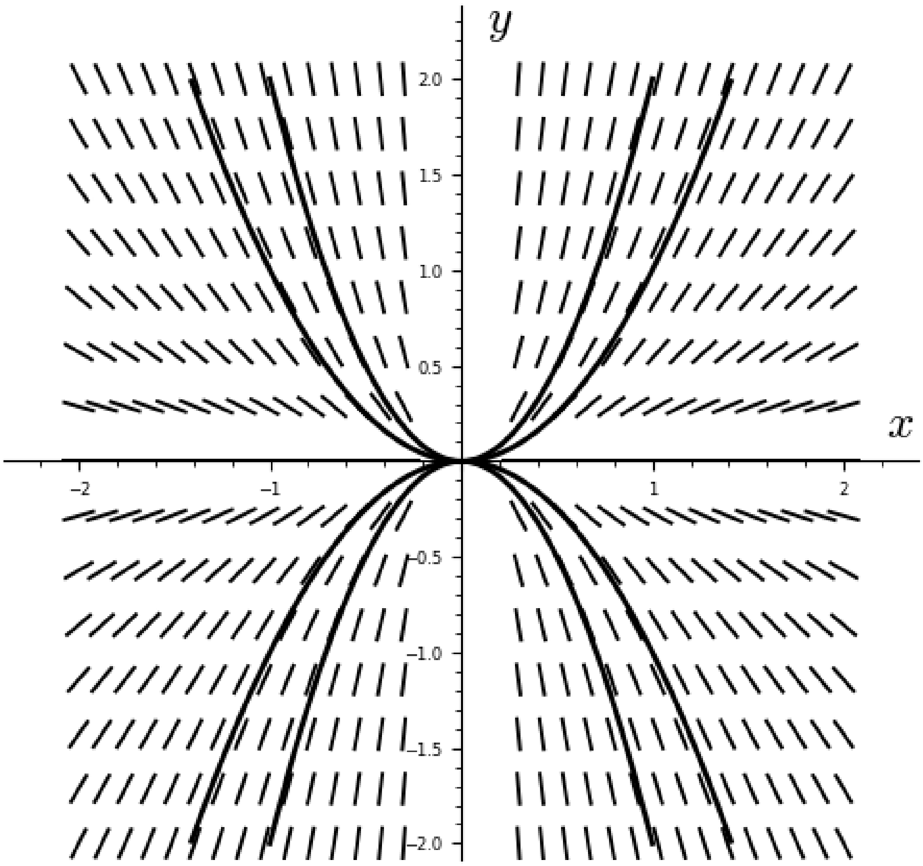

Figure 5: The direction field gives the solution curves.

(The video dicusses the construction of the direction field for the differential equation $y' = 2y/x$ and obtaining the solution curves from the direction field.) V

6.3 The general situation and further examples

If a is a parameter then the family of curves described by an equation such as

$$y = f(a, x) \tag{1}$$

is referred to as a one-parameter family. Differentiation with respect to x gives

$$\frac{dy}{dx} = f'(a, x) \tag{2}$$

Eliminating the parameter a between (1) and (2) gives a first-order differential equation which, in many cases, is equivalent to the original equation in the sense that the curves (i.e. the solutions of the differential equation) can be obtained from the direction field and that it may be possible to obtain (1) from this differential equation.

When we are given a differential equation of the form

$$\frac{dy}{dx} = F(x, y) \tag{3}$$

we can normally sketch the direction field and, by joining up neighbouring line segments, we can obtain a graphical representation of the curves which represent solutions to this differential equation.

Example 6.1 Sketch the direction field for the differential equation

$$\frac{dy}{dx} = \sin(xy)$$

and hence obtain graphical representation of the solutions for $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.

Solution At each point (x, y) the gradient of any solution is $\sin(xy)$. Therefore, at a representative sample of points (x, y) satisfying $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$ we draw small line segments with gradient $\sin(xy)$. The diagram below gives the resulting direction field. (Figure 6.)

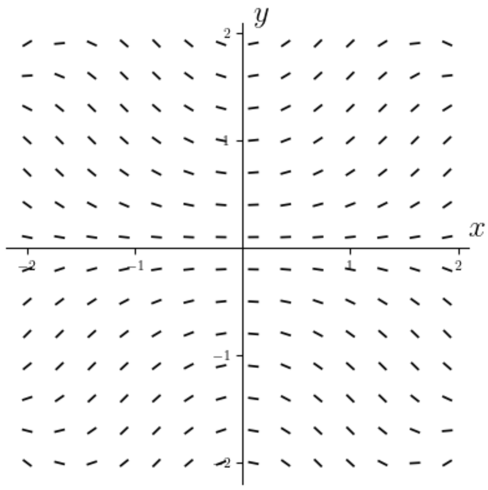

Figure 6: The direction field.

If we now join up neighbouring line segments we obtain the following diagram which represents some solutions of the differential equation. (Figure 7.)

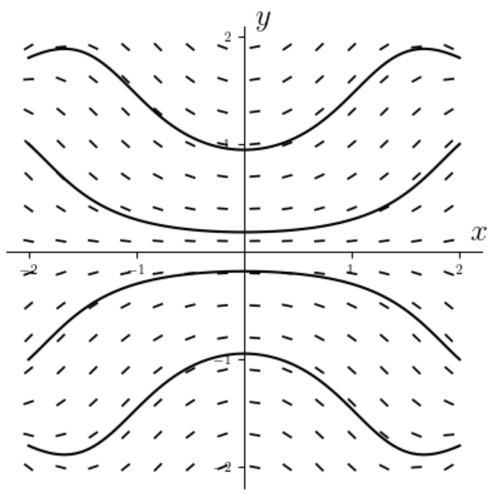

Figure 7: The solution curves.

(The video deals with the example $y' = sin(xy)$.) V

6.4 Isoclines

When plotting a direction field by hand it is often useful to first plot the isoclines. Literally, an **isocline** is a line of **equal gradient**.

For a differential equation of the form

$$\frac{dy}{dx} = F(x, y)$$

the isoclines are the curves given by the equation

$$F(x, y) = c$$

for different values of c.

Once the isoclines have been obtained and drawn, it is easy to sketch the direction field. Along the isocline $F(x, y) = c$ we simply sketch line segments all having the **uniform** gradient c.

Returning to the example of

$$\frac{dy}{dx} = \frac{2y}{x}$$

the isoclines are the lines

$$\frac{2y}{x} = c$$

or, in this case,

$$y = (\frac{c}{2})x$$

The diagram below (figure 8) shows these isoclines. Along each one of these isoclines line segments with gradient c have been drawn. For example, in the case of $c = 2$, the isocline is $y = x$ and along this line the line segments all have gradient 2.

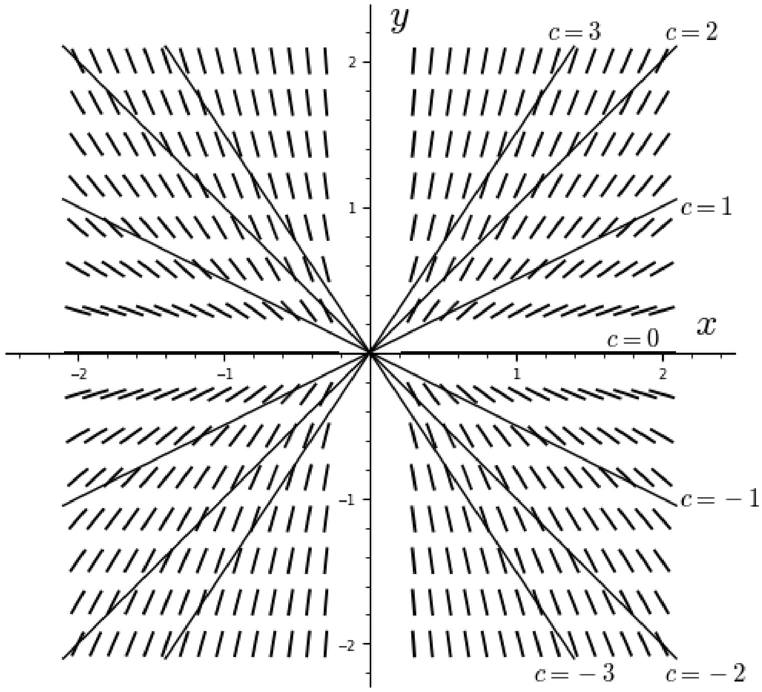

Figure 8: Isoclines and direction field for $y' = 2y/x$.

Example 6.2 Draw the isoclines for the differential equation

$$\frac{dy}{dx} = x^2 + y^2$$

Hence plot the direction field. Using the direction field, sketch the solution $y = y(x)$ to the differential equation which satisfies the requirement $y(0) = 1$.
Solution The isoclines are curves of the form

$$x^2 + y^2 = c$$

i.e. circles of radius $\sqrt{c}$ centred at (0,0). The diagram below shows the isoclines corresponding to $c = \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ with the direction field superimposed. Along each circle $x^2 + y^2 = c$ line segments with gradient c have been drawn. By inspection, the solution $y = y(x)$ which satisfies $y(0) = 1$ has been drawn using the direction field as a guide. (Figure 9.)

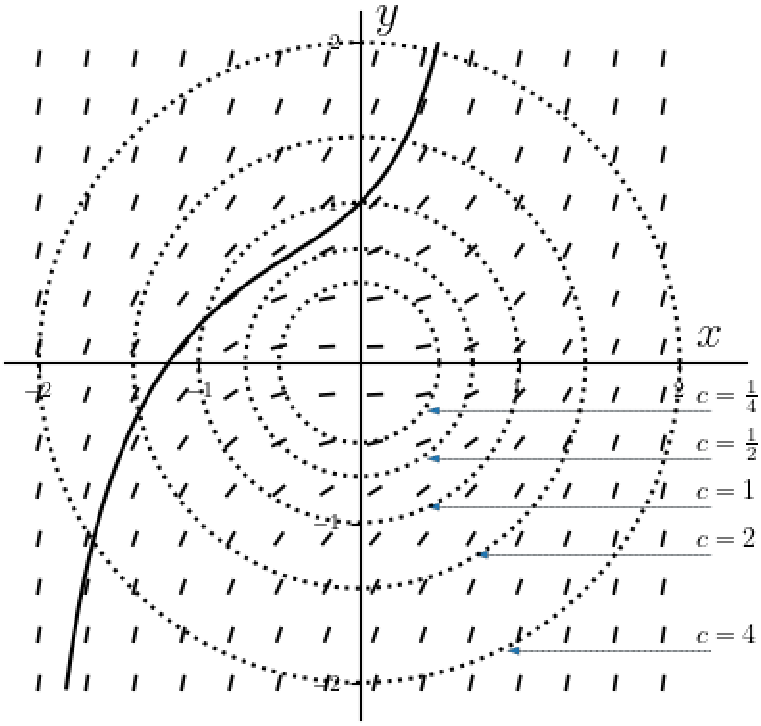

Figure 9: Isoclines, direction field, and solution satisfying $y(0) = 1$ for the equation $y' = x^2 + y^2$.

As we have seen in the above examples the method of direction fields enables us to gain some insight into the solution of a differential equation of the form

$$\frac{dy}{dx} = F(x, y)$$

We can sketch the solution curves and so get a rough numerical approximation to the solution $y = y(x)$ satisfying a particular condition of the form $y(x_0) = y_0$, where x_0 and y_0 are given values. It is also clear that for "reasonable" functions $F(x, y)$ the equation

$$\frac{dy}{dx} = F(x, y)$$

will indeed have some solutions. In the next section we look at one very elementary numerical method for obtaining an approximate solution. The method is based on joining-up the line segments of the direction field in the "obvious" way.

(The video covers the example $y' = x^2 + y^2$ described above. V
You should now try some examples involving:

- *obtaining first order differential equations from one parameter families of curves,*
- *obtaining the isoclines and direction field for given first order differential equations,*
- *using the isoclines to sketch solution curves.)* EX

7 Euler's Numerical Method

The method enables us to obtain a numerical solution to a differential equation

$$y' = F(x, y)$$

given an initial condition: $y = y_0$ when $x = x_0$.

The solution is obtained in a progressive fashion. Let us consider the first step. The gradient of the solution curve at (x_0, y_0) will be $F(x_0, y_0)$. If we move to a neighbouring point on the solution curve with x-coordinate $x_1 = x_0 + h$ then we can estimate its y-coordinate, y_1, from the following sketch.

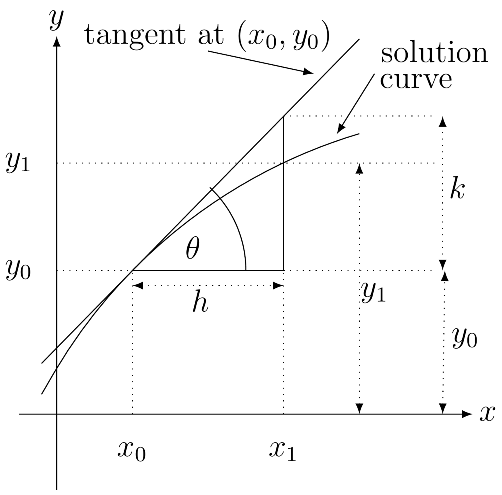

Figure 10: Estimating y_1.

If h is small then a good approximation to y_1 is the quantity $y_0 + k$. But

$$\tan\theta = \frac{k}{h}$$

and $\tan\theta$ is the gradient of the solution curve at (x_0, y_0) . Thus

$$F(x_0, y_0) = \frac{k}{h}$$

and so

$$k = hF(x_0, y_0)$$

Therefore, approximately,

$$y_1 = y_0 + hF(x_0, y_0)$$

If we now repeat the procedure, moving from (x_1, y_1) to (x_2, y_2) along the solution curve where $x_2 = x_1 + h = x_0 + 2h$, we obtain the approximation

$$y_2 = y_1 + hF(x_1, y_1)$$

In general, if $x_n = x_0 + nh$ we find that the corresponding y-coordinate on the solution curve which passes through (x_0, y_0) is given (approximately) by

$$\boxed{y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad (n \geq 1)}$$

Example 7.1 Obtain a numerical solution to the differential equation

$$\frac{dy}{dx} = x^2 + y^2$$

subject to the initial condition that $y(0) = 1$. Use a step length $h = 0.1$ and tabulate the values of y for x between 0 and 1.

Solution Put $x_0 = 0$, $y_0 = y(x_0) = y(0) = 1$, and $h = 0.1$. Then with $h = 0.1$,

$$y_1 \approx y_0 + h(x_0^2 + y_0^2) = 1 + 0.1(0^2 + 1^2) = 1.1$$

Similarly with $x_1 = x_0 + h = 0.1$,

$$y_2 \approx y_1 + h(x_1^2 + y_1^2) \approx 1.1 + 0.1((0.1)^2 + (1.1)^2) = 1.222$$

Likewise with $x_2 = x_0 + 2h = 0.2$,

$$y_3 \approx y_2 + h(x_2^2 + y_2^2) \approx 1.222 + 0.1((0.2)^2 + (1.222)^2) \approx 1.375 \quad \text{(to 3dp)}$$

Proceeding in this way we obtain the table

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1	1.1	1.222	1.375							

You might try completing the table. ○

Of course solutions obtained in this way are only approximations. You will see from the diagram that if the solution curve has large curvature then the error introduced at each stage (the difference between y_1 and $(y_0 + k)$) might be significant. Clearly the error will tend to grow at each step. More advanced methods try to overcome these problems and to provide some estimate of the error. References to such methods are given in the bibliography.

(The video covers the general theory and the example $y' = x^2 + y^2$, $y(0) = 1$ given above. V

You should now try some examples involving the use of Euler's method to obtain numerical solutions to first order differential equations.) EX

8 Separable equations

It may happen that in the equation

$$\frac{dy}{dx} = F(x, y)$$

we can express $F(x, y)$ as the product of two functions $f(x)$ and $g(y)$, the former involving only x and the latter involving only y. The equation can then be written as

$$\frac{dy}{dx} = f(x)g(y) \tag{4}$$

To solve this we gather together all the y - terms on one side and all the x - terms on the other ("separating the variables") thus

$$\frac{dy}{g(y)} = f(x)dx \tag{5}$$

we then integrate:

$$\int \frac{dy}{g(y)} = \int f(x)dx + c \tag{6}$$

There are a few points to note.

1. Firstly, strictly speaking, (5) is illegal because we have broken-up the composite symbol $\frac{dy}{dx}$. However (6) can be obtained directly from (4) by dividing by $g(y)$ and then integrating both sides with respect to x. My advice is to go from (4) to (5) to (6) without undue concern.

2. Secondly, we only need one constant in (6). It is true that we have done two integrations. If we introduce a separate constant for each then (6) might appear as

$$\int \frac{dy}{g(y)} + a = \int f(x)dx + b$$

   which can be re-written as

$$\int \frac{dy}{g(y)} = \int f(x)dx + c$$

   where $c = b - a$.

3. Thirdly, we are rather assuming that $g(y) \neq 0$. You need to keep an eye on this. Clearly you might not get a correct answer at points where $g(y) = 0$.

4. Fourthly, the method will work if either f or g are constants.

Example 8.1 Solve
$$\frac{dy}{dx} = e^{x+y}$$
Solution Write the differential equation as

$$\frac{dy}{dx} = e^x e^y$$

Separating the variables gives

$$\frac{dy}{e^y} = e^x dx$$

and so
$$\int e^{-y} dy = \int e^x dx + c$$
Therefore
$$-e^{-y} = e^x + c$$
hence
$$e^{-y} = -e^x - c$$
and so
$$-y = \log_e(-e^x - c)$$
i.e.
$$y = -\log_e(-e^x - c)$$
Plainly the solution is only going to be valid if c is chosen so that $-e^x - c > 0$ (why?). Note also that the constant c is not just tagged onto the end. It would be incorrect to write

$$y = -\log_e(-e^x) + \text{constant}$$

The constant has to be introduced as soon as the integration is performed. ○

Example 8.2 Solve

$$y' + x^3y^2 = y^2 \sin x$$

given that $y(0) = 1$.

Solution First we find the general solution and then we evaluate the constant using the fact that $y = 1$ when $x = 0$. We write the equations as

$$\frac{dy}{dx} = y^2 \sin x - x^3y^2$$

i.e.

$$\frac{dy}{dx} = y^2(\sin x - x^3)$$

Thus the equation **is** separable. Separating the variables gives

$$\frac{dy}{y^2} = (\sin x - x^3)dx$$

Hence

$$\int \frac{dy}{y^2} = \int (\sin x - x^3)dx + c$$

i.e.

$$-y^{-1} = -\cos x - \frac{x^4}{4} + c$$

The condition $y(0) = 1$ gives

$$-1^{-1} = -\cos 0 - \frac{0^4}{4} + c$$

i.e.

$$-1 = -1 + c$$

Hence $c = 0$. Therefore we have

$$-y^{-1} = -\cos x - \frac{x^4}{4}$$

i.e.

$$y = \frac{1}{\cos x + \frac{x^4}{4}}$$

A couple of comments are in order.

1. Firstly, when we separated the variables we rather assumed $y \neq 0$. The solution just obtained never takes the value zero but $y = 0$ is a solution of the differential equation (by $y = 0$ here we mean y to be zero for all x). Of course $y = 0$ does not satisfy the condition $y(0) = 1$.

2. Secondly, the solution obtained will not be valid whenever $(\cos x + \frac{x^4}{4})$ is zero. ○

In order to determine whether an equation is separable or not, you should firstly manipulate it into a form where the derivative appears on its own on the left hand side whilst the right hand side contains only x, y and constant terms. This puts the equation into the form

$$\frac{dy}{dx} = F(x, y)$$

If the right hand side now only contains x or only contains y or is a constant then the job is complete and the equation is separable. If, as is more likely, the right hand side contains both x and y then the next task is to factorise it. If this can be done in such a way that one of the factors contains all the x terms and the other factor contains all the y terms then the equation is separable. If it is impossible to separate the variables in this way then the equation is not separable. Note that most first order equations are **not** separable, i.e. they cannot be expressed in the form

$$\frac{dy}{dx} = f(x)g(y)$$

Example 8.3 The differential equation

$$(\sin y)^2 \frac{dy}{dx} + \cos x \cos y = \cos(x + y)$$

can be rewritten as

$$\frac{dy}{dx} = \frac{\cos(x + y) - \cos x \cos y}{(\sin y)^2}$$

Using the identity

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

we obtain

$$\frac{dy}{dx} = -\frac{\sin x \sin y}{(\sin y)^2} = -\frac{\sin x}{\sin y} = (-\sin x)(\frac{1}{\sin y})$$

This is of the form

$$\frac{dy}{dx} = f(x)g(y)$$

and so the equation is separable. ○

Example 8.4 The differential equation

$$y' - x^2 = y^2$$

can be rewritten as

$$y' = x^2 + y^2$$

The right hand side is not factorisable and so the equation is not separable. (In fact even if you use complex factors and write $x^2 + y^2 = (x + iy)(x - iy)$, the equation is non-separable because the two factors do not separate x from y.) ○

(The video deals with the general technique of separation of variables and covers the example $y' = e^{x+y}$ described above. There is a section on how to recognize separable equations at the end of the third video, after we have dealt with linear equations. V

You should now try solving some separable equations.) EX

9 Linear Equations

The first order differential equation

$$\frac{dy}{dx} = F(x, y)$$

is said to be linear if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{7}$$

where $P(x), Q(x)$ are functions of x alone (or are constants). This form of the equation is called the linear form because it is linear in y (this means that it does not involve $y^2, y^3, \log_e(\frac{dy}{dx}), e^y, \sin(\frac{dy}{dx})$, etc.,etc.; only y on its own and in $\frac{dy}{dx}$).

The method of solution is based on the observation that the differential equation

$$\frac{d}{dx}(R(x)y) = S(x) \tag{8}$$

has an easy solution, namely

$$R(x)y = \int S(x)dx + c$$

i.e.

$$y = \frac{\int S(x)dx + c}{R(x)} \tag{9}$$

The left-hand side of (8) can be expanded by the product rule to give

$$R(x)\frac{dy}{dx} + R'(x)y$$

where R' denotes the derivative of R. Thus (8) can be written as

$$R(x)\frac{dy}{dx} + R'(x)y = S(x)$$

or even as

$$\frac{dy}{dx} + \frac{R'(x)}{R(x)}y = \frac{S(x)}{R(x)} \tag{10}$$

provided $R(x)$ is non-zero. We reiterate: equation (10) is merely a re-write of equation (8); it has the same easy solution given by (9).

If we now compare (10) with the standard linear form (7):

$$\frac{dy}{dx} + \frac{R'(x)}{R(x)}y = \frac{S(x)}{R(x)} \qquad (10)$$

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (7)$$

we see they are the same provided that

$$\frac{R'(x)}{R(x)} = P(x) \tag{11}$$

and

$$\frac{S(x)}{R(x)} = Q(x) \tag{12}$$

Thus the standard linear form (7) will have the easy solution given by (9) provided we choose $R(x)$ and $S(x)$ to satisfy equations (11) and (12).

Of these two equations (11) is the more complicated. We can write it as

$$\frac{1}{R}\frac{dR}{dx} = P(x)$$

This equation is separable:

$$\frac{dR}{R} = P(x)dx$$

and so

$$\int \frac{dR}{R} = \int P(x)dx + k$$

i.e.

$$\log_e R = \int P(x)dx + k$$

i.e.

$$R = e^{\int P(x)dx + k}$$

We only require a particular solution (any one will do) so, for convenience we set $k = 0$ and obtain

$$R(x) = e^{\int P(x)dx} \tag{13}$$

$S(x)$ is then obtained easily from (12):

$$S(x) = Q(x)R(x)$$

i.e.

$$S(x) = Q(x)e^{\int P(x)dx} \tag{14}$$

Round about this stage students normally get lost so a summary may help.

Summary	To solve $\frac{dy}{dx} + P(x)y = Q(x)$
Step 1	Calculate $R(x) = e^{\int P(x)dx}$ ($R(x)$ is called the **integrating factor**)
Step 2	Multiply the given differential equation by $R(x)$. The left-hand side can then be written as $$\frac{d}{dx}(R(x)y)$$ so the equation can be expressed as $$\frac{d}{dx}(R(x)y) = Q(x)R(x)$$ Solving this by integration we obtain $$R(x)y = \int Q(x)R(x)dx + c$$

Note that the format $y' + P(x)y = Q(x)$ is important. Thus the equation $y' - xy = x^3$ has $P(x) = -x$, **not** $P(x) = x$. If you do make a mistake with the sign like this then your resulting 'solution' will generally be far removed from the correct solution.

Example 9.1 Solve

$$y' + xy = e^{-x^2/2}$$

Solution Here $P(x) = x$, so the integrating factor is

$$R(x) = e^{\int x dx} = e^{x^2/2}$$

Multiplying the given differential equation by this factor gives

$$\frac{d}{dx}(e^{x^2/2}y) = e^{-x^2/2}e^{x^2/2} = e^0 = 1$$

Integrating we obtain

$$e^{x^2/2}y = \int 1dx = x + c$$

Hence

$$y = \frac{x+c}{e^{x^2/2}} = (x+c)e^{-x^2/2}$$

○

Example 9.2 Solve

$$(\cos x)\frac{dy}{dx} + (\sin x)y = x(\cos x)^2$$

Solution Firstly we divide by $\cos x$ to get the equation into standard form:

$$\frac{dy}{dx} + \frac{\sin x}{\cos x}y = x\cos x \qquad (15)$$

The integrating factor is

$$\begin{aligned} R(x) &= e^{\int \frac{\sin x}{\cos x}dx} \\ &= e^{\int \tan x dx} \\ &= e^{-\log_e \cos x} \\ &= \frac{1}{e^{\log_e \cos x}} \\ &= \frac{1}{\cos x} \end{aligned}$$

Multiplying (15) by this factor enables us to write the equation as

$$\frac{d}{dx}(\frac{1}{\cos x}y) = \frac{x\cos x}{\cos x} = x$$

Hence, integrating we obtain

$$\frac{y}{\cos x} = \int x dx + c = \frac{x^2}{2} + c$$

Therefore

$$y = (\frac{x^2}{2} + c)\cos x$$

Before leaving this example we remark that the inclusion of a constant in the formula for $R(x)$ would have no effect, i.e. if we used

$$R(x) = e^{\int P(x)dx + k}$$

we could write this as

$$R(x) = e^{\int P(x)dx}.e^k$$

Since both sides of the standard from are multiplied by this quantity we can cancel the numerical factor e^k. This will happen to any integrating factor and so you need never bother with the constant of integration in the formula for $R(x)$. ○

In order to determine whether or not an equation is linear you must compare it with the standard form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Note that $\frac{dy}{dx}$ appears (on its own) on the extreme left. The other term on the left hand side contains all the remaining references to y and it consists of y times a function of x (or y times a constant). The right hand side contains no y terms.

Example 9.3 As we saw towards the end of the previous section, the differential equation

$$(\sin y)^2\frac{dy}{dx} + \cos x\cos y = \cos(x + y)$$

can be manipulated into the form

$$\frac{dy}{dx} = -\frac{\sin x}{\sin y}$$

Getting all the y terms onto the left hand side gives

$$\frac{dy}{dx} + \frac{\sin x}{\sin y} = 0$$

This is non-linear because y appears as $\dfrac{1}{\sin y}$ and not simply as y. ○

Example 9.4 The differential equation

$$y' - x^2 = y^2$$

can be rewritten as

$$y' - y^2 = x^2$$

Again this is non-linear because the second term on the left hand side contains a factor y^2 rather than just y. ○

Example 9.5 The differential equation

$$yy' - xy = y^2 \sin x$$

can be divided by y and rearranged to give

$$y' - y \sin x = x$$

(provided you are willing to discard the solution $y = 0$). Comparing this with

$$y' + P(x)y = Q(x)$$

we see that the (rewritten) equation is linear. Indeed $P(x) = -\sin x$ (note the minus sign!) and $Q(x) = x$. ○

Example 9.6 The differential equation

$$y' = (x + 1)(y + 1)$$

can be written as

$$y' - (x + 1)y = x + 1$$

This is in the standard form with $P(x) = -(x+1)$ and $Q(x) = x+1$, and so is linear. In fact the equation is also separable - look at the original form. ○

To conclude this section we observe that most first order equations are not linear, i.e. they cannot be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Coupling this with what we said in an earlier section, most first order equations are **neither** separable **nor** linear. A few equations are both separable and linear and in these cases it is up to you which method you use to solve them. In the next section we look at some equations which are neither separable nor linear but which can be reduced to one or other of these forms by suitable substitutions.

(The video covers the general theory of linear equations and the example $(\cos x)y' + (\sin x)y = x(\cos x)^2$ given above. The final video section describes how to recognize separable and linear equations and covers the four examples immediately preceding this note. V

You should now try some examples of linear equations. In addition, you should make sure that you can correctly identify separable and linear equations.) EX

10 Other types of equation

(The videos do not cover this section.)

We look at some equations which are neither separable nor linear but which can be reduced to one or other of these forms by means of a suitable substitution.

10.1 Homogeneous Equations

A differential equation

$$\frac{dy}{dx} = F(x, y)$$

is said to be homogeneous[2] if $F(x, y)$ can be expressed in terms of the ratio y/x. That is to say, we can write the equation as

$$\frac{dy}{dx} = G(\frac{y}{x})$$

where G is some function.

Example 10.1 If

$$F(x, y) = \frac{x^2 + y^2}{2x^2}$$

then we can express $F(x, y)$ as

$$F(x, y) = \frac{1 + (\frac{y}{x})^2}{2}$$

If we put $v = y/x$ and $G(v) = (1 + v^2)/2$ then $F(x, y) = G(y/x)$ and so the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$$

[2] Unfortunately the same word is also used for a different property -see the notes on second order differential equations.

is homogeneous. We can write it as

$$\frac{dy}{dx} = \frac{1 + (\frac{y}{x})^2}{2}$$

Of course, most equations are **not** homogeneous - for example there is no way of expressing $\sin(x + y)$ in terms of y/x. (You should be convinced by the fact that $\sin(1 + 1) \neq \sin(2 + 2)$ even though 1/1=2/2). Therefore the equation

$$\frac{dy}{dx} = \sin(x + y)$$

is **not** homogeneous.

To solve a homogeneous equation

$$\frac{dy}{dx} = G(\frac{y}{x}) \tag{16}$$

we substitute $v = y/x$ so that $y = vx$. Differentiating this by the product rule gives

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{17}$$

Substituting this and $v = y/x$ into (16) then produces

$$v + x\frac{dv}{dx} = G(v)$$

i.e.

$$\frac{dv}{dx} = \frac{1}{x}(G(v) - v) \tag{18}$$

This latter equation is separable. We can solve it for v, and then obtain y from the equation $y = vx$.

Example 10.2 Solve

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$$

Solution We express the equation as

$$\frac{dy}{dx} = \frac{1 + (\frac{y}{x})^2}{2}$$

We substitute $v = y/x$ so that $y = vx$ and hence

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

The equation can then be expressed as

$$v + x\frac{dv}{dx} = \frac{1+v^2}{2}$$

i.e.

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{x}\left[\frac{1+v^2}{2} - v\right] \\ &= \frac{1}{x}\left[\frac{1-2v+v^2}{2}\right] \\ &= \frac{(1-v)^2}{2x} \end{aligned}$$

Separating the variables gives

$$2\frac{dv}{(1-v)^2} = \frac{dx}{x}$$

and so

$$2\int \frac{dv}{(1-v)^2} = \int \frac{dx}{x} + c$$

Integrating we obtain

$$\frac{2}{1-v} = \log_e x + c$$

Therefore

$$\frac{1-v}{2} = \frac{1}{\log_e x + c}$$

hence

$$v = 1 - \frac{2}{\log_e x + c}$$

Finally, $y = vx$ gives

$$y = x\left[1 - \frac{2}{\log_e x + c}\right]$$ ○

(You should now try some examples involving homogeneous equations.) EX

10.2 Nearly-homogeneous equations (1st type) (Sometimes called linear fractional equations).

These are equations which can be expressed in the form

$$\frac{dy}{dx} = H\left(\frac{ax + by + c}{ex + fy + g}\right) \tag{19}$$

where a, b, c, e, f, g are constants; c and g are not both zero and $af \neq be$; H is some function of one variable. Note that if c and g were both zero then we could write

$$\frac{dy}{dx} = H\left(\frac{a + b\frac{y}{x}}{e + f\frac{y}{x}}\right) \tag{20}$$

and the equation would be homogeneous. It is the presence of c and g which prevent the homogeneity. We will make substitutions that effectively remove the c and the g and convert (19) to something closely resembling (20)

Firstly we solve the algebraic simultaneous linear equations

$$\begin{aligned} ax + by + c &= 0 \\ ex + fy + d &= 0 \end{aligned}$$

for the variables x and y. The condition $af \neq be$ ensures that these equations have a unique solution. Suppose that this solution is $x = \alpha$ and $y = \beta$. That is,

$$a\alpha + b\beta + c = 0 \tag{21}$$

$$e\alpha + f\beta + g = 0 \tag{22}$$

We now return to the differential equation and substitute $X = x - \alpha$ and $Y = y - \beta$. Then $\frac{dX}{dx} = 1$, $\frac{dY}{dy} = 1$ and so $\frac{dy}{dx} = \frac{dY}{dX}$. We obtain

$$\begin{aligned} \frac{dY}{dX} &= H\left(\frac{ax + by + c}{ex + fy + g}\right) \\ &= H\left(\frac{a(X + \alpha) + b(Y + \beta) + c}{e(X + \alpha) + f(Y + \beta) + g}\right) \\ &= H\left(\frac{aX + bY + (a\alpha + b\beta + c)}{eX + fY + (e\alpha + f\beta + g)}\right) \\ &= H\left(\frac{aX + bY}{eX + fY}\right) \end{aligned}$$

because by (21) and (22) the $(\dots)$ terms are both zero. Finally we obtain

$$\frac{dY}{dX} = H\left(\frac{a + b\frac{Y}{X}}{e + f\frac{Y}{X}}\right) \tag{23}$$

which is, as promised, very similar to (20). This equation is homogeneous; we solve it by substituting $v = Y/X$. Having solved (23) we replace X by $x - \alpha$ and Y by $y - \beta$ and hence obtain the solution to (19).

Example 10.3 Solve

$$\frac{dy}{dx} = \frac{x + 2y - 5}{2x - y}$$

Solution Firstly we solve the simultaneous equations

$$\begin{aligned} x + 2y - 5 &= 0 \\ 2x - y &= 0 \end{aligned}$$

These have solution $x = 1$ and $y = 2$. We now substitute $X = x - 1$ and $Y = y - 2$ in the differential equation. This gives

$$\frac{dY}{dX} = \frac{X + 2Y}{2X - Y} = \frac{1 + 2\frac{Y}{X}}{2 - \frac{Y}{X}}$$

Substituting $v = Y/X$ gives $Y = vX$ and so

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Hence

$$v + X\frac{dv}{dX} = \frac{1 + 2v}{2 - v}$$

i.e.

$$X\frac{dv}{dX} = \frac{1 + 2v - 2v + v^2}{2 - v} = \frac{1 + v^2}{2 - v}$$

Separating the variables gives

$$\frac{2 - v}{1 + v^2}dv = \frac{dX}{X}$$

and so

$$\int \left(\frac{2}{1 + v^2} - \frac{v}{1 + v^2} \right) dv = \int \frac{dX}{X} + k$$

Therefore

$$2 \arctan v - \frac{1}{2}\log_e(1 + v^2) = \log_e X + k$$

Replacing v by Y/X, Y by $(y - 2)$ and X by $(x - 1)$ then gives

$$2 \arctan(\frac{y - 2}{x - 1}) - \frac{1}{2}\log_e(1 + (\frac{y - 2}{x - 1})^2) = \log_e(x - 1) + k$$

This can be simplified slightly using properties of the $\log_e$ function:

$$2\arctan(\frac{y-2}{x-1}) - \frac{1}{2}\log_e((x-1)^2+(y-2)^2) + \frac{1}{2}\log_e((x-1)^2) = \log_e(x-1)+k$$

which gives

$$2\arctan(\frac{y-2}{x-1}) - \frac{1}{2}\log_e((x-1)^2+(y-2)^2) = k$$

because

$$\frac{1}{2}\log_e((x-1)^2) = \log_e(x-1)$$

○

(You should now try one or two equations of the above type.) EX

10.3 Nearly-homogeneous equations (2nd Type)

These are equations which can be expressed in the form

$$\frac{dy}{dx} = H(\frac{ax+by+c}{ex+fy+g})$$

where a, b, c, e, f, g are constants; c and g are not both zero and $\boldsymbol{af = be}$; H is some function of one variable.

As before, if c and g are both zero then the equation would be homogeneous. However, the condition $af = be$ ensures that the quantities

$$(ax+by) \quad \text{and} \quad (ex+fy)$$

are constant multiples of one another. If we substitute $z = ax + by$ then the equation can be reduced to separable form (this applies whether or not c and g are both zero).

Example 10.4 Solve

$$\frac{dy}{dx} = \frac{2x+3y+6}{4x+6y+3}$$

Solution Put $z = 2x + 3y$. Then

$$\frac{dz}{dx} = 2 + 3\frac{dy}{dx}$$

and so

$$\frac{dy}{dx} = \frac{1}{3}(\frac{dz}{dx} - 2)$$

Hence the differential equation may be written as

$$\frac{1}{3}(\frac{dz}{dx} - 2) = \frac{z+6}{2z+3}$$

Therefore

$$\frac{dz}{dx} = \frac{3z + 18 + 4z + 6}{2z + 3} = \frac{7z + 24}{2z + 3}$$

Separating the variables gives

$$\frac{2z + 3}{7z + 24} dz = dx$$

and so

$$\int \frac{2z + 3}{7z + 24} dz = \int dx + k$$

To perform the left-hand integral we divide:

$$\begin{array}{rl} & \quad \frac{2}{7} \\ 7z + 24 & \overline{\big|\ 2z + 3} \\ & -(2z + \frac{48}{7}) \\ \cline{2-2} & \quad (3 - \frac{48}{7}) \end{array}$$

Since $3 - \frac{48}{7} = -\frac{27}{7}$, the remainder is $-\frac{27}{7}$. Hence

$$\frac{2z + 3}{7z + 24} = \frac{2}{7} - \frac{27}{7} . \frac{1}{7z + 24}$$

Therefore

$$\int \frac{2z + 3}{7z + 24} dz = \frac{2}{7} z - \frac{27}{7} . \frac{1}{7} . \log_e (7z + 24)$$

Hence

$$\frac{2}{7} z - \frac{27}{49} \log_e (7z + 24) = x + k$$

Replacing z by $2x + 3y$ gives

$$\frac{2}{7}(2x + 3y) - \frac{27}{49} \log_e (14x + 21y + 24) = x + k$$

(You should now try one or two equations of the above type.) EX

10.4 Bernoulli's Equation

The general form of Bernoulli's equation is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{24}$$

where $P(x),\ Q(x)$ are functions of x alone (or are constants) and $n \neq 0$ or 1.

If we consider (24) with $n = 0$ we see that the equation is then linear. If we consider it with $n = 1$ then by algebraic manipulation we can convert it to both the standard separable and linear forms. The real interest is what happens when $n \neq 0$ or 1.

We can solve Bernoulli's equation (24) with the substitution

$$z = y^{1-n}$$

Differentiation gives

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

and so

$$\frac{dy}{dx} = \frac{1}{1-n}y^n\frac{dz}{dx}$$

If we substitute this into (24) we get

$$\frac{1}{1-n}y^n\frac{dz}{dx} + P(x)y = Q(x)y^n$$

and dividing by y^n then produces

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)y^{1-n} = Q(x)$$

from which, with y^{1-n} replaced by z, we get

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

This is a linear equation which can be solved for z by the integrating factor method. Having obtained a solution for z we can replace z by y^{1-n} and obtain an expression for y.

Example 10.5 Solve

$$\frac{dy}{dx} + \frac{y}{x} = xy^2$$

given that $y = 1$ when $x = 1$.

Solution Here $n = 2$ in the Bernoulli general form. We put $z = y^{1-2} = y^{-1}$. Rather than go through the general working above it is easier to convert this to $y = 1/z$. Then differentiation gives

$$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

and the differential equation becomes

$$-\frac{1}{z^2}\frac{dz}{dx} + \frac{1}{xz} = \frac{x}{z^2}$$

i.e.

$$\frac{dz}{dx} - \frac{z}{x} = -x$$

This is a first order linear equation (in z) with integrating factor

$$R(x) = e^{\int -\frac{1}{x}dx} = e^{-\log_e x} = \frac{1}{e^{\log_e x}} = \frac{1}{x}$$

The differential equation can therefore be expressed as

$$\frac{d}{dx}(\frac{1}{x}z) = -\frac{x}{x} = -1$$

Integrating, we obtain

$$\frac{z}{x} = \int -1 dx = -x + c$$

Therefore $z = x(c - x)$, and so $y^{-1} = x(c - x)$, i.e.

$$y = \frac{1}{x(c - x)}$$

In this problem we have the extra information to determine c, namely that $y = 1$ when $x = 1$. This gives

$$1 = \frac{1}{c - 1}$$

and so $c = 2$. Finally therefore

$$y = \frac{1}{x(2 - x)}$$ ○

(You should now try one or two equations of the above type.) EX

11 Differential Equations as Mathematical Models

(See section 5 for reference to the videos.) V

Our models are all concerned with birth/death processes.

11.1 Exponential Growth

Imagine a population (such as bacteria) growing in a favourable environment (such as a culture medium with plentiful food supplies). We ignore the possibility of death and concentrate on the initial phase of growth. Let p denote the population size at time t. We expect the rate of growth to be proportional to the population size, i.e.

$$\frac{dp}{dt} = kp \qquad (k > 0, \text{ a constant})$$

This differential equation is both separable and linear. Whichever way you solve it, the solution can be expressed as

$$p = Ae^{kt}$$

Here A represents the population at time $t = 0$. (See figure 11.)

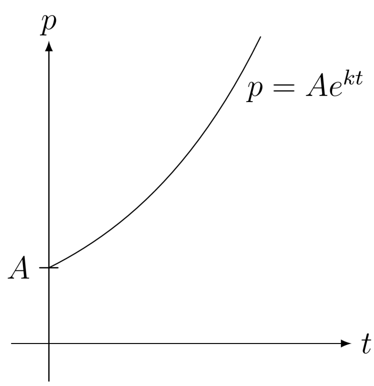

Figure 11: Exponential growth.

11.2 Restricted growth

This describes a late phase of growth (still without death) in a restricted environment. The population approaches an upper limit P beyond which it cannot grow due to insufficient resources for reproduction. We expect the rate of growth to be proportional to (P − population size), i.e.

$$\frac{dp}{dt} = k(P - p) \qquad (k > 0, \text{ a constant})$$

This differential equation is also both separable and linear. It has solution

$$p = P - Ae^{-kt}$$

where $P - A$ represents the population at time $t = 0$. (See figure 12.)

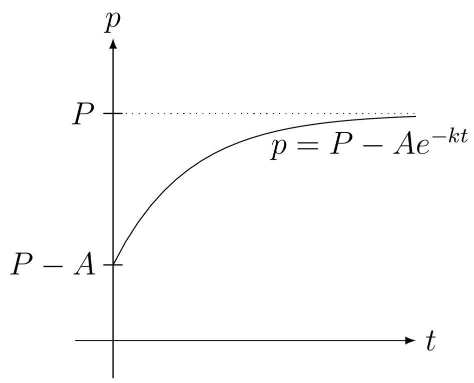

Figure 12: Restricted growth.

11.3 Logistic Equation

This starts from the assumption that the rate of increase of population is the difference between the birth rate (b) and the death rate (d), i.e.

$$\frac{dp}{dt} = b - d$$

To proceed further we examine the fertility rate $f = b/p$ and the mortality rate $m = d/p$. As the population p increases we might expect f (the number of births per head) to decline due to competition for resources and m (the number of deaths per head) to increase (for similar reasons). The simplest form for declining f is

$$f = A - Bp \qquad (A, B > 0, \text{ constants})$$

Likewise the simplest form for increasing m is

$$m = C + Dp \qquad (C, D > 0, \text{ constants})$$

These give

$$b = fp = (A - Bp)p, \qquad d = mp = (C + Dp)p$$

and so

$$\begin{aligned} \frac{dp}{dt} &= p\left[(A - C) - (D + B)p\right] \\ &= p\left[\alpha - \beta p\right] \end{aligned}$$

where $\alpha = A - C$ and $\beta = D + B$ are constants. Separation of variables and partial fractions give the solution to this equation as

$$p = \frac{\frac{\alpha}{\beta}}{Ee^{-\alpha t} + 1}$$

where E is a constant determined by reference to the population at time $t = 0$. (See figure 13.)

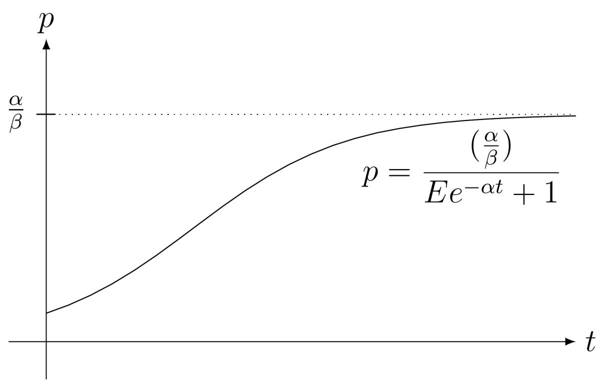

Figure 13: Solution of the logistic equation.

For simple experiments (e.g. growth of yeast in a culture medium) and for suitable choice of the constants α, β this solution is in good agreement with the experimental data.

(At the end of this section, check that you can solve the three differential equations we obtained as mathematical models.)

12 Summary

When you have completed this package, you should be able to do the things listed below:

1. recognise a **first order differential equation,**
2. sketch the **isoclines** and **direction field** for a first order differential equation,
3. obtain an approximate numerical solution using **Euler's method**,
4. recognise **separable** and **linear** equations,
5. **solve separable equations** by separating the variables,
6. **solve linear equations** by computing an appropriate integrating factor,
7. solve some other types of first order equations by reducing them to either separable or linear form,
8. understand how first order equations arise from **one-parameter families of curves**,
9. explain how first order equations can be used to **model birth/death processes**.

13 Bibliography

For textbooks covering basic differentiation and integration see, for example, one of the following (although there are dozens of other suitable textbooks many of which are in the University library).

Jeffrey, A. Mathematics for Engineers and Scientists, Van Nostrand, 1989.

Thomas, G. B. and Finney, R. L. Calculus and Analytic Geometry, Addison-Wesley, 1988.

Larson, R. E., Hostetler, R. P. and Edwards, B. H. Calculus, D. C. Heath and Company, 1990.

Gilbert, J. Guide to Mathematical Methods, Macmillan, 1991.

Anton, H. Calculus with Analytic Geometry, Wiley, 1992.

There are many textbooks which cover first order differential equations. Indeed, some of the books listed above give an elementary treatment. Those listed below are devoted exclusively to the subject of differential equations and they give a great deal more detail.

Sanchez, D. A., Allen, R. C. and Kyner, W. T. Differential Equations (second edition), Addison-Wesley, 1988.

Zill, D. G. A first course in Differential Equations with Applications (fourth edition), Prindle, Weber, Schmidt - Kent Publishing Company, 1989.

Boyce, W. E. and DiPrima, R. C. Elementary Differential Equations and Boundary Value Problems (fifth edition), Wiley, 1992.

Picard's theorem mentioned in section 4 of these notes is proved in appendix 1 of the book by Sanchez et al. and in chapter 2 of the book by Boyce et al. Zill does not offer a proof. All three volumes give details of other, better, numerical techniques than Euler's method. Of the three books listed, you would probably find the volume by Boyce et al. to be too advanced for general use. All three volumes cover a much wider range of topics than this package - for example second order differential equations, Laplace Transforms, partial differential equations and Fourier series. In addition to these three books there are very many other textbooks covering first order equations and, again, many of these are in the University library. You need never be short of an alternative approach or more questions to try!

14 Appendix - Video Summaries

There are three videos associated with the topic of first order differential equations. The presenter is Mike Grannell from the Department of Mathematics at the University of Central Lancashire. We suggest that you read the subsection about the videos in the preamble to these notes. This makes some suggestions about how you should approach viewing the videos.

Video title: First Order Differential Equations (part 1).
(33 minutes)
Summary

1. **Examples** of differential equations.
2. **What is meant by a solution**

   Explicit, implicit and numerical solutions.
3. **The order of a differential equation**

   The number of constants in a solution.
4. **How differential equations arise**

   Growth of bacteria.
5. **One-parameter families of curves**

   The family $y = ax^2$, the corresponding differential equation $y' = 2y/x$.
6. **Direction Fields**

   General method. The particular case of the differential equation $y' = 2y/x$. Obtaining the solution curves from the direction field. A further example: $y' = \sin(xy)$. Existence of solutions.
7. **Isoclines**

   General method. The particular case of the differential equation $y' = x^2 + y^2$.

Video title: First Order Differential Equations (part 2).
(22 minutes)
Summary

1. **Euler's numerical method**

   General theory. The particular case of the differential equation $y' = x^2 + y^2$, $y(0) = 1$.

2. **Separable equations**

   General theory. The example $y' = e^{x+y}$.

Video title: First Order Differential Equations (part 3).
(25 minutes)
Summary

1. **Linear Equations**

   The standard form $\frac{dy}{dx} + P(x)y = Q(x)$. Integrating factors. The example $(\cos x)y' + (\sin x)y = x(\cos x)^2$.

2. **Determining which equations are separable and which are linear**

   The examples

$$\begin{aligned}
(\sin y)^2 y' + \cos x \cos y &= \cos(x+y) \\
y' - x^2 &= y^2 \\
yy' - xy &= y^2 \sin x \\
y' &= (x+1)(y+1)
\end{aligned}$$